NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
NOVEMBER 1986

Primes

A *prime number* is a natural number that has exactly two factors, itself and 1. The pyramid below is called a *prime pyramid*. Each row in the pyramid begins with 1 and ends with the number that is the row number. In each row, the consecutive numbers from 1 to the row number are arranged so that the sum of any two adjacent numbers is a prime.

For example, look at row 5:

1) It must contain the numbers 1, 2, 3, 4, and 5.
2) It must begin with 1 and end with 5.
3) The sum of adjacent pairs must be a prime number.
4) 1 + 4 = 5, 4 + 3 = 7, 3 + 2 = 5, and 2 + 5 = 7.

*	Row 1
1 2	Row 2
1 2 3	Row 3
1 2 3 4	Row 4
1 4 3 2 5	Row 5
1 4 3 2 5 6	.
1 _ _ _ _ _ 7	.
1 _ _ _ _ _ _ 8	.
1 2 3 4 7 6 5 8 9	
1 _ _ _ _ _ _ _ _ 10	
1 _ _ _ _ _ _ _ _ _ 11	
1 _ _ _ _ _ _ _ _ _ _ 12	
1 _ _ _ _ _ _ _ _ _ _ _ 13	
1 _ _ _ _ _ _ _ _ _ _ _ _ 14	
1 _ _ _ _ _ _ _ _ _ _ _ _ _ 15	Row 15

- Supply the missing numbers in this prime pyramid.
- Can you extend the prime pyramid beyond row 15?
- What patterns do you see in your solutions?
- What is your solution strategy for completing the pyramid?

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Prime Start

Euclid (born about 300 B.C.), a Greek mathematician famous for his geometry, showed that the number of primes is infinite. Eratosthenes (born in 230 B.C.), also a Greek, was the chief librarian at the University of Alexandria in Egypt. He developed a method for finding primes that is still in use today.

Sieve of Eratosthenes

Start with a 10 × 10 array of the numbers 1 through 100. The number 1 is crossed out because it is not a prime. Why not? ____________________

Circle 2, the first prime; then cross out the remaining multiples of 2 using a horizontal slash (—).

What visual pattern do the multiples of 2 make in the grid? ____________________

What's the next number that hasn't been crossed out? ______

Circle 3, the next prime; then cross out the remaining multiples of 3 using a vertical slash (|).

What visual pattern is made by the multiples of 3? ____________________

What is the next prime number that hasn't been crossed out? ______

Circle the 5 and cross out the remaining multiples of 5 using the diagonal slash (/).

In a similar manner, circle the next prime, 7, and cross out all its remaining multiples using the reverse diagonal slash (\ ).

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

At this point, except for 2, 3, 5, and 7, you have sifted out all the multiples of 2, 3, 5, and 7.

1. Are any multiples of 11, the next prime, not yet crossed out? ______

2. Why is this the case? ____________________

____

The numbers that are not crossed out are the primes less than 100. Make a list of these primes.

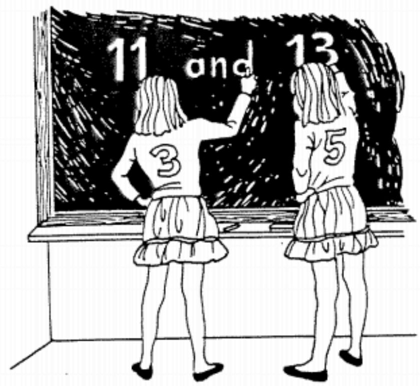
2, 3, 5, 7, 11, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____, ____,

____, ____, ____, and ____.

Twin Primes

Several pairs of primes in the list of primes less than 100 have a difference of 2. For example, the pairs 3 and 5, 5 and 7, and 11 and 13 each have a difference of 2. These pairs are called *twin primes*. Complete the list of all twin primes less than 100. Also, find the sums and products of these twin primes.

Twin Primes	Sums	Products
3 and 5	______	______
5 and 7	______	______
11 and 13	______	______
____ and ____	______	______
____ and ____	______	______
____ and ____	______	______
____ and ____	______	______
____ and ____	______	______



1. Do you see a pattern in the column of sums? Can you *prove* a fact about the *sums* of twin primes?

2. Do you see a pattern in the column of products? Can you *prove* a fact about the *products* of twin primes?

3. Examine the primes larger than 5 in your list. What *different* digits appear in the units position of these primes? ________________ Will any prime larger than 100 have a different ending than the ones you have found? ______

The number 13 is a prime, and 31, the reverse of 13, is also a prime. 13 is called an *emirp* (*prime* spelled backward) because its reverse is a *different prime*. 31 is also an emirp. But 11 is not an emirp. Why not? ________________

4. List all the emirps less than 100. ________________

Prime Concerns

Over the centuries we have learned a great deal about prime numbers. But in all this time no one has discovered a simple formula that will produce all the primes starting with 2. Many attempts have been made, and no doubt will continue to be made, to find such a formula. One such attempt produced the following:

$$p_n = n^2 - n + 41$$

p_n is a prime number for $n = 1$ through $n = 40$. For example, $p_1 = 41$, $p_2 = 43$, and $p_3 = 47$. Use your calculator or write a computer program to find additional values of p_n for $n = 4, \ldots, 40$. What is p_{41}? ______ Is it prime or composite? Why? ________________

5. What are some other values of $n > 41$ for which p_n is a composite number? ________________

6. Is p_n prime for some values of $n > 41$? If so, list some of them. ________________

Can You . . .

- replace each blank with a "+" or "−" to get an equality relation?

17 = 1 ___ 2 ___ 3 ___ 5 ___ 7 ___ 11 ___ 13 ___ 13

19 = 1 ___ 2 ___ 3 ___ 5 ___ 7 ___ 11 ___ 13 ___ 17

23 = 1 ___ 2 ___ 3 ___ 5 ___ 7 ___ 11 ___ 13 ___ 17 ___ 19 ___ 19

29 = 1 ___ 2 ___ 3 ___ 5 ___ 7 ___ 11 ___ 13 ___ 17 ___ 19 ___ 23

- find some emirps that contain three, four, or more digits?
- in Eratosthenes' sieve, name the primes whose multiples *must* be crossed out to find all primes less than 200? 300? 500?
- write a computer program based on Eratosthenes' sieve to find all primes less than 1000?
- find a string of at least 1 million consecutive numbers that are all composite? For example, 24, 25, 26, 27, 28 is a string of five consecutive numbers that are all composite. Identify your string by naming the first number and the one-millionth number in the string. (*Hint:* Use factorials!)

Did You Know That . . .

- a conjecture states that the number of twin primes is infinite? No one has been able to prove or disprove this conjecture.
- J. P. Kulik spent twenty years, unassisted, computing a factor table of the numbers from 1 to 100 000 000? He completed his monumental work in 1867. It filled eight volumes, but volume 2 is now missing from the collection at the Vienna Royal Academy.
- a special kind of prime number, *Mersenne numbers,* are named after the French priest and amateur mathematician Marin Mersenne? In 1644, he published his (incomplete) list of primes that satisfied the rule $M_p = 2^p - 1$, where p is prime: p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257. It took 304 years to resolve the errors in his short list.
- in 1978, the twenty-fifth Mersenne prime, $2^{21\,701} - 1$, was found by two eighteen-year-old students, Laura Nickel and Curt Noll, using a CYBER-174 computer at California State University at Hayward?
- the largest known Mersenne prime, $2^{216\,091} - 1$, consists of 65 050 digits and was discovered in 1985 in Houston, Texas, on a Cray X-MP supercomputer by scientists at Chevron Geosciences Company? This find will probably be declared the thirtieth Mersenne prime.
- Christian Goldbach, 1690–1764, conjectured that every even number greater than 4 is equal to the sum of two prime numbers? His conjecture remains unproved today.

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Extension—THE FIRST 500 PRIMES

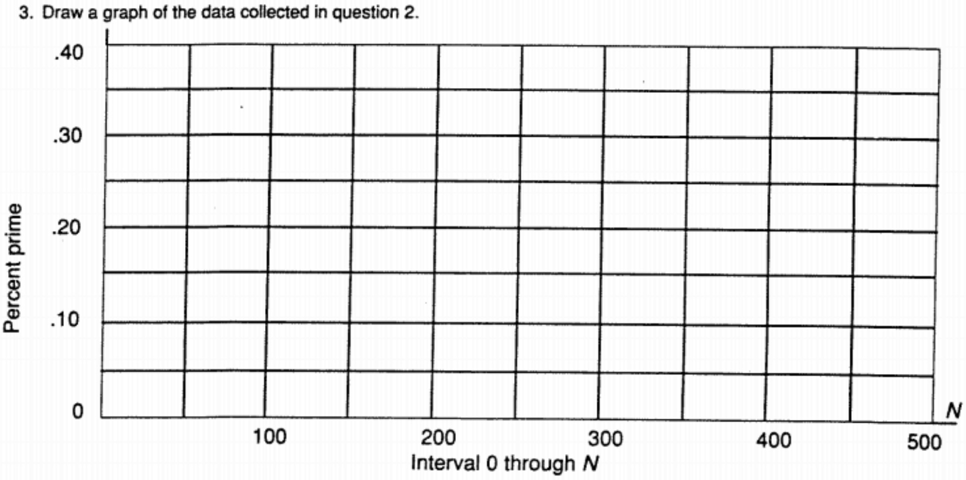
2	101	233	383	547	701	877	1049	1229	1429	1597	1783	1993	2161	2371	2579	2749	2957	3187	3373
3	103	239	389	557	709	881	1051	1231	1433	1601	1787	1997	2179	2377	2591	2753	2963	3191	3389
5	107	241	397	563	719	883	1061	1237	1439	1607	1789	1999	2203	2381	2593	2767	2969	3203	3391
7	109	251	401	569	727	887	1063	1249	1447	1609	1801	2003	2207	2383	2609	2777	2971	3209	3407
11	113	257	409	571	733	907	1069	1259	1451	1613	1811	2011	2213	2389	2617	2789	2999	3217	3413
13	127	263	419	577	739	911	1087	1277	1453	1619	1823	2017	2221	2393	2621	2791	3001	3221	3433
17	131	269	421	587	743	919	1091	1279	1459	1621	1831	2027	2237	2399	2633	2797	3011	3229	3449
19	137	271	431	593	751	929	1093	1283	1471	1627	1847	2029	2239	2411	2647	2801	3019	3251	3457
23	139	277	433	599	757	937	1097	1289	1481	1637	1861	2039	2243	2417	2657	2803	3023	3253	3461
29	149	281	439	601	761	941	1103	1291	1483	1657	1867	2053	2251	2423	2659	2819	3037	3257	3463
31	151	283	443	607	769	947	1109	1297	1487	1663	1871	2063	2267	2437	2663	2833	3041	3259	3467
37	157	293	449	613	773	953	1117	1301	1489	1667	1873	2069	2269	2441	2671	2837	3049	3271	3469
41	163	307	457	617	787	967	1123	1303	1493	1669	1877	2081	2273	2447	2677	2843	3061	3299	3491
43	167	311	461	619	797	971	1129	1307	1499	1693	1879	2083	2281	2459	2683	2851	3067	3301	3499
47	173	313	463	631	809	977	1151	1319	1511	1697	1889	2087	2287	2467	2687	2857	3079	3307	3511
53	179	317	467	641	811	983	1153	1321	1523	1699	1901	2089	2293	2473	2689	2861	3083	3313	3517
59	181	331	479	643	821	991	1163	1327	1531	1709	1907	2099	2297	2477	2693	2879	3089	3319	3527
61	191	337	487	647	823	997	1171	1361	1543	1721	1913	2111	2309	2503	2699	2887	3109	3323	3529
67	193	347	491	653	827	1009	1181	1367	1549	1723	1931	2113	2311	2521	2707	2897	3119	3329	3533
71	197	349	499	659	829	1013	1187	1373	1553	1733	1933	2129	2333	2531	2711	2903	3121	3331	3539
73	199	353	503	661	839	1019	1193	1381	1559	1741	1949	2131	2339	2539	2713	2909	3137	3343	3541
79	211	359	509	673	853	1021	1201	1399	1567	1747	1951	2137	2341	2543	2719	2917	3163	3347	3547
83	223	367	521	677	857	1031	1213	1409	1571	1753	1973	2141	2347	2549	2729	2927	3167	3359	3557
89	227	373	523	683	859	1033	1217	1423	1579	1759	1979	2143	2351	2551	2731	2939	3169	3361	3559
97	229	379	541	691	863	1039	1223	1427	1583	1777	1987	2153	2357	2557	2741	2953	3181	3371	3571

1. How many numbers from 1 through 25 are prime?
   Express the answer as a percent.

2. Use the table of primes. Find the percent of the numbers in each interval that are prime.

Interval	1–50	1–100	1–150	1–200	1–250	1–300	1–350	1–400	1–450	1–500
Percent prime	___	___	___	___	___	___	___	___	___	___

3. Draw a graph of the data collected in question 2.



4. As the number *N* increases, what appears to happen to the percent of prime numbers in the interval from 1 to *N*?