# 8 Fraction Book

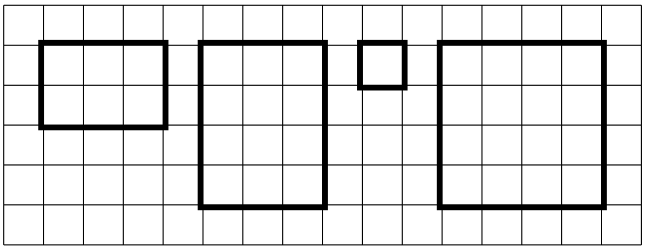
## 8.1 About this part

This book is intended to be an enjoyable supplement to the standard text and workbook material on fractions. Understanding why the rules are what they are, and why they work, is this books primary goal. Beginning chapters are written at an advanced second grade or third grade level. Later materials may follow immediately, or be used also in fourth, fifth, and sixth grades.
Material is developed in a logical sequence and presupposes only knowledge of basic arithmetic of small whole numbers. Some review of previous concepts is built into those that follow.
Each child will need a soft pencil with a good eraser, a straight-edge, and plenty of evenly lined graph paper. An adult guide should slowly read and reread each page or concept, allowing the children time to make the figures on a separate sheet of graph paper. The idea is that each child makes as many figures as possible – those shown in the book, those asked by the book, and as many of their own conception as they can. The pace should be leisurely, and the children should be given plenty of time and hints to complete all the questions and exercises in each chapter. Also the adult guide or, preferably, the children themselves should make up supplementary exercises like those in the particular chapter. Each child should be comfortable with the concepts in the chapter before going on to the next. There is little point in proceeding further until the individual indicates readiness.
These materials were developed in, and have been used for several years by, the Open Classroom Program of the Salt Lake School District.

Herb Clemens

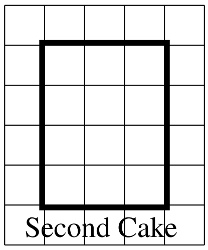
## 8.2 Pieces of Cake

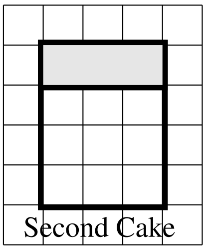
Here are some cakes:



Some of these cakes are bigger than others. The first cake has 6 pieces, the second has 12 pieces and the third has only one piece. How many pieces does the fourth one have?
One thing about all our cakes – they are all cut into pieces for us. We won't be allowed to cut up any of the pieces into smaller ones. NO MORE CUTTING!
Suppose we're sitting around the dinner table, everybody has dealt with their vegetables, and now it's time for dessert. Were having cake for dessert and everybody likes cake a lot! Nobody is feeling particularly generous, so were going to have to give exactly the SAME amount of cake to each person at the table.
Let's suppose there are four people at the table. If we bring the first cake from the kitchen and serve it, then each of the four persons at the table gets a piece and there are two pieces left over. We want to eat all of that cake while it's nice and fresh, but, remember, nobody's feeling particularly generous, so you say that we had better cut each of those last two pieces in half and give half a piece to each person. But we cant do that! We had made a strict rule that we cant cut the pieces into smaller ones: NO CUTTING!
So we'd better take these two left over pieces back to the kitchen. But that's not good either. Everybody at the table loves the cake, and so everybody is going to be very unhappy knowing that there is cake still on the plate but they cant have any of it.
We're in trouble! What can we do so that everybody is happy? What would you do?

Here is an idea – let's not serve the first cake at all. Lets leave it in the kitchen where no one can see it, and nobody will even know it's there. Instead, we'll serve the second cake:

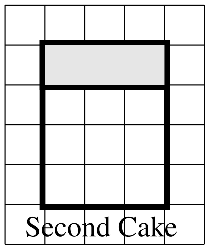



This is how much cake each person gets.

Now everything works out just fine! Each person gets three pieces of cake, so nobody's jealous, and there's no cake left over, so nobody's unhappy at seeing left-over cake that they cant have.

We say that the part of the cake that each person gets is one-fourth of the cake. One-fourth is how much of the cake each person gets when there are four people at the table, and each gets the same amount.

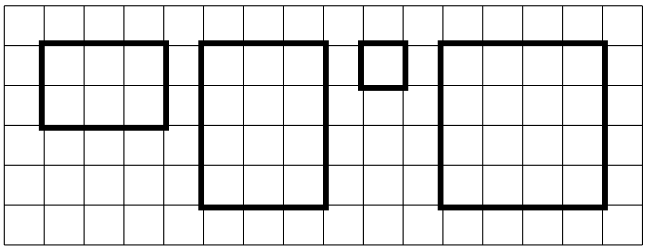


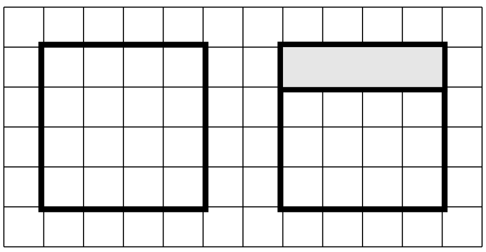

One-fourth of the cake.

Instead of writing out "one-fourth of the cake" each time, we'll make things easier by using numbers. We'll write:

$$\frac{1}{4}$$

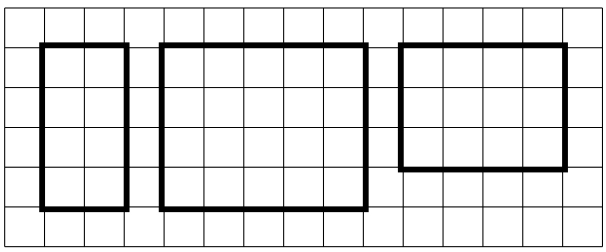
We could have chosen to serve a different cake instead of the second one. Remember the four cakes we had to choose from?



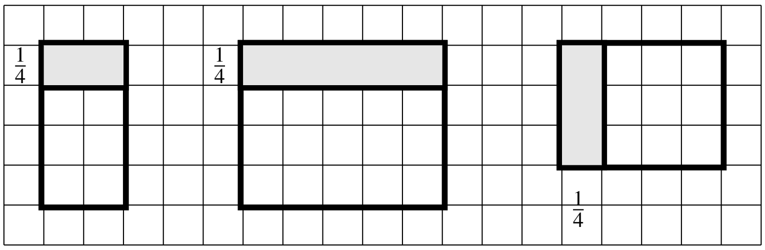


If we had used the last cake, each person would get four pieces, again with nothing left over. Again, each person would get one-fourth of the cake, because one-fourth is how much each person gets when there are four people at the table.

Here are some other cakes that would work for four people:
Can you shade in one-fourth of each of these cakes? That is, can you shade in how much of each cake would one person get if there are four people at the table? Try it, then go on for a bunch of pictures of one-fourth.



Let's make a whole page of one-fourths:

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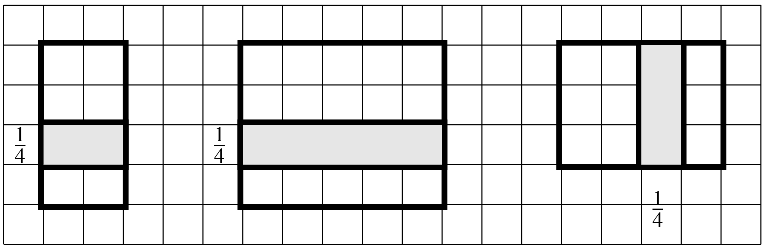


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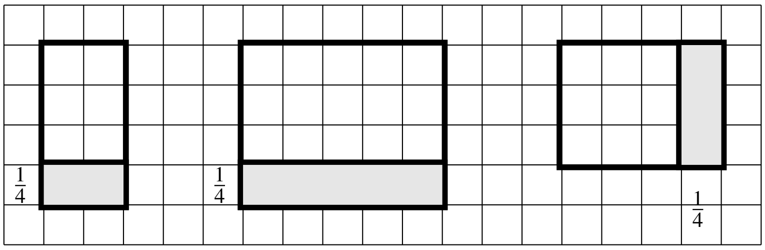
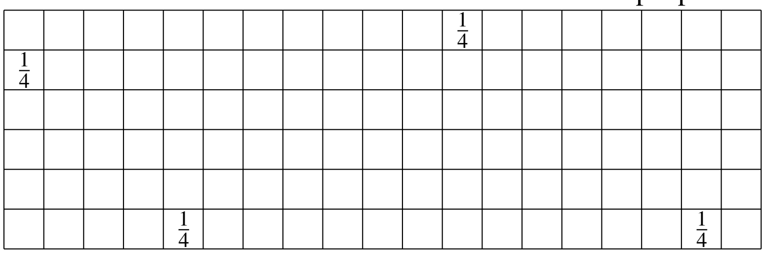
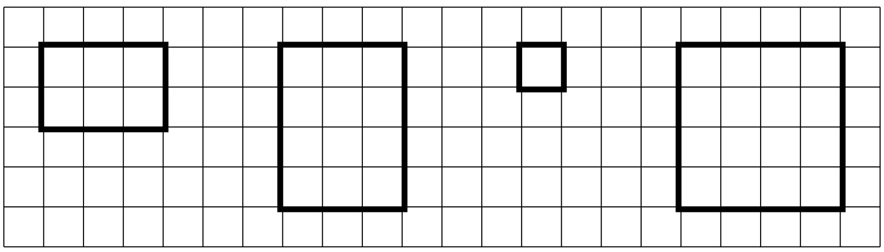



Or else (draw some cakes that would work when there are four people at the table):



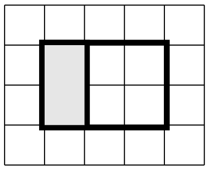

## 8.3 Dinner for Three

It is another day in another house, but someone has been baking again:

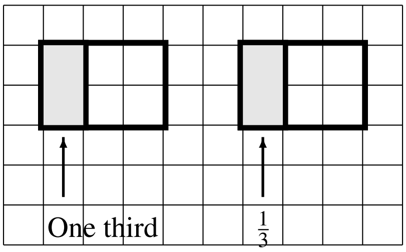


But today at this house there are three people at the table. Which of these cakes can we serve in this house today? Remember, we want to give the same number of pieces to each person at the table, and we don't want to have any pieces of the cake leftover. Also, these cakes are already cut into square pieces for us. We can't cut the pieces any more. No cutting! So, which of these cakes can we serve?

If you answered that we could serve the first cake, you would be right. If you answered that we could serve the second cake, you would be right. And if you answered that we could serve either of the first two cakes, you would be exactly right! Suppose we serve the first cake. Each person would get two pieces.

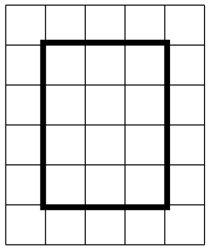


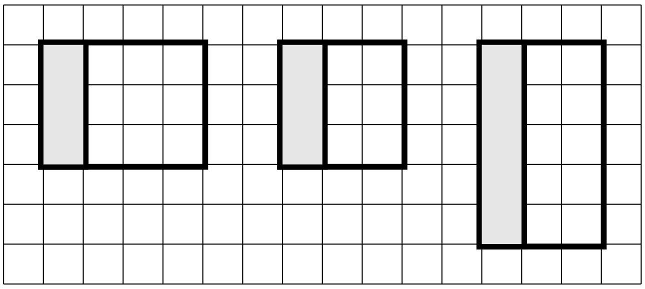
This is how much of the cake we give to each of the 3 people at the table




We say that the amount each person gets is one third of the cake. One third is the part of the cake that each person gets when there are three people around the table and each person gets equal amount of cake.

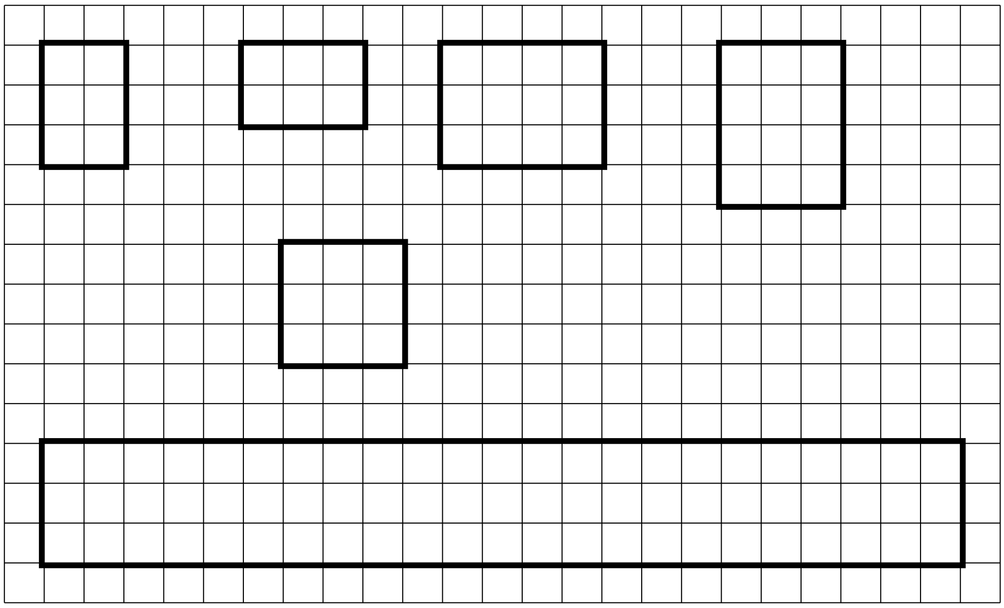
Suppose we had used the second cake instead
How many pieces of cake should each of the three people at the table get? Shade in one third of this cake.



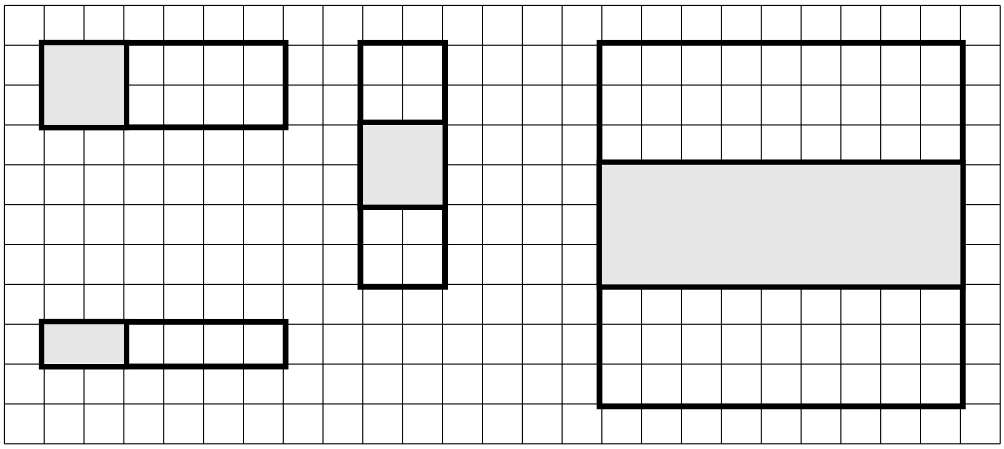


Here are some more pictures of one third.

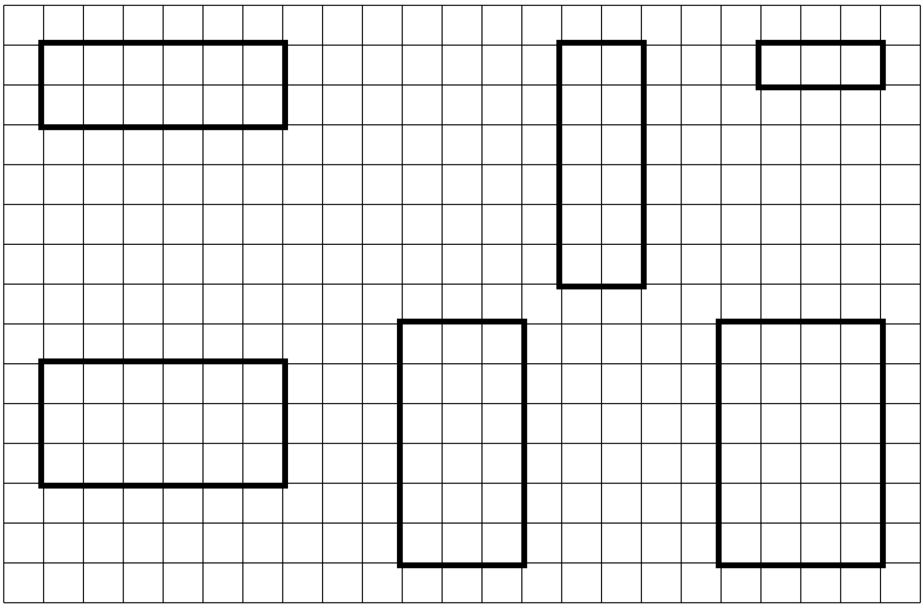
Remember, a part of the cake is called one third if it how much of the cake one person would get when there are three people at the table. Shade in one third of each of these cakes:



So now you have many different pictures of one third!
Remember the only thing it takes to be one third is to be how much one person gets when there are three people at the table. So here are some more pictures of one third:

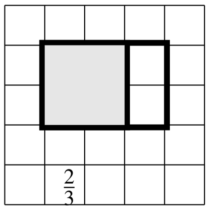


Now you shade in one third of each cake:

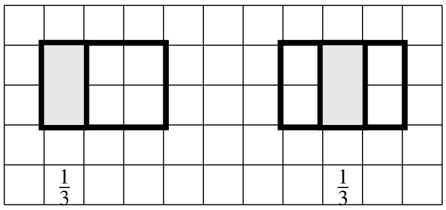


These are all pictures of one third (if you did them correctly). The cakes have many sizes and shapes. But they all have one thing in common. They are cakes which you can divide up equally between three people.

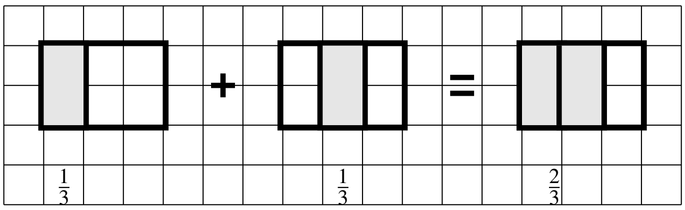
Now we know about one third. That's how much cake one person gets when there are three people at the table. It's easy to say what two thirds is. Two thirds of the cake is how much two people get altogether when there are three people at the table:



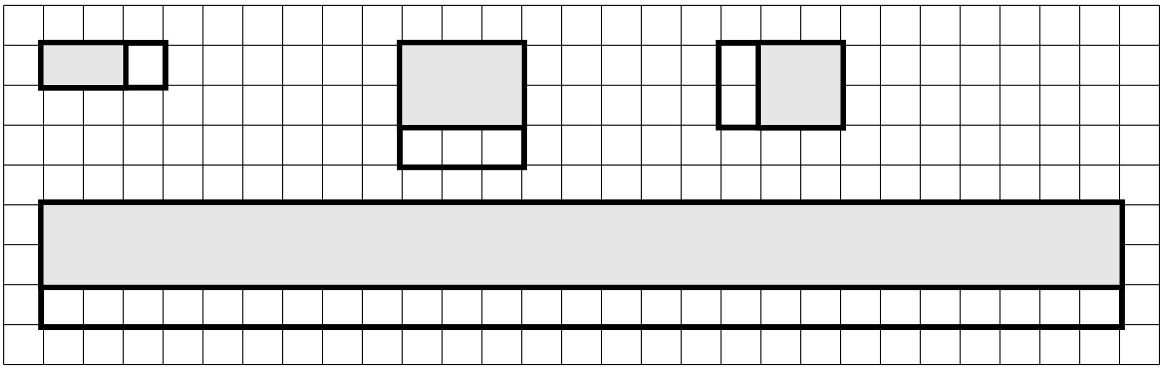

Each person gets one third of the cake:




So if we put together the cake that two people get:

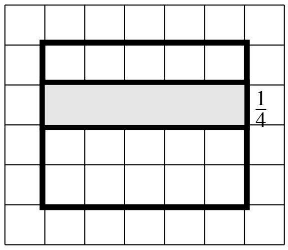



Two thirds of the cake is how much of the cake two people would get altogether when there are three people at the table.
Here are some other pictures of two thirds:

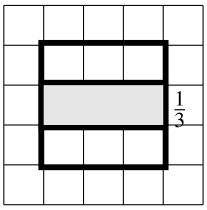


### 8.4 Fractions

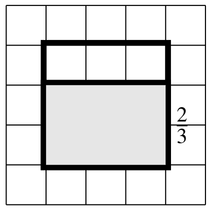
These parts of the cake we've been shading in are called fractions of the cake. Up to now we've seen one fourth of a cake:




and one third of a (different) cake:

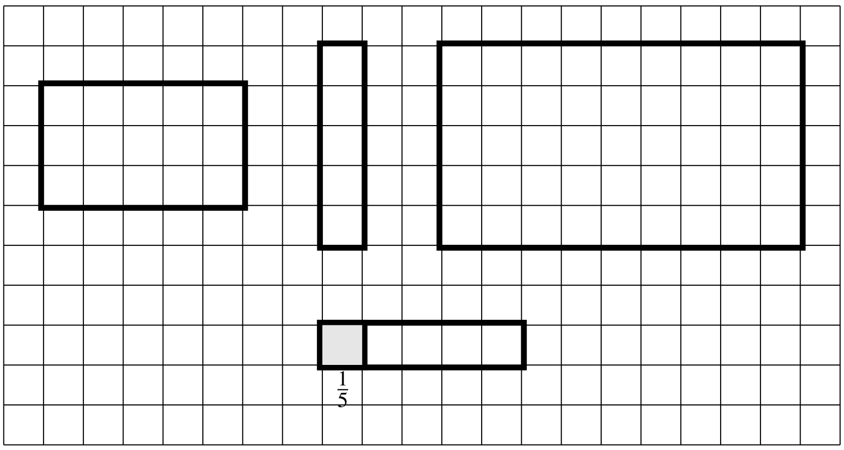



and two thirds of a cake:

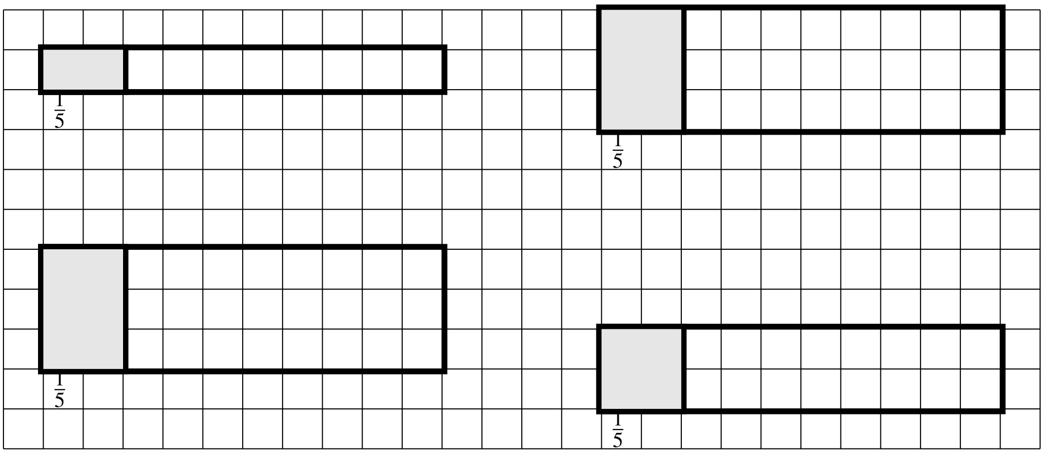



Now it'ss not too hard to make a lot of other fractions.

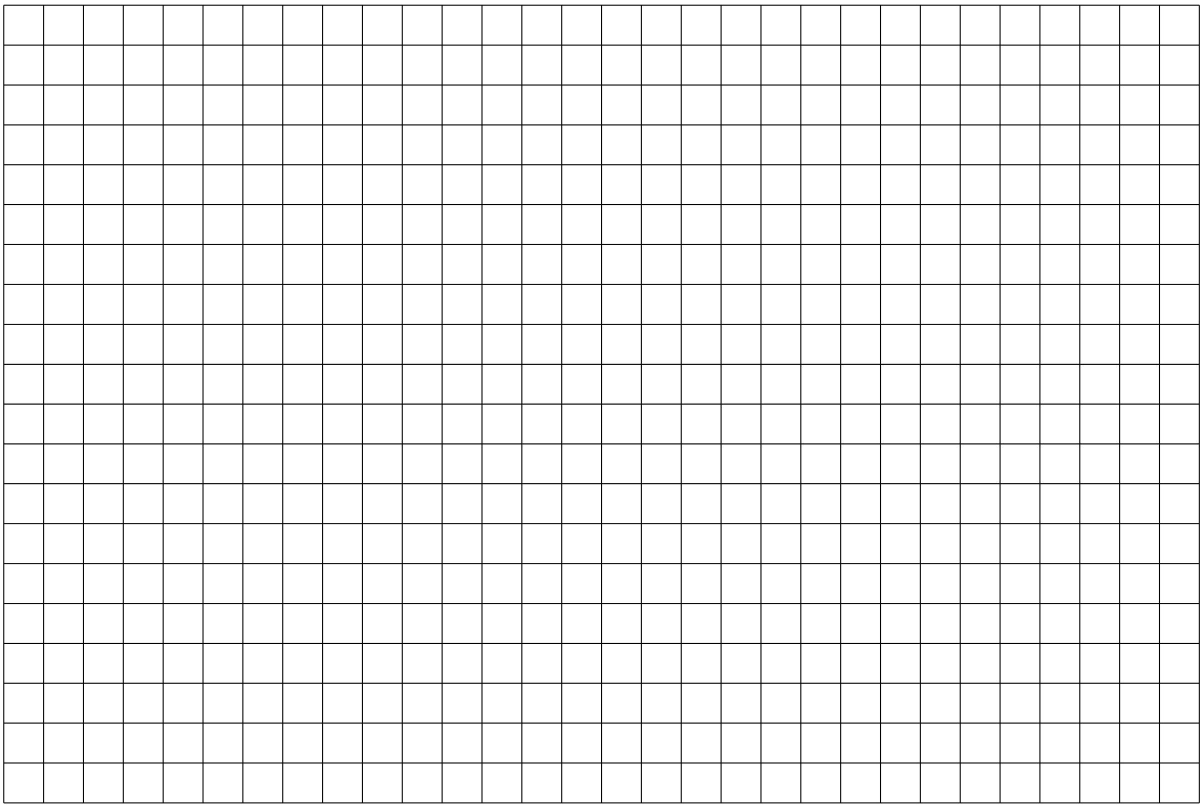
Suppose there are five people at the table. What size of cake would work? Here are some answers:
In each of these cakes, shade in what one person would get if there were five people at the table. That is, shade in one fifth of the cake.



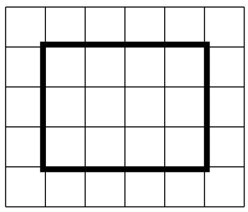

Remember, all it takes to be one fifth of a cake is to be how much one person gets when there are five people at the table. Here are some other pictures of one fifth:




Why don’t you draw some of your own pictures of one fifth below?



Maybe when you were drawing right now you remembered that some cakes just won’t work when there are five people at the table. Here is a cake that won’t work:



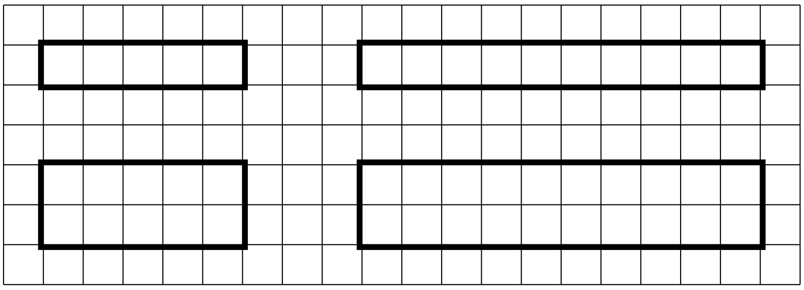
This cake has 12 pieces. If we want to use it to serve desserts to five people, we start out by giving each person two pieces. How many pieces have we served so far?

$$2+2+2+2+2=10$$

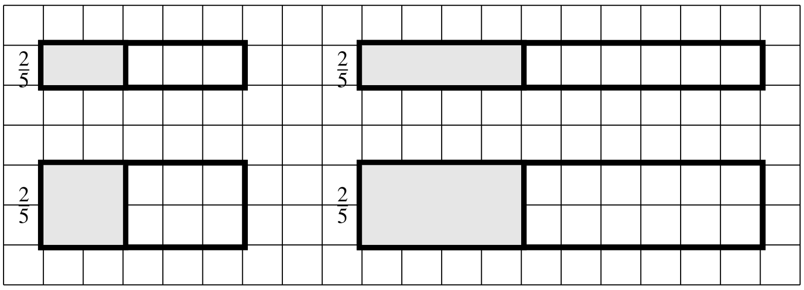
But there's cake leftover, and we can't use it, because there is not enough leftover to give each person another piece. Let's try again. Suppose we start off giving each person three pieces:

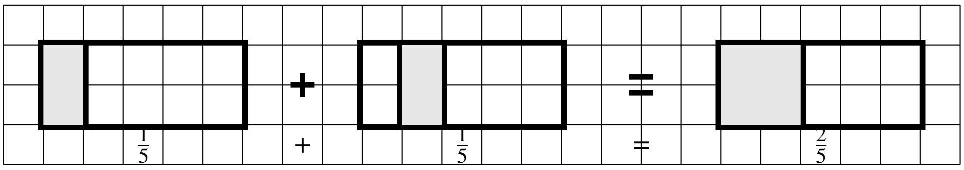
$$3+3+3+3+3=15$$

No, we can't do that – there's not enough cake.
But there are plenty other cakes that will work for five people:

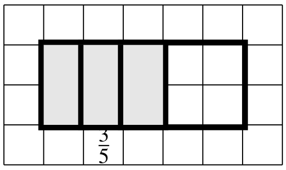


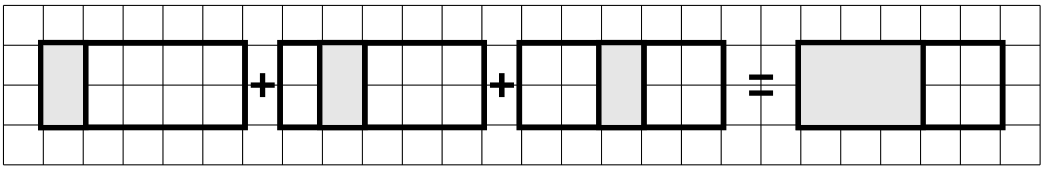
We could shade in one fifth of each of these cakes, but we're not going to do it. We're going to shade in two fifths of each cake. We're going to shade in how much of each cake two people would get altogether if there were five people at the table:



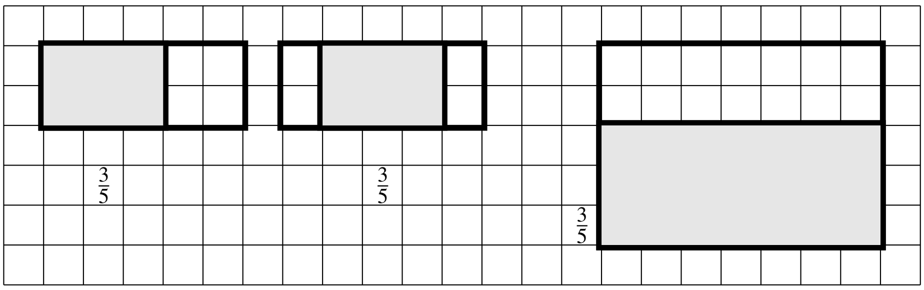




We can go on with this. Let's do three-fifths – that's how much *three* people get altogether when there are five people at the table:

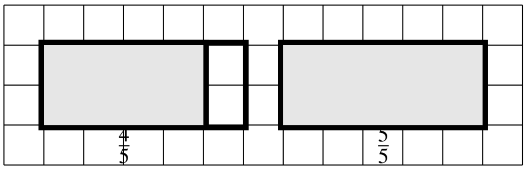



Here are some other pictures of three-fifths. Remember, a part of the cake is called three-fifths whenever it is how much of the cake *three* people get altogether when there are *five* people at the table:



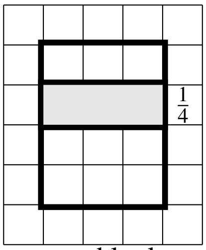
And here are some other fractions:



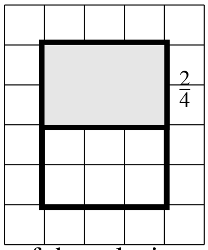

Five-fifths of the cake is how much five people get altogether when there are five people at the table. Five-fifths of the cake is the whole cake!

## 8.5 ...and More Fractions

Let's keep going:




How much cake one person gets when there are four people at the table.

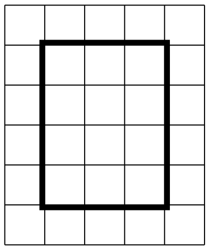


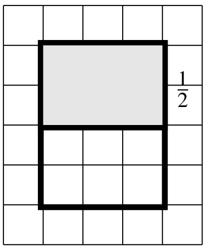

How much cake two people get altogether when there are four people at the table.

Take a good look at two-fourths of the cake – two-fourths of the cake is exactly one-half of the cake. We can write

$$\frac{2}{4} = \frac{1}{2}$$

Let's check this. Here's the cake. Suppose there are only two people at the table. How much does one of them get?





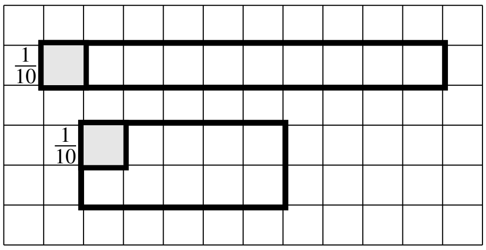

How much one person gets when there are two people at the table. So this is one-half of the cake.

Look at this picture of one-half of the cake. It is exactly he same as the picture of two-fourths of the cake that we drew higher up on the page. That's why we can write:

$$\frac{1}{2} = \frac{2}{4}$$

Maybe, when you first saw $\frac{1}{2}$, you wanted to say "one-twoth". After all, $\frac{1}{4}$ is one-fourth, and $\frac{1}{5}$ is one-fifth. Well, it really does make sense to say "one-twoth," but for some reason people decided to say "one-half" instead. Since they decided that many hundreds of years ago, and since everybody says "one-half" now, we had better say "one-half" too.

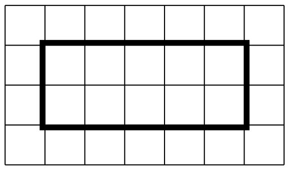
Here are some more fractions:




One-tenth of the cake, that's how much one person gets when there are ten people at the table.

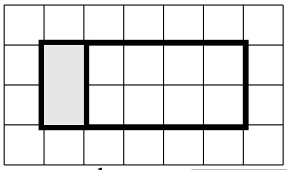
This last cake has a different shape from the last one just above it, but each of those two cakes has exactly ten pieces in it. Each of the two cakes has exactly ten pieces in it. Each of the two cakes has exactly the same number of pieces. So it really doesn't matter which of the two cakes we serve. As long as there are ten people at the table, each person will get exactly one piece.

Let's look at this last cake again:

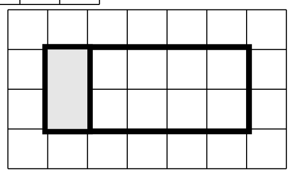


There are ten people at the table. How much of this cake will two people get altogether? Well, the answer is:

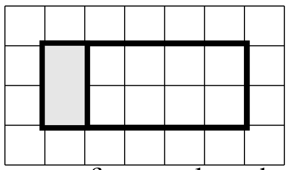
"This much!"



So "this much" must be two-tenths, because that's how much two people get altogether when there are ten people at the table. But this picture already has another name. Look back through some of the pages you have already studied. This may help you remember.



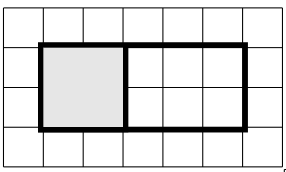
This is also a picture of one-fifth! Why?



That's because it is how much of the cake one person would get when there are five people at the table. Two-tenths of the cake is the same-sized part of the cake as one-fifth of the cake. So we can write

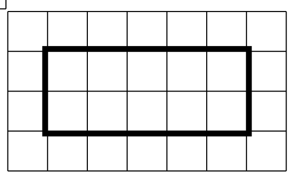
$$\frac{2}{10} = \frac{1}{5}$$

Let's do one more. Here's a picture of four-tenths of the cake:

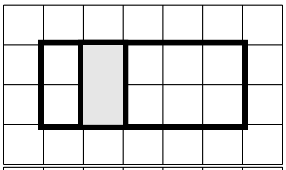


Why is this four-tenths?

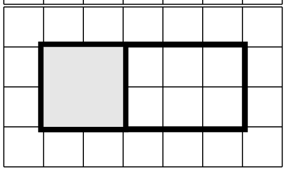
But if we take the *same* cake



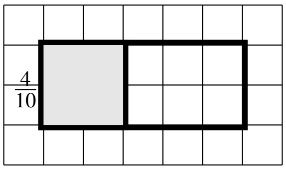
and now suppose there are only five people at the table; each person gets this much:

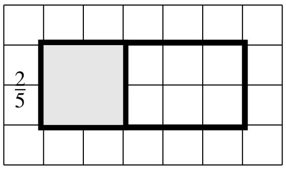


So *two* people altogether get twice as much:



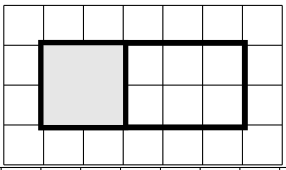
They get two-fifths of the cake. So, four-tenths of the cake is the same amount as two-fifths of the same cake.



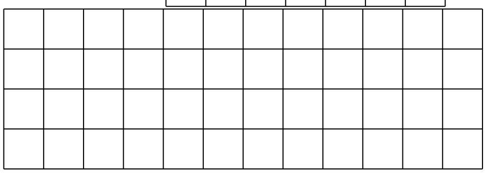




So we can write $\frac{4}{10} = \frac{2}{5}$.

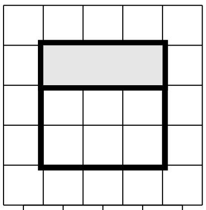
This picture shows why $\frac{4}{10} = \frac{2}{5}$.



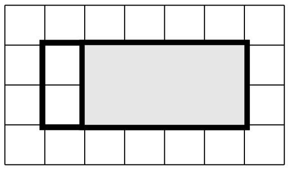
Can you draw a picture that shows why $\frac{6}{10} = \frac{3}{5}$?



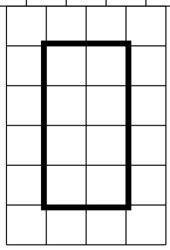
Fill in the pictures or the equal fractions:



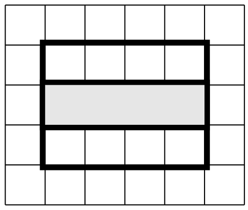
$\frac{3}{9} =$



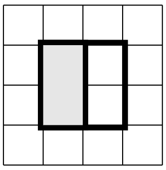
$\frac{8}{10} =$



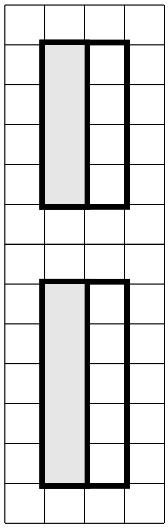
$\frac{1}{4} =$



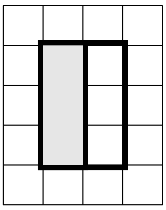
$=$



$\frac{1}{2} =$



$\frac{1}{2} =$

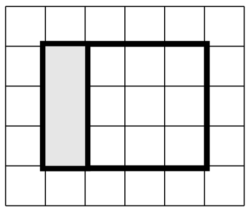


$\frac{1}{2} =$

$\frac{8}{10} =$

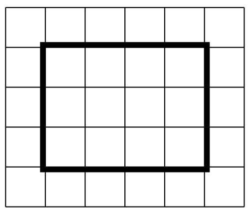
## 8.6 Different Ways to Do the Same Thing

Here's a fraction of a cake:

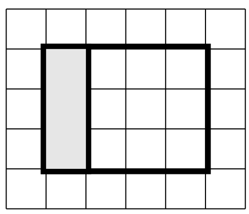


Which fraction of the cake is it? Well, if there are four people at the table, we would say that this is one-fourth of the cake, since it is how much of the cake one person would get if there were four people at the table.

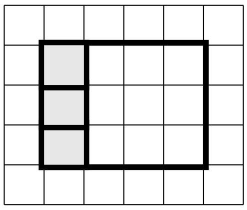
But let's look at the same cake again:



Suppose there are twelve people at the table. We can use this same cake to feed twelve people – we would simply give one piece to each person. Now look back at the picture at the top of the page



When there are 12 people at the table, what fraction is this a picture of? It is a picture of three-twelfths, because it is a picture of how much three people would get if there were 12 people at the table.



But the picture of three-twelfths of this cake is exactly the same as the picture of one-fourth of this cake at the top of the page. This means that

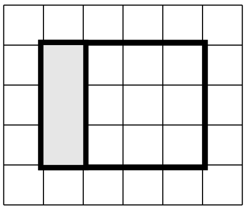
$$\frac{1}{4} \text{ of the cake}$$

and

$$\frac{3}{12} \text{ of the cake}$$

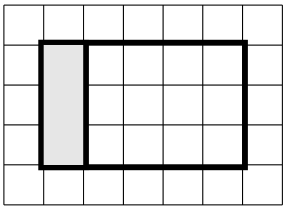
are exactly the same part of the cake!
Let's say the whole thing this way:

"The picture



tells us that $\frac{1}{4} = \frac{3}{12}$."

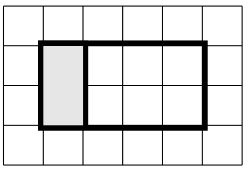
Now let's do another one:
What does this picture tell us?



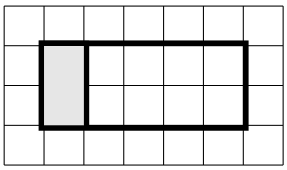
If you said,

$$\frac{1}{5} = \frac{3}{15},$$

you are right! 15 is the number of pieces in the whole cake. (Remember, when you are counting the number of pieces in the whole cake, you have to count the shaded pieces, too. The shaded pieces and the unshaded ones are all part of the cake.)
Here are two more pictures. Try to figure out what each one tells us:



=



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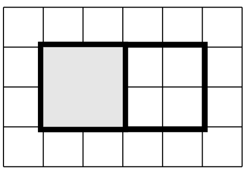
How about these?



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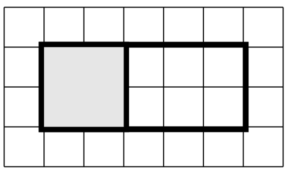

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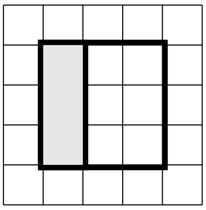
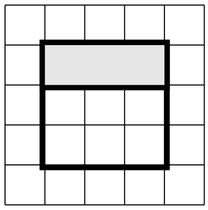
How about these two pictures?

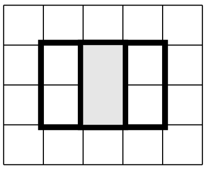




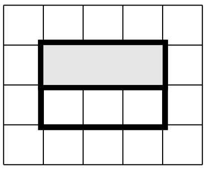
Each of these pictures tells us the same thing, namely

$$\frac{1}{3} = \frac{3}{9}.$$

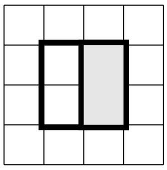
Out of the nine squares in the cake, we have shaded in three, so the shaded part is $\frac{3}{9}$.
But we have shaded in how much cake each person gets if there are three people at the table, so the shaded part of the cake is $\frac{1}{3}$.
Here are some other pictures. Fill in what they say about fractions:



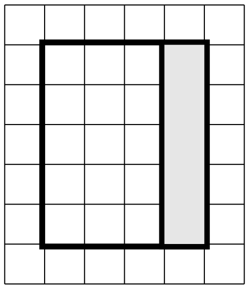
$\frac{1}{3} =$



$\frac{1}{2} =$

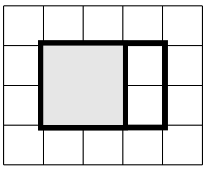


$\frac{1}{2} =$

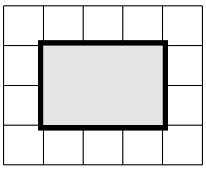


$=$

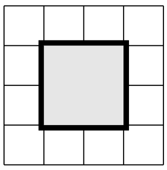
We can do the same thing for some of the other fractions we have been talking about:



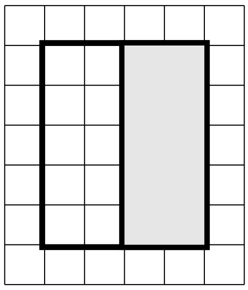
$=$



$1 = \frac{6}{6}$

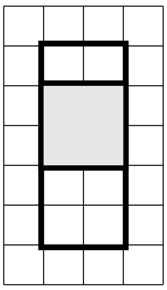


$=$

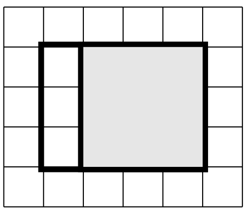


$=$

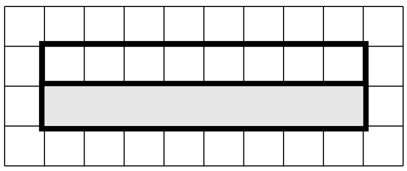
Here are some more for you to do:



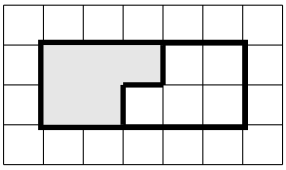
$=$



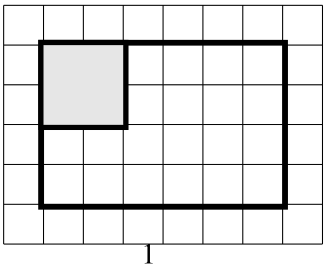
$=$



$=$



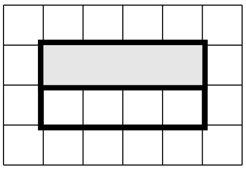
$\frac{1}{2} =$



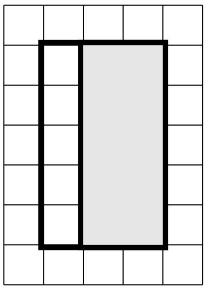
$\frac{1}{6} =$

Why is this a picture of $\frac{1}{6}$?

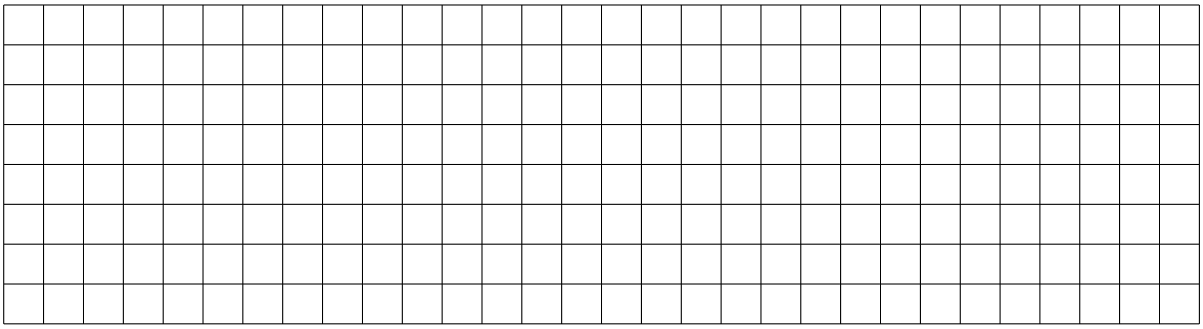
Now let's try to go in the opposite direction. If I ask you to draw a picture of $\frac{1}{2} = \frac{4}{8}$, you can draw:



For $\frac{2}{3} = \frac{10}{15}$, you can draw:



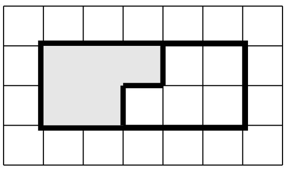
Make pictures for:



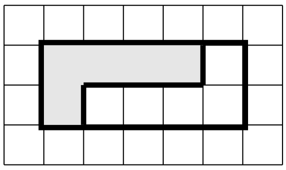
$\frac{3}{4} = \frac{6}{8}$ $\frac{3}{4} = \frac{9}{12}$ $\frac{3}{4} = \frac{12}{16}$ $\frac{3}{4} = \frac{15}{20}$

## 8.7 How much, not Where

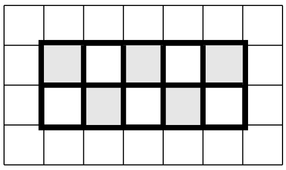
In the last lesson, we saw that



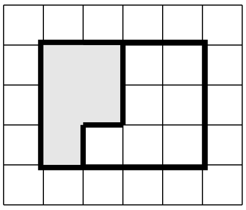
was a picture of one-half because it was how much of the cake one person would get if there were two people at dinner, and that it is also a picture of five-tenths because it is how much of the cake five people would get altogether if there were 10 people at dinner. So

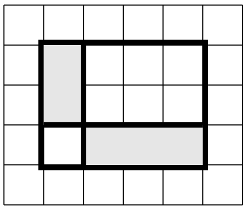


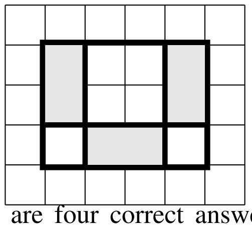
is also a picture of one-half because it is how much of the cake one person would get if there were two people at dinner, and it is also a picture of five tenths because it is how much of the cake five people would get altogether if there were 10 people at dinner. And



is also a picture of one-half because it is how much of the cake one person would get if there were two people at dinner, and it is also a picture of five-tenths because it is how much of the cake five people would get altogether if there were 10 people at dinner.
*Conclusion:* It doesn't really matter how the pieces of cake are arranged in the cake pan, only how much of the cake we have.
What fractions of the cake are shown in the pictures below?

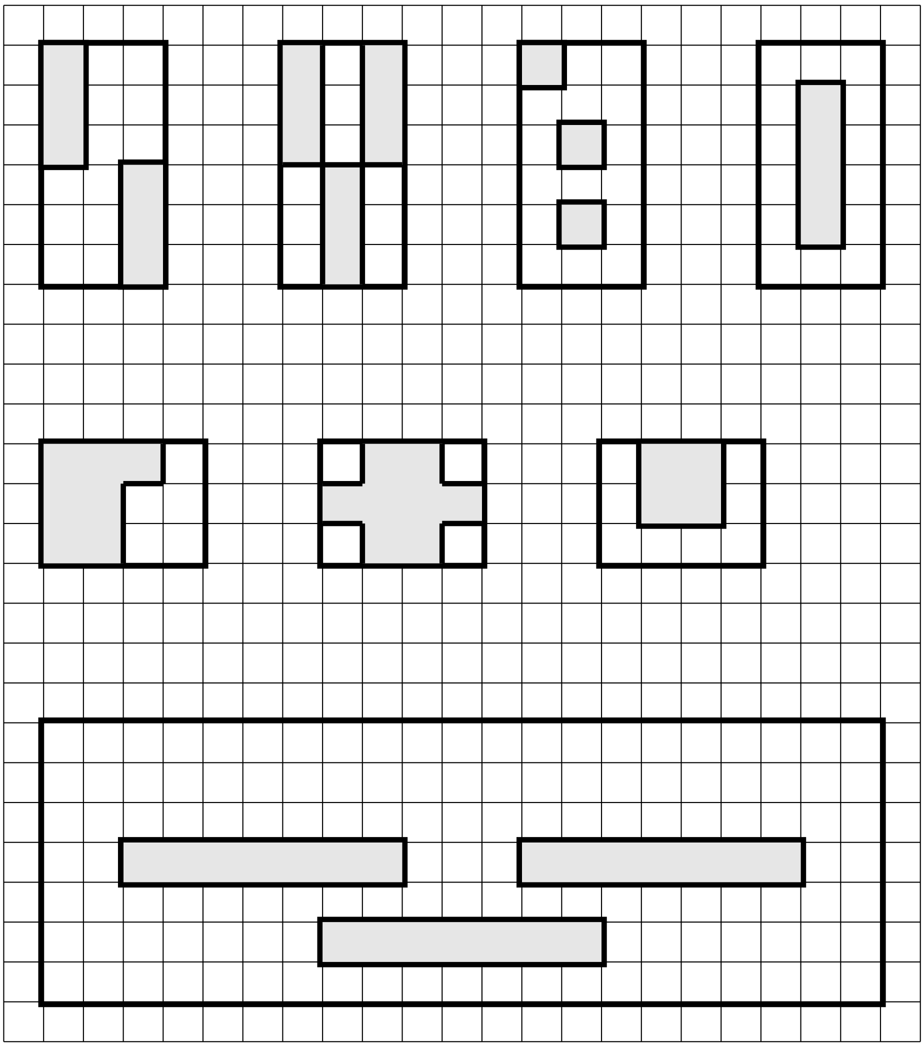






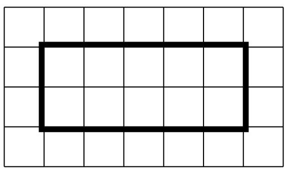
There are four correct answers to this one. Can you find them all?

Here are some more pictures of fractions for you to identify. Remember, our first picture had two answers, one-half and five-tenths, since both of these fractions represent the same part of the cake. Try to give all possible answers in the problems below:

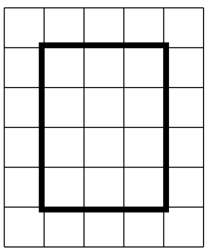


## 8.8 Different Parts of the Same Cake

We are now ready to try something new. Suppose it is the middle of the afternoon, and we have to get a cake ready for dessert tonight, but we don't know whether there will be *four* people or *five* people at dinner. What size cake should we bake now so that it can be divided up equally among all the people at the dinner table?

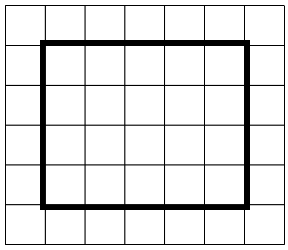
We might try this cake: 

This cake would work out fine if there are five people at dinner, but, if there are only four, we are in trouble!

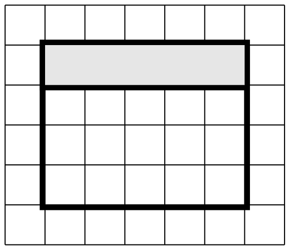
If instead we bake this cake: 

we’re in great shape if there are four people at the table tonight. But this time there will be big trouble if five come.

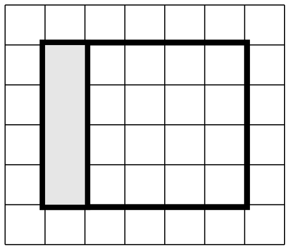
Well, there’s a way to solve this problem. We will bake this cake:



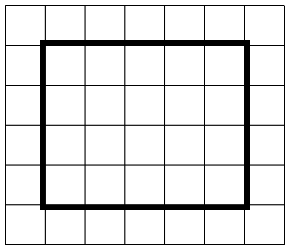
That way, if four people come to dinner, we can give each one this much:



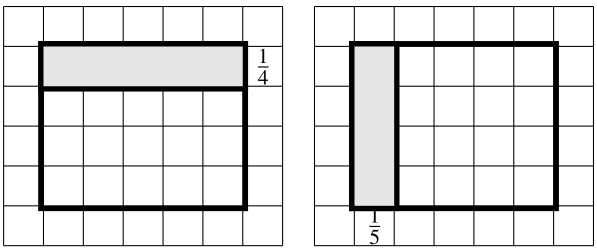
If instead five people happen to turn up for dinner, the *same* cake will still work. We will just give each person this much:



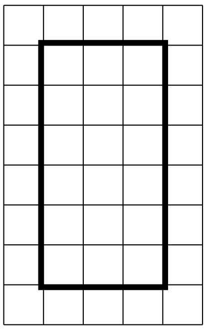
Another way to say it is this: In the *same* cake



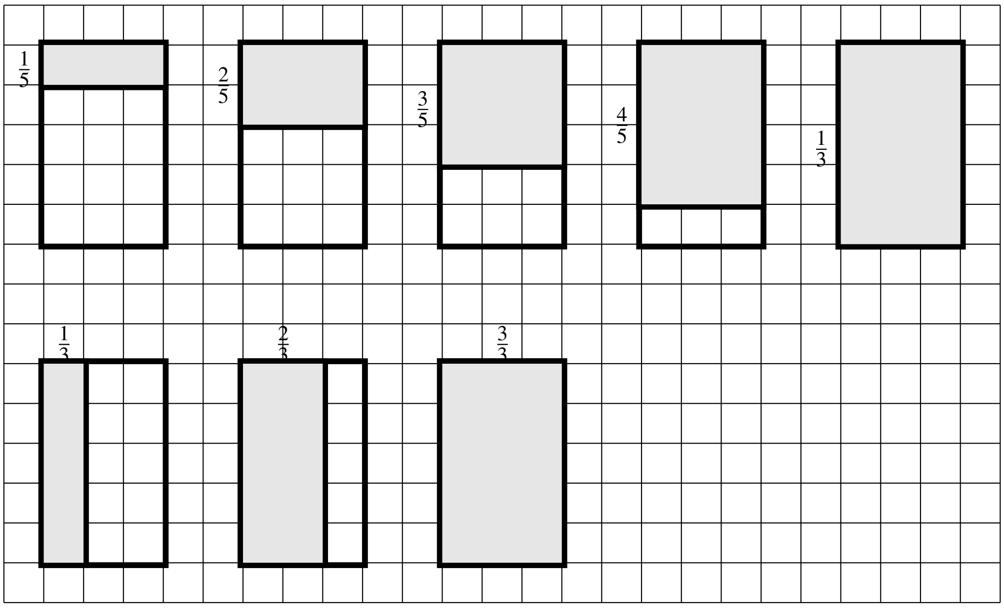
we can draw pictures of two *different* fractions:



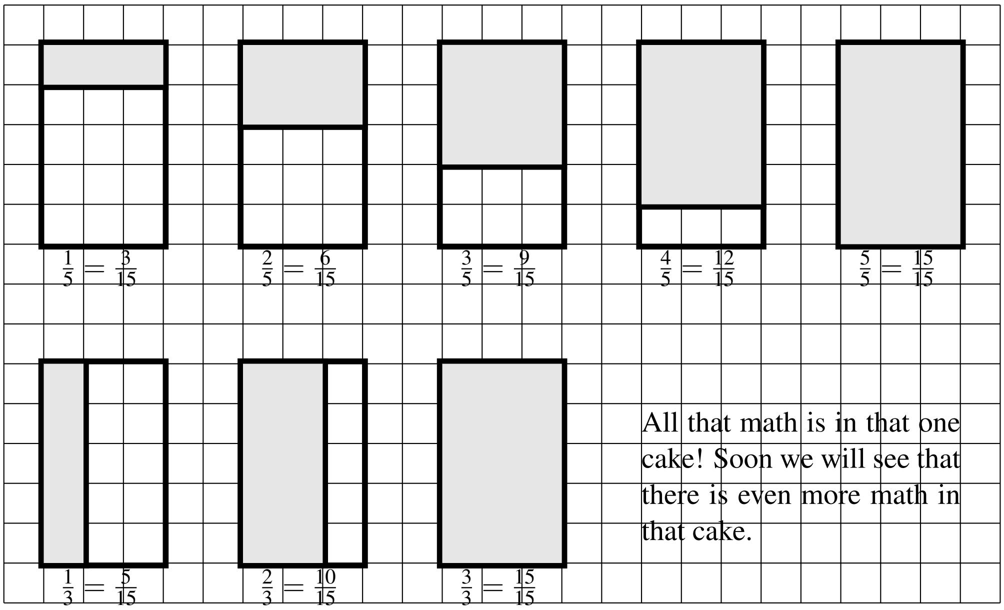
Let's do another one. Suppose there will be either three people or five people for dinner tonight. We'll go ahead and bake this cake:



Look at all the fractions we can make in this same cake:




And there is more math in each of these cakes:



Now it is your turn. What cake would you bake if you didn't know whether there would be three or four people for dinner tonight?

In this space write all the math you can about the cake you just drew:

Now do a cake you can divide into halves and fifths. Do all the math you can in copies of that cake:

On your own sheet of graph paper, pick other pairs of numbers and make cakes that work for both of the numbers. Do as much math as you can in each cake.

## 8.9 Adding Fractions of the Cake

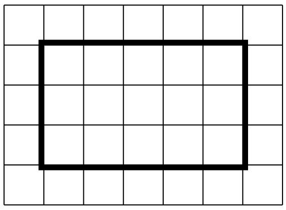
All of a sudden, believe it or not, we are ready to do this math problem:

$$\frac{2}{5} + \frac{1}{3} = ?$$

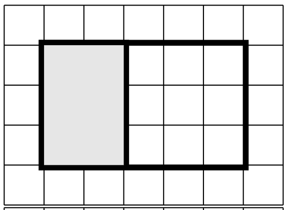
First of all, you might ask, “What does this problem mean?” Let’s put the problem in words:

> How much of the cake do you get if you put two-fifths of a cake and one-third of the same sized cake together in one cake pan?

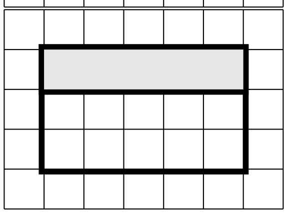
Two-fifths is how much two people get if there are five people at the table, and one-third is how much one person gets if there are three people at the table, so we will need a cake that works for five people and for three people. That is, we will need a cake that works for both fifths and thirds.



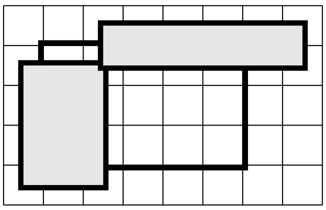
First let’s show two-fifths of this cake.



Now let’s show one-third of the same-sized cake:



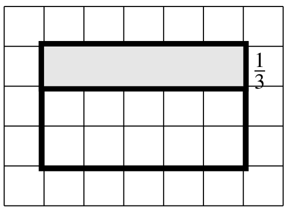
Now suppose we try to put both of these parts of the cake in the same cake pan:



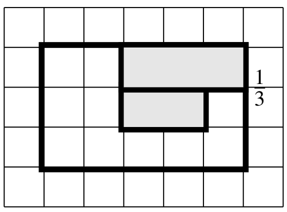
Trouble! We’re going to mess up the cake if we put pieces on top of each other!

What should we do? (Remember, it’s not how the cake is arranged in the pan, but how much cake there is that counts.)

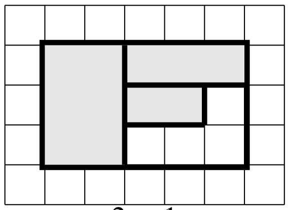
This is one-third of the cake:




but so is this:

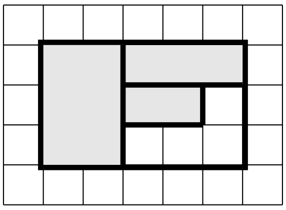


So we will use this second picture of one-third since it leaves room to fit the two-fifths of the cake in the same pan:



$$\frac{2}{5}+\frac{1}{3}$$

So how much of the cake do we get altogether?



This cake works when 15 people come to dinner.

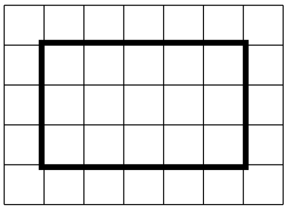
We have a picture of eleven-fifteenths since we have a picture of how much of the cake eleven people get altogether if there are fifteen people at the table! So:

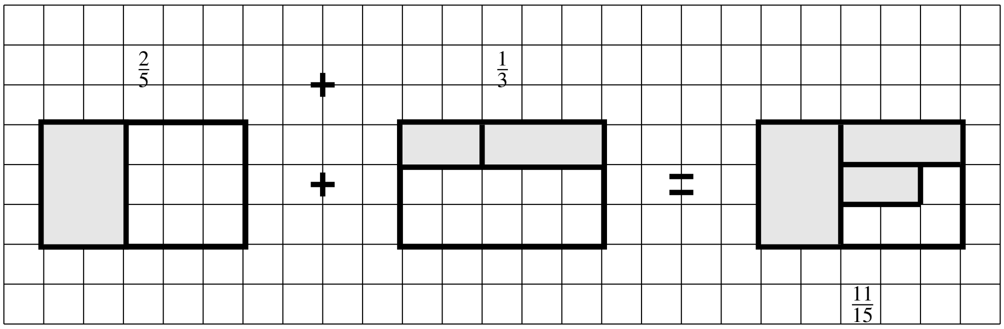
$$\frac{2}{5}+\frac{1}{3}=\frac{11}{15}.$$

Let's run the whole thing by again:

$$\frac{2}{5}+\frac{1}{3}=?$$

What cake can I use to draw both fifths and thirds?

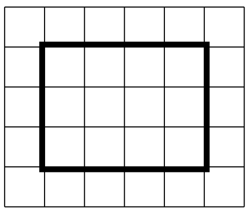


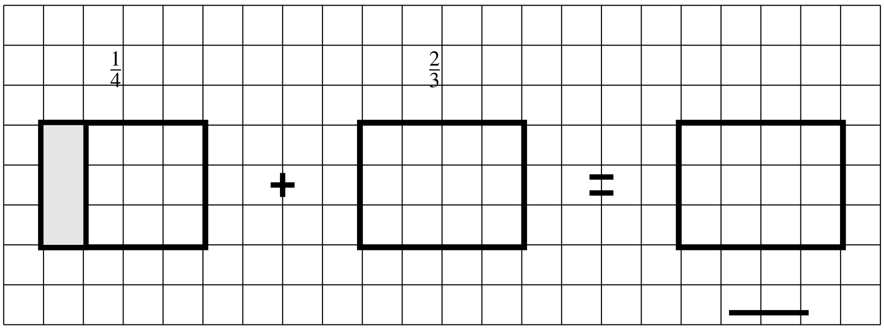



Here's another problem. You finish it up:

$$\frac{1}{4}+\frac{2}{3}=?$$

Cake for fourths and thirds:

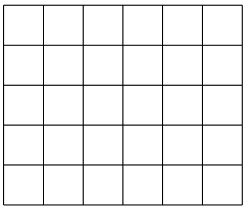


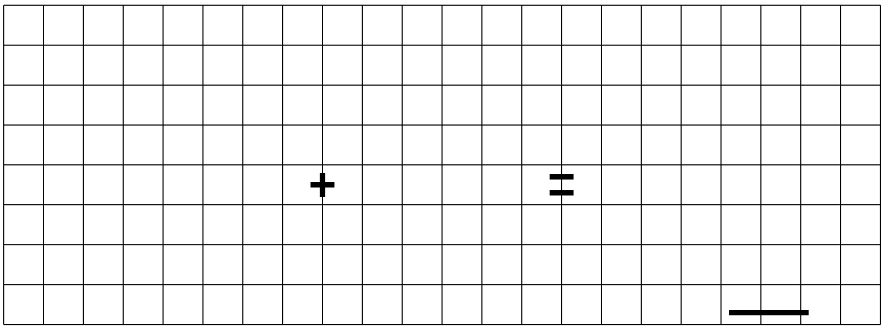



Here's another problem. This time you decide which cake will work:

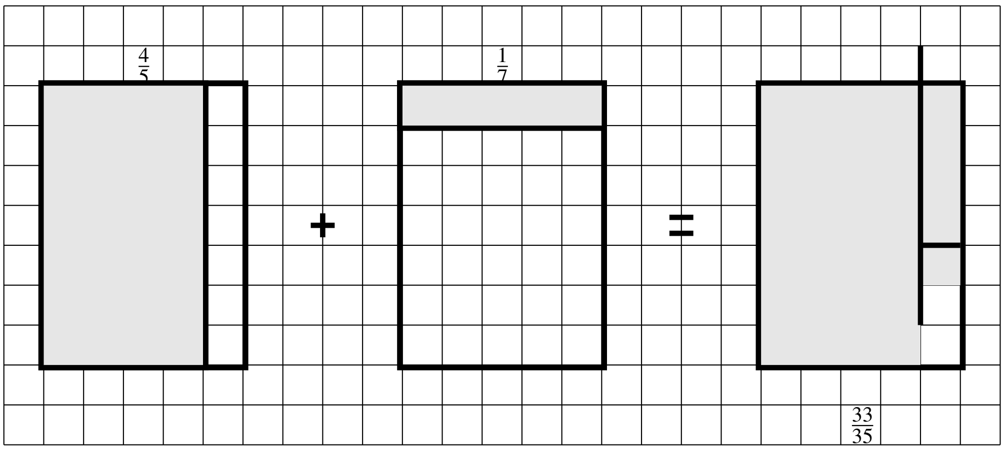
$$\frac{1}{2} + \frac{1}{3} = ?$$

Cake for halves and thirds:





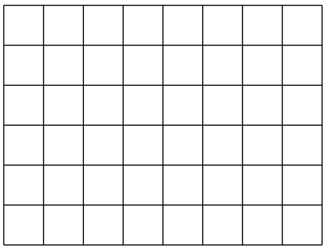
Now we'll give you a rest and do one for you:

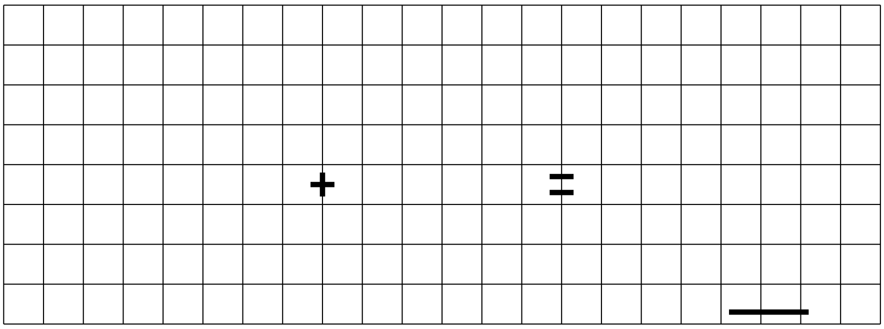



Now you try these by yourself:

$$\frac{3}{5} + \frac{1}{6} = ?$$

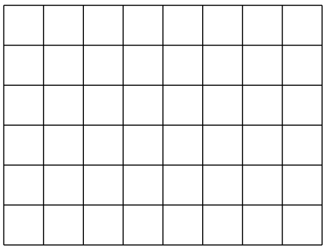
What shape of cake will work?

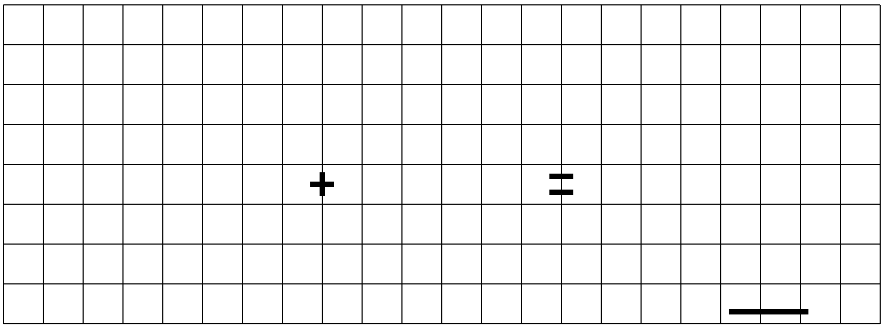





$$\frac{1}{6}+\frac{1}{2}=?$$

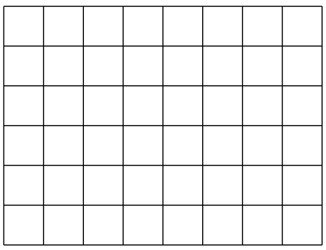
What shape of cake will work?

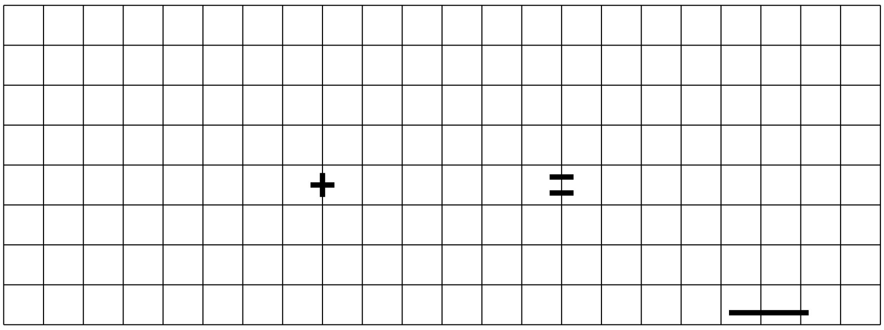





$$\frac{2}{3}+\frac{1}{5}=?$$

What shape of cake will work?



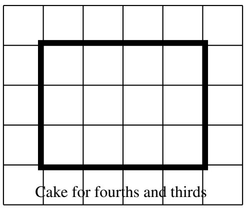


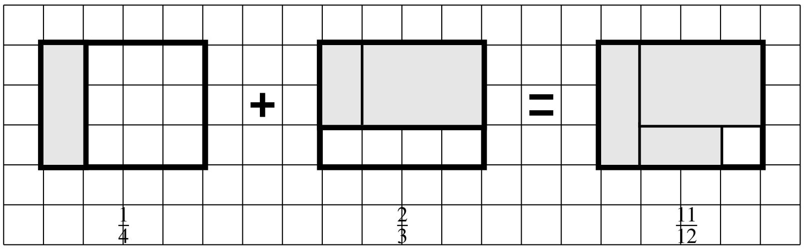
## 8.10 Different Levels

Let's do one of the adding problems at three different levels.

**Level 1**

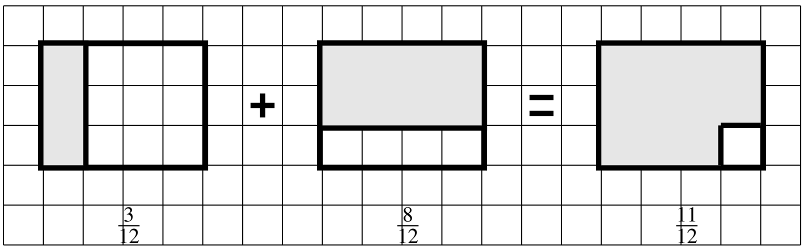
$$\frac{1}{4} + \frac{2}{3} = ?$$



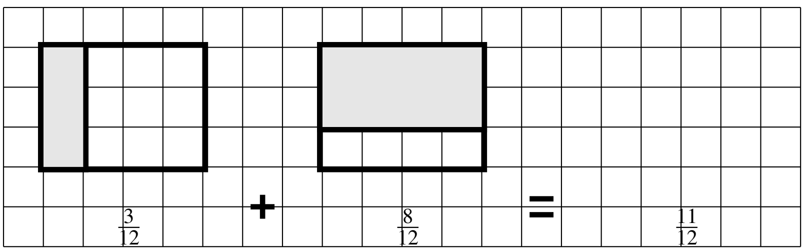

**Level 2**

$$\frac{1}{4} + \frac{2}{3} = ?$$




**Level 3**

$$\frac{1}{4} + \frac{2}{3} = ?$$



Now you do this problem at each of the three levels:

$$\frac{1}{3}+\frac{3}{7}=?$$

**Level 1:**

**Level 2:**

**Level 3:**

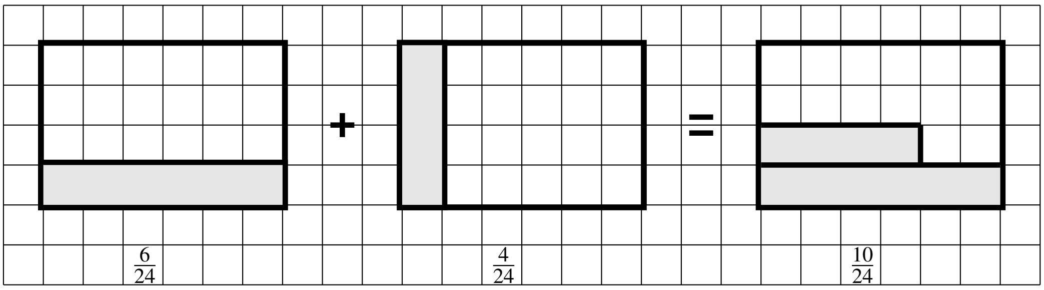
On your own graph paper, do some more problems like these at all three levels.

## 8.11 Sometimes Smaller Cakes Will Do

Try this problem:

$$\frac{1}{4}+\frac{1}{6}=?$$

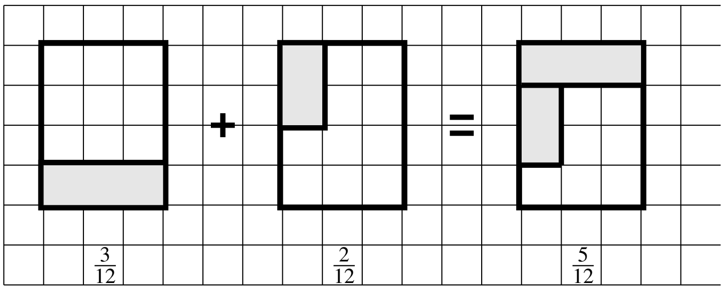
That's not so hard to do:



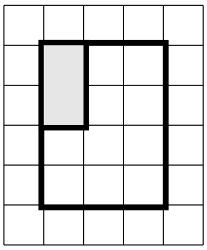

$$\frac{1}{4}+\frac{1}{6}=\frac{10}{24}$$

But there is a smaller, easier cake we can use for this same problem:

$$\frac{1}{4}+\frac{1}{6}$$




Wait a minute! First of all, why is



a picture of one-sixth? Well, this smaller cake can be divided up evenly if there are six people at the table, and the shaded part is how much of the cake one person would get if there were six people at the table. So it is a picture of one-sixth.

Second of all, it looks like we got two different answers to the same problem.

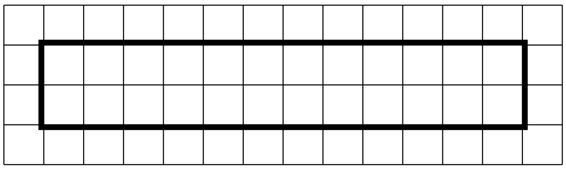
$$\frac{1}{4}+\frac{1}{6}=\frac{10}{24}$$

$$\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$$

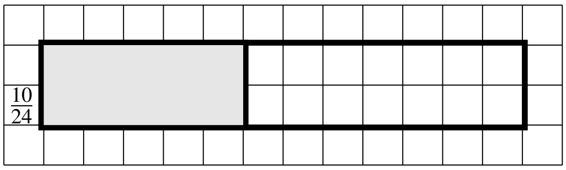
But that's o.k., too, because

$$\frac{10}{24} = \frac{5}{12}$$

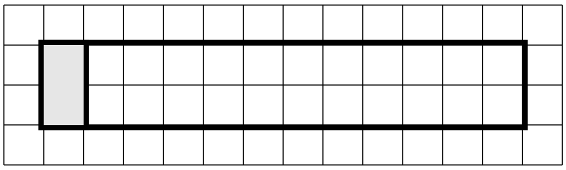
Let's see why. We want to see why five-twelfths of a cake is the same amount as ten twenty-fourths of that same cake.



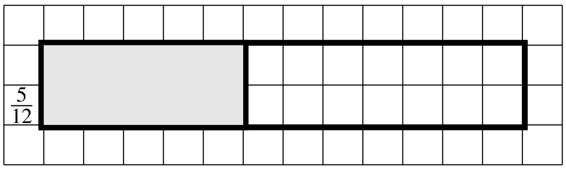
There are twenty-four little squares in this cake. So each little square is one twenty-fourth of the cake, since it is how much of the cake one person would get if there were twenty-four persons at the table. So here is ten twenty-fourths of the cake:




Here we have one-twelfth of the cake:



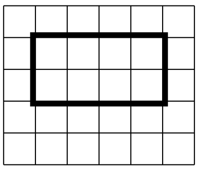
since it is how much of the cake each person gets if there are twelve people at the table. So here is five-twelfths of the same cake:

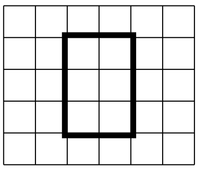


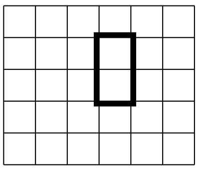

So, five-twelfths and ten twenty-fourths are the same part of a cake!
Here's another problem like the last one:

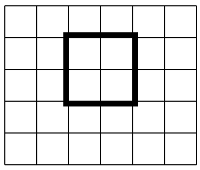
$$\frac{1}{2} + \frac{1}{4} = ?$$

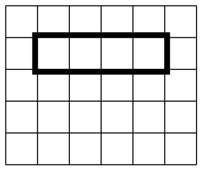
Which of these cakes will work for this problem?


First Cake


Second Cake

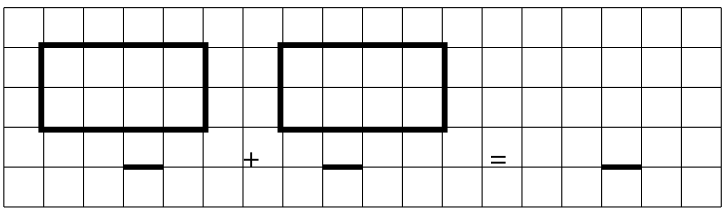

Third Cake


Fourth Cake

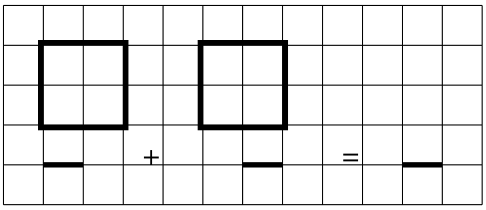

Fifth Cake

The correct answers are the first, fourth, or fifth cake. In fact, either the fourth or fifth cake is the smallest cake that will work for this problem. Do this problem using the first cake:

$$\frac{1}{2} + \frac{1}{4} = ?$$



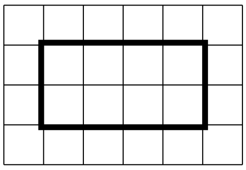
Now do the same problem with the fourth cake:




If you did these problems correctly, you got six-eighths the first time and three-fourths the second time. Why is

$$\frac{6}{8} = \frac{3}{4}?$$

Explain in in this cake:

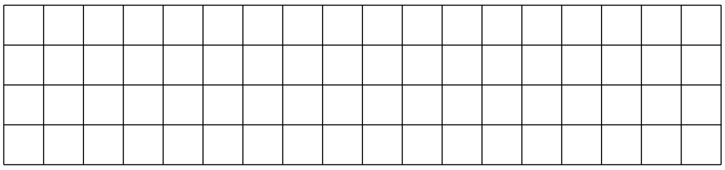


Remember, when we compare fractions, we must do it in the same cake (or, at least, in the same sized cake). The same for adding or anything else – the question is always about two fractions of the same (sized) cake.

Choose the smallest cake you can in which to do each problem, and then do the problem.

$$\frac{1}{2} + \frac{1}{6}$$

What is the smallest cake that works for two people or six people?



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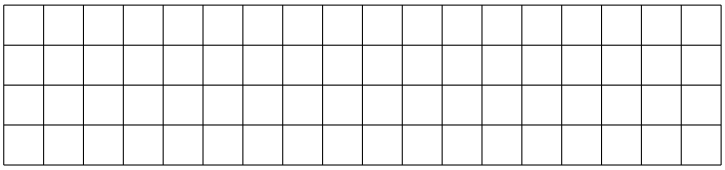
$$\frac{1}{6} + \frac{1}{9}$$

Don't do this problem in a fifty-four piece cake



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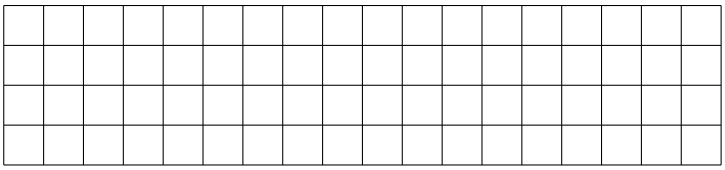
$$\frac{2}{7} + \frac{3}{7}$$

What is the smallest cake that works for seven people or seven people?



Whatever the bottom numbers are in the two fractions in the problem, the total number of pieces in the cake has to be a multiple of each.

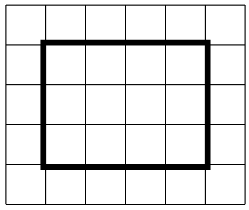
Smallest multiple of both 2 and 6 is 6. Smallest multiple of both 6 and 9 is 18. Smallest multiple of both 7 and 7 is 7.

## 8.12 Subtracting Fractions

This last thing that we learned is called adding fractions. It is a big step and you'll probably have to go over it several times. But once you get it, it makes several other things really easy. So it is worth the work. One thing that is easy once you learn to add fractions is to *subtract* them:

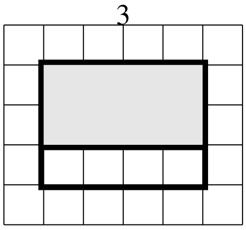
$$\frac{2}{5} - \frac{1}{4} = ?$$

We need a cake that we can divide into thirds and also into fourths. This will do:



So we make pictures of each fraction of the same-sized cake:

$\frac{2}{3}$ - $\frac{1}{4}$ = ?



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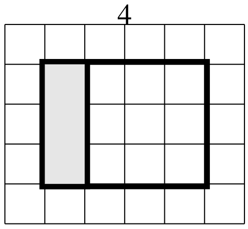


= ?

Two-thirds of this cake is eight pieces out of twelve pieces in the whole cake.

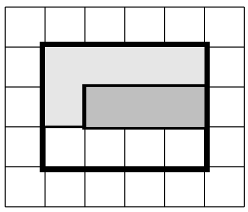
One-fourth of this cake is three pieces out of twelve pieces in the whole cake

If someone takes away one-fourth of the cake from two-thirds of the cake, that means taking away three of eight pieces.

$\frac{8}{12}$ - $\frac{3}{12}$ = $\frac{5}{12}$

This is almost like adding. We do the same steps. But, instead of adding the eight pieces to the three pieces like we do when we add fractions, we take three pieces away from the eight pieces.

$$\frac{2}{3} - \frac{1}{4} = ?$$

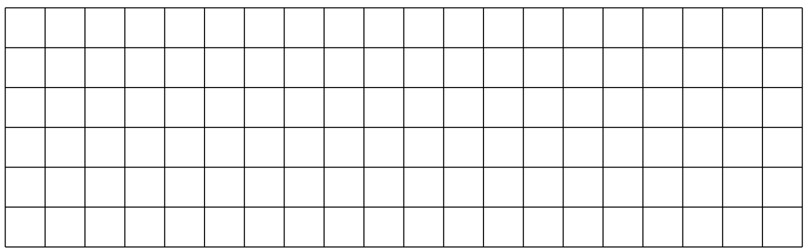


What's left is five-twelfths of the cake.

Now you try these subtraction problems.

$$\frac{3}{4} - \frac{1}{5}$$

What size cake can you divide into both fourths and fifths?



$$\frac{3}{4} - \frac{1}{2}$$

What size cake can you divide into both fourths and halves? What is the smallest cake you can use?

---

$$\frac{3}{4} - \frac{1}{6}$$

What size cake can you divide into both fourths and sixths? What is the smallest cake you can use?

---

$$\frac{5}{6} - \frac{1}{6}$$

What size cake can you divide into both sixths and sixths? What is the smallest cake you can use?

Of course, when we subtract, everything is not exactly the same as when we are adding. For example, when we were adding, the order in which we wrote the two fractions didn't make any difference. The problem:

$$\frac{2}{5} + \frac{1}{3} = ?$$

is the same as the problem

$$\frac{1}{3} + \frac{2}{5} = ?$$

But when we are subtracting, the problem

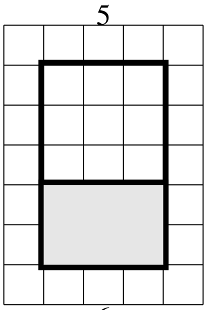
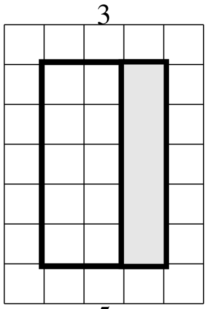
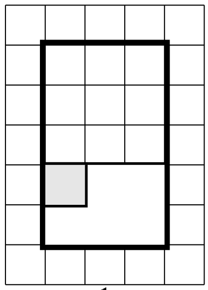
$$\frac{2}{5} - \frac{1}{3} = ?$$

is very different from the problem

$$\frac{1}{3} - \frac{2}{5} = ?$$

Let's try these two problems:

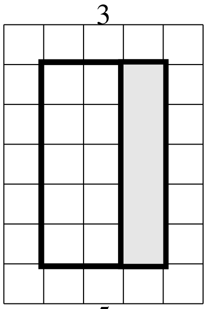
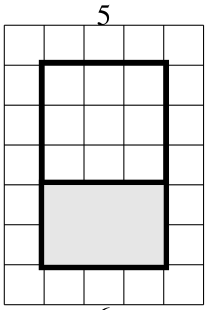
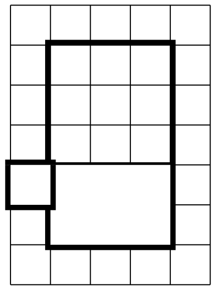
$\frac{2}{5} - \frac{1}{3} = ?$

 -  = 

$\frac{6}{15} - \frac{5}{15} = \frac{1}{15}$

Now try:

$\frac{1}{3} - \frac{2}{5} = ?$

 -  = 

$\frac{5}{15} - \frac{6}{15} = ?$

We are asked to take away more than we have! We take away all the cake there was, and we still owe one-fifteenth of a cake!

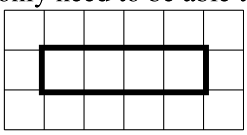
When you start using negative numbers, the answer to this last problem will be $-\frac{1}{15}$.

## 8.13 When the Answer is More than One Cake

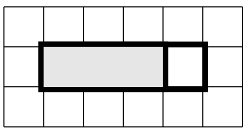
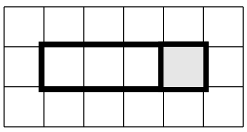
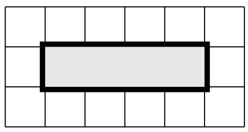
Let's go back to adding fractions:

$$\frac{3}{4} + \frac{1}{4} = ?$$

For this problem, we only need to be able to divide the cake into fourths.

This cake will do: 

$\frac{3}{4} + \frac{1}{4} = ?$

 +  = 

$\frac{3}{4} + \frac{1}{4} = \frac{4}{4}$

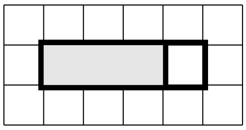
$\frac{3}{4} + \frac{1}{4} =$ 1 cake

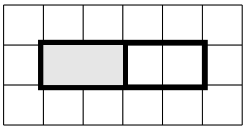
Now suppose the problem is

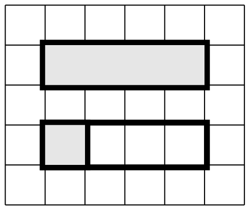
$$\frac{3}{4} + \frac{1}{2} = ?$$

We can still use the same cake, since it can be divided into both fourths and halves.

$\frac{3}{4} \quad + \quad \frac{1}{2} \quad = \quad ?$







$\frac{3}{4} \quad + \quad \frac{2}{4} \quad = \quad \frac{5}{4} = 1 + \frac{1}{4}$ cakes.

Here, when we put all the cake together, the answer turns out to be more than one whole cake. We start with three-fourths of a cake. We want to add two more fourths of the same sized cake to it. What we get is $3+2=5$ fourths. Four-fourths makes one whole cake, so we have one more fourth left over. So, all together we have:

$$\begin{aligned} \text{five} - \text{fourths} &= \text{one and one} - \text{fourth} \\ \frac{5}{4} &= 1 + \frac{1}{4} \end{aligned}$$

Instead of writing $1+\frac{1}{4}$, people usually write $1\frac{1}{4}$. So if you see $1\frac{1}{4}$, it means $1+\frac{1}{4}$.
So remember, if you see something like

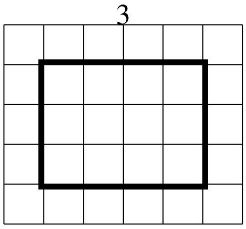
$$12\frac{5}{8}$$

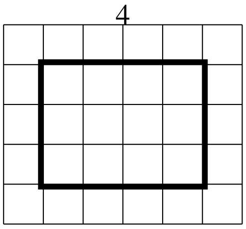
it means

$$12 + \frac{5}{8}$$

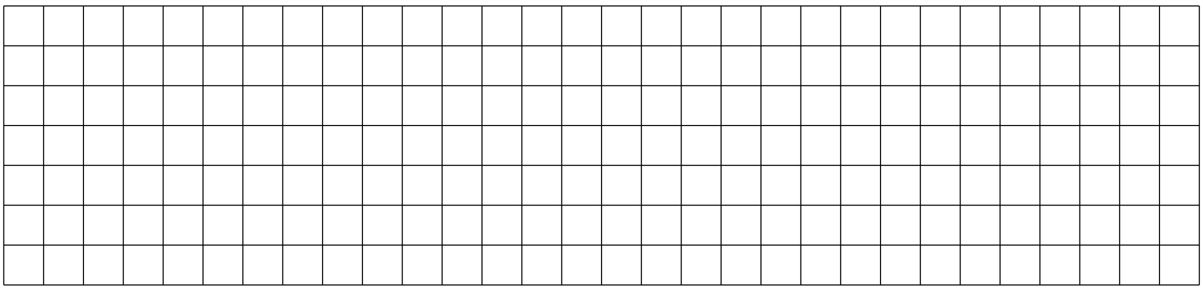
(twelve whole cakes plus five-eighths of another cake).
Now you try these problems:

$\frac{2}{3} \quad + \quad \frac{3}{4} \quad = \quad ?$

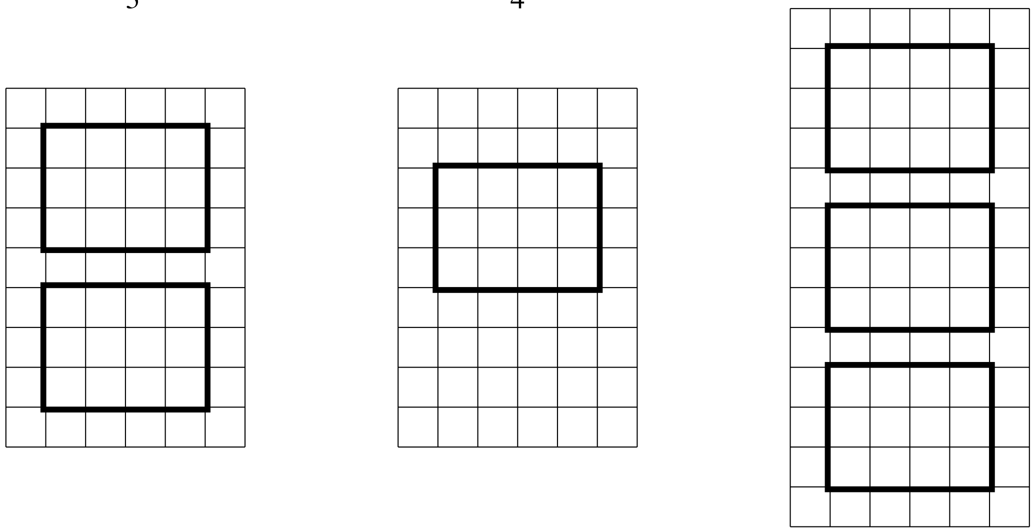




$$\frac{5}{8} + \frac{3}{5} = ?$$



$$1\frac{2}{3} \quad + \quad \frac{3}{4} \quad = \quad ?$$



Suppose now we want to add $1\frac{2}{3}+5\frac{3}{4}$.
Let's stop for a moment and think what this means. The problem asks us to add all together:

$$1+\frac{2}{3}+5+\frac{3}{4}=?$$

So first let's do the easy part; let's add the one and the five:

$$1+5=6$$

So now all we have left to do is to add the $\frac{2}{3}$ and the $\frac{3}{4}$ to the 6 whole cakes.
To do that, add the $\frac{2}{3}$ and the $\frac{3}{4}$ together first (we already did this!):

$$\begin{aligned} \frac{2}{3}+\frac{3}{4} &= ? \\ \frac{8}{12}+\frac{9}{12} &= \frac{17}{12}=1\frac{5}{12} \end{aligned}$$

Finally, we add this $1\frac{5}{12}$ to the 6 we had before:

$$6+1\frac{5}{12}=7\frac{5}{12}$$

We've added everything together:

$$1+\frac{2}{3}+5+\frac{3}{4}=1+5+\frac{2}{3}+\frac{3}{4}=7\frac{5}{12}$$

We could have written this problem in columns like we do for addition of whole numbers. Then it works just like addition of whole numbers does:

$$\begin{array}{rrr} 1\frac{2}{3} & 1\frac{8}{12} & 1\frac{8}{12} \\ +5\frac{3}{4} & +5\frac{9}{12} & +5\frac{9}{12} \\ \hline & 6\frac{17}{12} & 7\frac{5}{12} \end{array}$$

"Five-twelfths and carry the one."

Now you do these problems, first with pictures of the fractions, then by the column method:

$$2\frac{2}{3}+1\frac{1}{2}=?$$

$$3\frac{1}{4}+7\frac{7}{8}=?$$

$$114\frac{1}{4}+988\frac{7}{8}=?$$

## 8.14 Borrowing

We can also do subtraction problems like we did for whole numbers:

$$\begin{array}{r} 5\tfrac{1}{4} \\ -1\tfrac{3}{4} \\ \hline \end{array}$$

We can't take $\frac{3}{4}$ from $\frac{1}{4}$, so we borrow 1 from the 5:

$$5\frac{1}{4} = 4+1+\frac{1}{4} = 4+\frac{4}{4}+\frac{1}{4} = 4+\frac{5}{4}$$

So our problem becomes:

$$\begin{array}{rl} 4\tfrac{5}{4} & \text{Now we can do the subtraction of the whole numbers} \\ -1\tfrac{3}{4} & \text{and the subtraction of the fractions separately!} \\ \hline 3\tfrac{2}{4} & = 3\tfrac{1}{2} \quad \text{Why?} \end{array}$$

Now you try a couple of these subtraction problems:

$$\begin{array}{r} 4\tfrac{2}{5} \\ -3\tfrac{3}{5} \\ \hline \end{array}$$

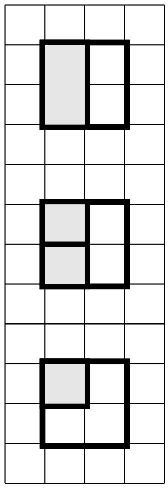
$$\begin{array}{r} 4\tfrac{2}{3} \\ -2\tfrac{4}{5} \\ \hline \end{array}$$

This is a problem about thirds and fifths. So to start, rewrite it...

$$\begin{array}{r} 4\tfrac{10}{15} \\ -2\tfrac{12}{15} \\ \hline \end{array}$$

What's next?

## 8.15 Fractions of Fractions



How much is half of a half? Well, let's draw a picture of a half that we can break up into halves:

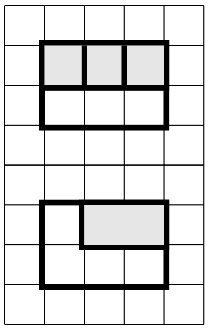
Suppose this is all the cake left in the house, half a cake. Suppose there are two people at the table tonight. How much cake should each one get?

Each person should get one piece:

What is this a picture of? It is a picture of one-fourth of a whole cake. Each person gets one-fourth of a whole cake. One-half of one-half of a cake is one-fourth of the whole cake!

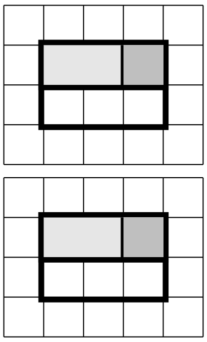
$$\begin{array}{ccc}
\text{one-half of one-half of the cake} & = & \text{one-fourth of the cake} \\
\frac{1}{2}\text{ (times) of }\frac{1}{2}\text{ of the cake} & = & \frac{1}{4}\text{ of the cake} \\
\frac{1}{2}\text{ times }\frac{1}{2} & = & \frac{1}{4} \\
\frac{1}{2}\times\frac{1}{2} & = & \frac{1}{4}
\end{array}$$

Let's do another one: How much is two-thirds of a half of a cake? We need a picture of half a cake we can break up evenly into thirds. This one will do:



So how much is two-thirds of that half cake? What fraction of the whole cake is this a picture of?

In short, here is two-thirds of one-half of a cake:



$$\begin{array}{cccc}
\frac{2}{3} & \text{of} & \frac{1}{2}\text{ of a cake} = & \\
 & = & \frac{2}{6}\text{ of the whole cake} & \\
\frac{2}{3} & \text{(times) of} & \frac{1}{2} & = \frac{2}{6} \\
\frac{2}{3} & \text{times} & \frac{1}{2} & = \frac{2}{6} \\
\frac{2}{3} & \times\text{ of} & \frac{1}{2} & = \frac{2}{6}
\end{array}$$

Now you do some problems:

$$\frac{2}{5} \times \frac{1}{2} =$$

What's a picture of one-half that you can divide evenly into fifths?

$$\frac{1}{7} \times \frac{1}{4} =$$

What's a picture of one-fourth that you can divide evenly into sevenths?

$$\frac{2}{7} \times \frac{1}{4} =$$

$$\frac{3}{7} \times \frac{1}{4} =$$

$$\frac{4}{7} \times \frac{1}{4} =$$

$$\frac{5}{7} \times \frac{1}{4} =$$

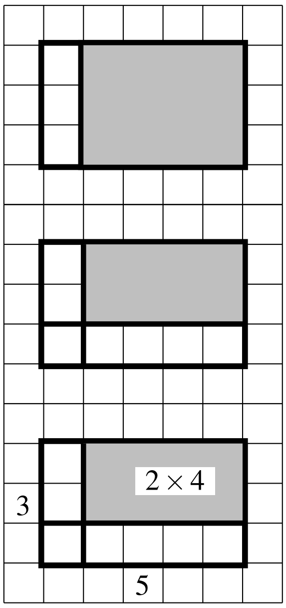
$$\frac{6}{7} \times \frac{1}{4} =$$

$$\frac{7}{7} \times \frac{1}{4} =$$

Now let's examine what we have been doing a little more closely:

$$\frac{2}{3} \times \frac{4}{5} = ?$$

We need a picture of four fifths that we can break up evenly into thirds:




$\frac{4}{5}$

$\frac{2}{3}$ of $\frac{4}{5}$

$\frac{2}{3}$ of $\frac{4}{5}$

Two-thirds of four-fifths is $2 \times 4$ pieces out of $3 \times 5$ pieces in the whole cake.

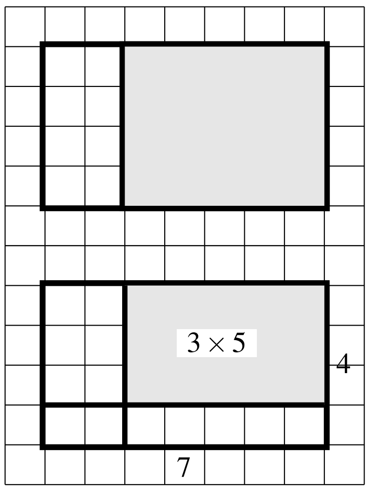
$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5}$$

Try another one: $\frac{3}{4} \times \frac{5}{7} = ?$

$\frac{5}{7}$

$\frac{3}{4} \times \frac{5}{7}$

$= \frac{3 \times 5}{4 \times 7}$




Now take three-fourths of the five-sevenths. How much of the whole cake is that?

Can you find the pattern in these problems? If so, you will discover an easy rule for multiplying fractions!

You can also do multiplication of complicated fractions by doing the problems in steps. Here are some examples. Later on, you will probably get lots of practice with problems like these:

$$\begin{aligned} 3 \times 2\frac{2}{3} &= 3 \times \left(2 + \frac{2}{3}\right) \\ &= (3 \times 2) + \left(3 \times \frac{2}{3}\right) \\ &= 6 + 2 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \frac{4}{5} \times 2\frac{2}{3} &= \frac{4}{5} \times \left(2 + \frac{2}{3}\right) \\ &= \left(\frac{4}{5} \times 2\right) + \left(\frac{4}{5} \times \frac{2}{3}\right) \\ &= \frac{8}{5} + \frac{8}{15} \\ &= \frac{24}{15} + \frac{8}{15} \\ &= \frac{32}{15} \\ &= 2\frac{2}{15} \end{aligned}$$

$$\begin{aligned} 1\frac{4}{5} \times 2\frac{2}{3} &= 1\frac{4}{5} \times \left(2 + \frac{2}{3}\right) \\ &= \left(1\frac{4}{5} \times 2\right) + \left(1\frac{4}{5} \times \frac{2}{3}\right) \\ &= ?+? \end{aligned}$$

To finish this problem, figure out $1\frac{4}{5} \times 2$, then figure out $1\frac{4}{5} \times \frac{2}{3}$, then add up the two answers you get. Can you do all that? If not, don't worry. It's a complicated problem. After you have done a lot of simpler ones, you'll probably be able to do it!

## 8.16 Dividing Fractions by Fractions

We have just talked about adding, subtracting, and multiplying fractions in this book. The story would not be complete if we did not also tell about dividing fractions. A division problem is just a question about multiplication:

$$5 \div 2 = ?$$

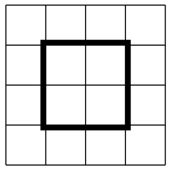
is just the question "What do you multiply 2 by to get 6?" Or, in other words, fill in the number in the box:

$$2 \times \boxed{\phantom{0}} = 6$$

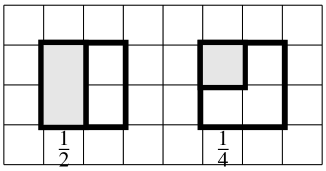
Or, finally, it is the same as, "How many twos in six?"
Well, the same is true if the numbers are fractions:

$$\frac{1}{2} \div \frac{1}{4} = ?$$

is just the question, "How many $\frac{1}{4}$s are there in $\frac{1}{2}$?"

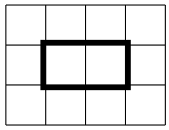


To answer this, we'll draw a cake that can show both fourths and halves.

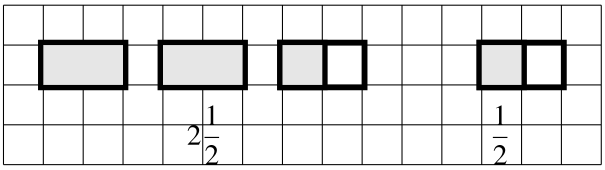



How many $\frac{1}{4}$s fit into $\frac{1}{2}$? The answer is clearly that two of the $\frac{1}{4}$s fit into the $\frac{1}{2}$. So $\frac{1}{2} \div \frac{1}{4} = 2$.

Try another one: $2\frac{1}{2} \div \frac{1}{2} = ?$



Here is a cake to show halves:




How many one-halves fit into two and one-half? The answer to this last problem is five. Five of those one-halves fit into two and one-half:

$$2\frac{1}{2} \div \frac{1}{2} - 5$$

Now you do these problems:

$$3\frac{1}{2} \div \frac{1}{4} =$$

$$\frac{3}{4} \div \frac{1}{8} =$$

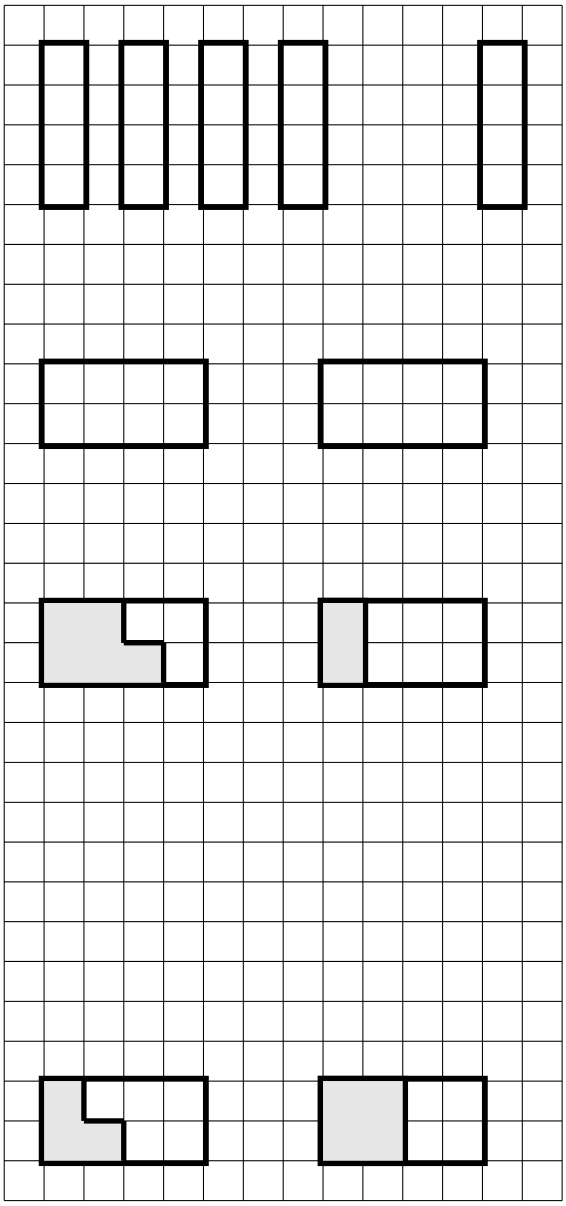
$$\frac{5}{8} \div \frac{1}{4} = 2\frac{1}{2}$$

We can fit $2\frac{1}{2}$ of those one-fourths in five-eighths.

$$\frac{3}{4} \div \frac{1}{2}$$

$$\frac{3}{8} \div \frac{1}{2} = \frac{3}{4}$$

There is $\frac{3}{4}$ of that one-half in three-eighths.



We always want to know how many of the second fractions fit into the first, or what part of the second fraction fits into the first. Do several more problems like these on your own graph paper.

There is another way to think about these division problems. Remember division is a question about multiplication.

$$6 \div 2 = ? \text{ is the question } 2 \times \square = 6$$

So

$$\frac{5}{8} \div \frac{1}{4} = ?$$

is the question

$$\frac{1}{4} \times \square = \frac{5}{8}$$

So to figure out the answer, we might reason as follows: "It's too hard to figure out what to multiply $\frac{1}{4}$ by to get $\frac{5}{8}$, so I'll first figure out what I multiply $\frac{1}{4}$ to get 1" :

$$\frac{1}{4} \times \frac{4}{1} = \frac{1 \times 4}{4 \times 1} = \frac{4}{4} = 1$$

Now that I multiplied $\frac{1}{4}$ up to 1, I'll keep going and multiply that 1 by something to get $\frac{5}{8}$:

$$1 \times \boxed{\phantom{0}} = \frac{5}{8}$$
$$1 \times \boxed{\frac{5}{8}} = \frac{5}{8}$$

Putting it all together:

$$\left(\frac{1}{4} \times \frac{4}{1}\right) \times \boxed{\frac{5}{8}} = \frac{5}{8}$$
$$\frac{1}{4} \times \frac{4}{1} \times \frac{5}{8} = \frac{5}{8}$$
$$\frac{1}{4} \times \boxed{\frac{4}{1} \times \frac{5}{8}} = \frac{5}{8}$$

So what goes in the box in the problem

$$\frac{1}{4} \times \boxed{\phantom{000}} = \frac{5}{8}$$

is

$$\frac{4}{1} \times \frac{5}{8}$$

$$\frac{5}{8} \div \frac{1}{4} = ?$$

is the question

$$\frac{1}{4} \times \boxed{\phantom{000}} = \frac{5}{8}$$

And the answer to the question is

$$\frac{4}{1} \times \frac{5}{8}$$

because

$$\frac{1}{4} \times \boxed{\frac{4}{1} \times \frac{5}{8}} = \frac{5}{8}$$

Now you try a couple:

$$\frac{3}{4} \div \frac{1}{2} = ?$$

$$\frac{1}{2} \times \boxed{- \times -} = \frac{3}{4}$$

$$\frac{4}{5} \div \frac{2}{3} = ?$$

$$\frac{2}{3} \times \boxed{- \times -} = \frac{4}{5}$$

Do you see the pattern? If you do, you will discover a quick way to divide fractions!