5.5 Additional Applications of Integration

Ex 1 For $D(q) = \dfrac{300}{(0.1q+1)^2}$ (\$/unit)

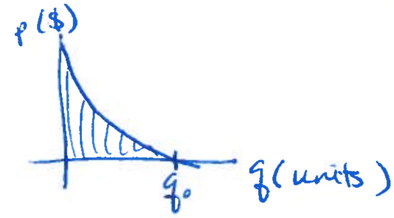
Find total \$ consumers are willing to spend to obtain 5 units of this commodity. (sketch curve)

Consumer Willingness to Spend

for up to q_0 units of a commodity is given by

$$WS = \int_0^{q_0} D(q)\,dq$$

where $p = D(q)$ = demand fn



5.5 (cont)

Ex 2 Given $p = D(q) = \sqrt{245 - 2q}$ (demand)

$p = S(q) = 5 + q$ (supply).

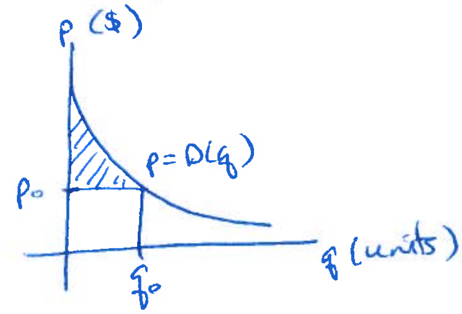
(a) Find equilibrium price p_e (where supply = demand)

(b) Find consumers' surplus + producers' surplus at equilibrium.

Consumers' Surplus

If q_0 units of a commodity are sold at price p_0 & if $p = D(q)$ = demand fn, then

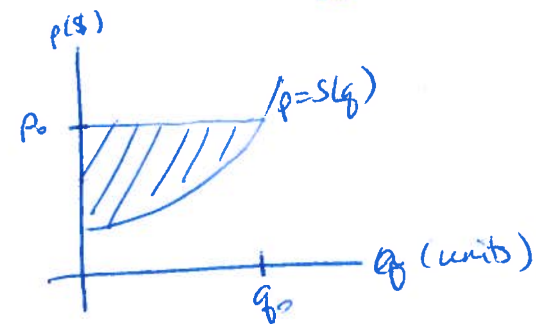
consumers' surplus $= CS = \int_0^{q_0} D(q)\,dq - p_0 q_0$



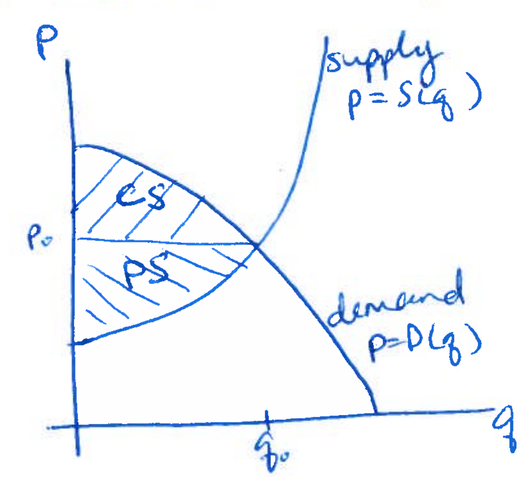
Producers' Surplus

$p = S(q)$ = supply fn.

$PS = p_0 q_0 - \int_0^{q_0} S(q)\,dq$



Equilibrium (for surplus)



5.5 (cont)

Ex 3 Money is transferred continuously into an acct at constant rate of \$2400/yr. The acct earns interest at annual rate of 6% (compounded continuously). How much will be in the acct at end of 5 yrs?

$(r = 0.06,\ f(t) = 2400,\ T = 5)$

Future Value of Income Stream

Suppose money is transferred continuously into an acct. over a time period $t \in [0, T]$ at a rate given by fn $f(t)$ & that the account earns interest at an annual rate r compounded continuously. Then the FV is given by

$$FV = \int_0^T f(t)\, e^{r(T-t)}\, dt$$

$$= e^{rT} \int_0^T f(t)\, e^{-rt}\, dt$$

6.3 Improper Integrals

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty} \int_a^N f(x)\,dx$$

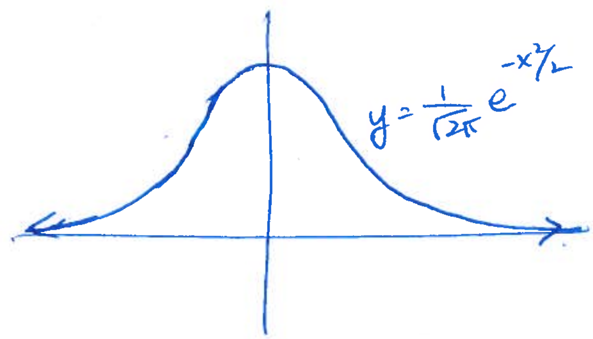
if limit exists, integral **converges**
if limit DNE, integral **diverges**.

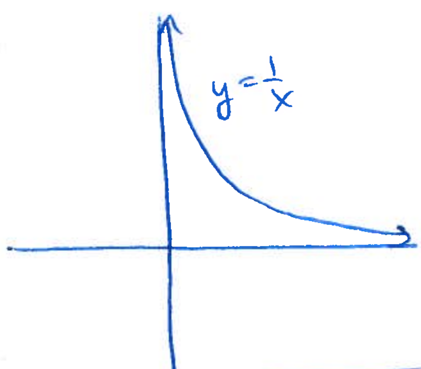
$$\int_{-\infty}^\infty f(x)\,dx = \int_c^\infty f(x)\,dx + \int_{-\infty}^c f(x)\,dx$$

If **both** $\int_c^\infty f(x)\,dx$ **and** $\int_{-\infty}^c f(x)\,dx$ converge, then $\int_{-\infty}^\infty f(x)\,dx$ converges. Otherwise, $\int_{-\infty}^\infty f(x)\,dx$ diverges.

(c is just a finite # of your choice, usually 0)

The real question is: is the area under an infinite curve finite or infinite? Strangely enough, sometimes it is finite, if the curve approaches the x-axis "fast enough."





Ex 1 Evaluate: $\int_1^\infty \frac{1}{x^4}\,dx$

6.3 (cont)

Ex 2 Evaluate.

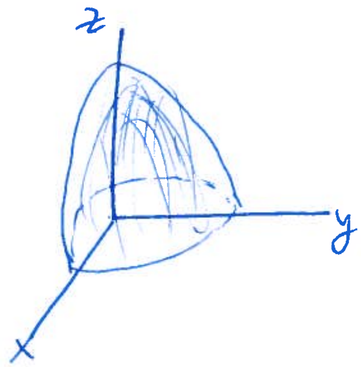
(a) $\displaystyle\int_0^\infty x e^{-x^2}\,dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^{3/2}}\,dx$

7.1 Functions of Several variables

$z = f(x,y)$ = a fn of 2 independent variables x and y & returns another number. (This fn can be graphed in 3-d.)

- so (x,y) ordered pair is an input; and z is output



ex
$z = f(x,y) = 9-(x^2+y^2)$

Ex 1 Compute the indicated fn value.

(a) $f(x,y) = \dfrac{e^{xy}}{2-e^{xy}}$ at $(1,0)$ and $(\ln 2, 2)$

(b) $f(x,y,z) = xye^{z} + xze^{y} + yze^{x}$ at $(1,1,1)$ and $(\ln 2, \ln 3, \ln 4)$

Ex 2 Describe domain, for $f(x,y) = \dfrac{x}{\ln(x+y)}$

Ex3 Sketch a level curve for
$f(x,y) = \ln(x^2+y^2)$ at $C=4$, and at $C=\ln 4$

A level curve

is basically a 2-d cross-section of a 3-d surface at a particular z-value.
(it's like a topographical map)

7.2 Partial Derivatives

If $z=f(x,y)$, $\frac{\partial z}{\partial x} = f_x(x,y)$ is called partial derivative of wrt (with respect to) x, and $\frac{\partial z}{\partial y} = f_y(x,y)$ is partial derivative of f wrt y.

$\frac{\partial z}{\partial x}$ = derivative of $z=f(x,y)$, pretending like x is the only variable + y is constant.

Ex 1 For $f(x,y) = \frac{xy^2}{x^2y^3+1}$, find f_x and f_y.

Note: In general, for nice fns, $f_{xy} = f_{yx}$ (i.e. their mixed partial derivatives are equal)

2nd order Partial Derivatives

$z=f(x,y)$

$$f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

(take derivative wrt x first & then y)

$$f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

(take derivative wrt y first + then x)

$$f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Ex 2 For $f(x,y) = (x-2y)^2 + (y-3x)^2 + 5$, find $f_x(0,-1)$ and $f_y(0,-1)$.

Ex 3 For $f(x,y) = x^2 y e^x$, find all second partial derivatives.

7.2 (cont)

Ex 4 If $z = x^{1/2} y^{1/3}$, $x = 2t$ and $y = 2t^2$, compute $\frac{dz}{dt}$.

Ex 5 weekly output

$Q(x,y) = 1175x + 483y + 3.1x^2y - 1.2x^3 - 2.7y^2$

$x = \#$ skilled workers, $y = \#$ unskilled workers

Right now, $x = 37$ and $y = 71$. What is approximate change in Q if we add 3 more skilled workers + decrease unskilled workers by 1?

Chain Rule for Partial Derivatives

Suppose $f(x,y) = z$ but $x = x(t)$ and $y = y(t)$.

Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Δz approximation

$z = f(x,y)$

$\Delta x =$ small change in x

$\Delta y =$ " " " y

Then

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

☆ This is 3-d analog of the differential approximation we've seen several times before this.