MATH 1210-004 MIDTERM 2 REVIEW(2.5-2.9,3.1-3.3,3.5)

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I.Summary of Theorems

2.5 Chain Rule

Theorem. *Chain rule*

Let $y = f(u)$, $u = g(v)$, *and* $v = h(x)$. *If* h *is differentiable at* x, g *is differentiable at* $v = h(x)$, *and* f *is differentiable at* $u = g(v)$ *then the composite function* $f \circ g \circ h$, *defined by* $(f \circ g \circ h)(x) = f(g(h(x)))$, *is differentiable at* x *and* $(f \circ g \circ h)'(x) = f'(g(h(x)))\, g'(h(x))\, h'(x)$.

2.6 Higher-Order Derivatives

The second derivative, f'', of a function f is the derivative of the first derivative $f''(x) = \frac{d}{dx}\left(f'(x)\right)$. The third derivative, f''', of a function f is the derivative of the second derivative $f'''(x) = \frac{d}{dx}\left(f''(x)\right)$. The other higher order derivatives are defined in a similar manner.

Notation for Higher-Order Derivatives

1st:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$,	$D_x[y]$
2nd:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$,	$D_x^2[y]$
3rd:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$,	$D_x^3[y]$
4th:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$,	$D_x^4[y]$
nth:	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$,	$D_x^n[y]$

2.7 Implicit Differentiation

Example

Suppose we want to differentiate the implicit function

$$y^2 + x^3 - y^3 + 6 = 3y$$

with respect x.

We differentiate each term with respect to x:

$$\frac{d}{dx}\left(y^2\right) + \frac{d}{dx}\left(x^3\right) - \frac{d}{dx}\left(y^3\right) + \frac{d}{dx}(6) = \frac{d}{dx}(3y)$$

Differentiating functions of x with respect to x is straightforward. But when differentiating a function of y with respect to x we must remember the rule given in the previous keypoint. We find

$$\frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx} + 3x^2 - \frac{d}{dy}\left(y^3\right) \times \frac{dy}{dx} + 0 = \frac{d}{dy}(3y) \times \frac{dy}{dx}$$

that is

$$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} = 3\frac{dy}{dx}$$

We rearrange this to collect all terms involving $\frac{dy}{dx}$ together.

$$3x^2 = 3\frac{dy}{dx} - 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

then

$$3x^2 = \left(3 - 2y + 3y^2\right)\frac{dy}{dx}$$

so that, finally,

$$\frac{dy}{dx} = \frac{3x^2}{3 - 2y + 3y^2}$$

This is our expression for $\frac{dy}{dx}$.

2.8 Related Rates

If a variable y depends on time t, then its derivative $\frac{dy}{dt}$ is called a **time rate of change**.

Example

Water leaking onto a floor forms a circular pool. The radius of the pool increases at a rate of 4 cm/min. How fast is the area of the pool increasing when the radius is 5 cm?

A = area of circle r = radius t = time

Equation: $A = \pi r^2$ Given rate: $\frac{dr}{dt} = 4$ Find: $\left.\frac{dA}{dt}\right|_{r=5}$

$$\left.\frac{dA}{dt}\right|_{r=5} = 2\pi r \cdot \frac{dr}{dt} = 40\pi \text{ cm}^2/\text{min}$$

2.9 Differentials and Approximations

Definition. Let $y = f(x)$ be a differentialble function of the independent variable x.

(1) Δx : an arbitrary increment in x.

(2) dx : the **differential of** x, $dx = \Delta x$

(3)Δy : the **actual change** in y as x changes from x to $x + \Delta x$, that is $\Delta y = f(x + \Delta x) - f(x)$

(4) dy : the **differential of** y, is defined by $dy = f^{'}(x)dx$

Example(on page 144 of the text) The side of a cube is measured 11.4cm with a possible error of $\pm 0.05cm$. Find the estimated absolute error and the relative error in the volume . Also estimate the volume of the cube.

Sol)

The volume V of a cube side x is $V = x^3$. Because $x = 11.4$ and $\boldsymbol{dx = \pm 0.05}$

then $V = (11.4)^3 \approx 1482\ cm^3$

Absolute error = $\boldsymbol{\Delta V \approx dV} = 3x^2dx = 3(11.4)^2(\pm 0.05) \approx \pm 19cm^3$,

Relative error= $\boldsymbol{\frac{\Delta V}{V} \approx \frac{dV}{V}} \approx \frac{\pm 19}{1482} \approx \pm 0.0128 = \pm 1.28\%$

Estimated volume= $\boldsymbol{V + \Delta V \approx V + dV} = (1482 \pm 19)cm^3$

If f is differentiable at x_0, using a tangent line f at x_0 the **linear approximation** to f at x_0 is $f(x) \approx L(x) = y = f(x_0) + f^{'}(x_0)(x - x_0)$. It is a good approximation to f when x is close to x_0.

3.1 Maxima and Minima

Theorem. *Critical Point Theorem*

Let f be defined on an interval I containing the point c. If $f(c)$ is an extreme value, then c must be a critical point; that is, either c is

*(i) an **end** point of I ;*

*(ii) a **stationary** point of f ; that is, a point where $f^{'}(c) = 0$; or*

*(iii) a **singular** point of f ; that is, a point where $f^{'}(c)$ does not exist*

3.2 Monotonicity and Concavity

Theorem. *Monotonicity Theorem*

Let f be continuous on an interval I and differentiable at every interior point of I.

(i) If $f'(x) > 0$ for all x interior to I , then f is increasing on I.

(ii) If $f'(x) < 0$ for all x interior to I , then f is decreasing on I.

Theorem. *Concavity Theorem*

Let f be twice differentiable on the open interval I.

(i) If $f''(x) > 0$ for all x in I , then f is concave up on I.

(ii) If $f''(x) < 0$ for all x in I , then f is concave down on I.

Definition. Let f be continuous at c. We call $(c, f(c))$ an **inflection point** of the graph of f if f is concave up on one side of c and concave down on the other side.

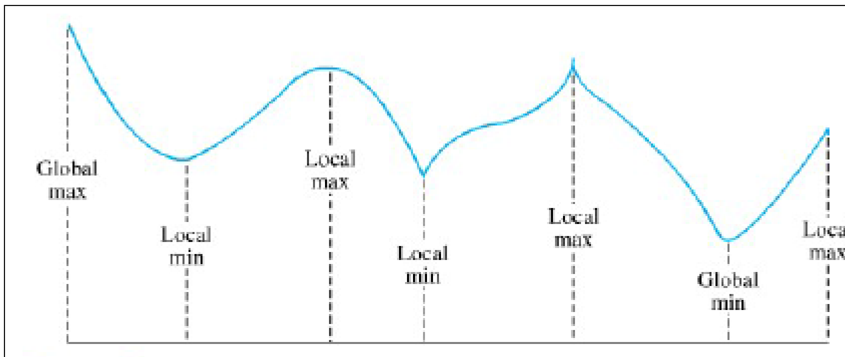
3.3 Local Extrema and Extrema on Open Intervals

Theorem. *Second Derivative Test*

Let f' and f'' exist at every point in an open interval (a, b) containing c , and suppose that $f'(c) = 0$.

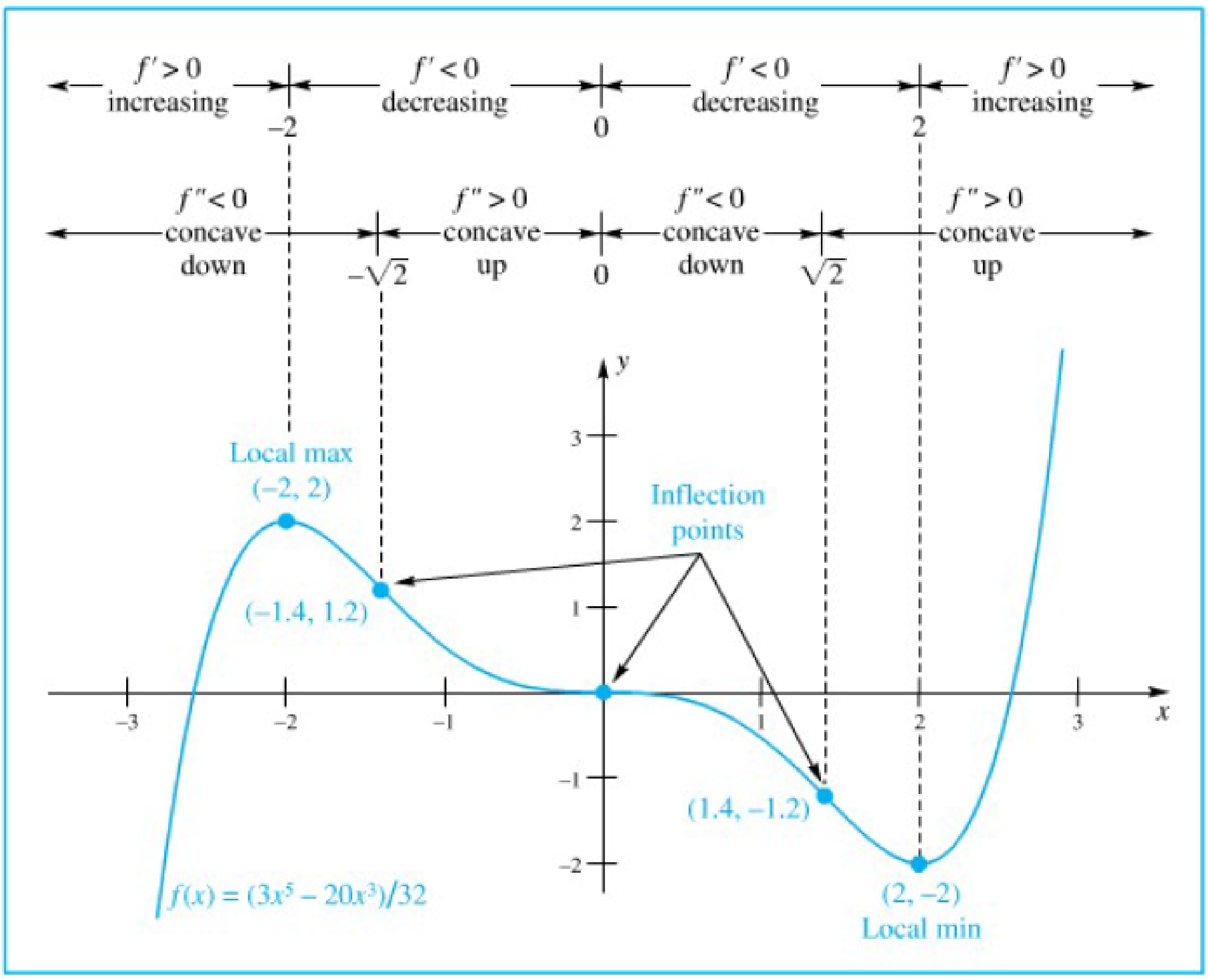
(i) If $f''(x) < 0$, then $f(c)$ is a local maximum value of f.

(ii) If $f''(x) > 0$, then $f(c)$ is a local minimum value of f.



3.5 Graphing Functions Using Calculus

Example(on page 179 in the text). Sketch the graph of $f(x) = \frac{3x^5 - 20x^3}{32}$.



II. Problems

Problem 1. Given that $f(3) = 5$, $f'(3) = 1$, $g(3) = 2$, $g'(3) = -2$ find the value of $\left((2f + 3g)^4\right)'(3)$.

Problem 2. Given that $f(e) = e$ and $f'(e) = \sqrt[4]{5}$. Find the derivative of $f\left(f\left(f\left(f(x)\right)\right)\right)$ at $x = e$.

Problem 3. Find the equation of the tangent line to $y = \left(x^2 + 1\right)^3 \left(x^4 + 1\right)^2$ at $x = 1$.

Problem 4. $\frac{d^n}{dx^n}(\cos x)$

Problem 5. From the top of a building $160ft$ high, a ball is thrown upward with an initial velocity of 64 ft/sec.

(1) When does it reach its maximum height?

(2) What is its maximum height?

(3) When does it hit the ground?

(4) With what speed does it hit the ground?

(5) What is its acceleration at $t = 2$?

Problem 6. For the implicitly defined curve $\sin\left(\frac{x^2 y\pi}{2}\right) = xy$, find the equation of the perpendicular line to the curve at the point $(1,\ 1)$.

Problem 7. Oil spilling from a ruptured tanker spreads in a circle on the surface of the ocean. The area of the spill increases at a rate of $9\pi\ m^2/min$. How fast is the radius of the spill increasing when the radius is $10m$?

Problem 8. A conical paper cup is 15 cm tall with a radius of 10 cm. The cup is being filled with water so that the water level rises at a rate of $2cm/sec$. At what rate is water being poured into the cup when the water level is $9cm$?

Problem 9. A spherical balloon is inflated so that its radius(r) increases at a rate of $\frac{2}{r}$ cm/sec. How fast is the volume of the balloon increasing when the radius is 4 cm?

Problem 10. Given the following functions, find dy and then compute its value at $x = \frac{\pi}{2}$ for $dx = 0.1$

(1) $y = \frac{1}{x}$

(2) $y = (\sin(2x) + \cos(2x))^3$

Problem 11. Use differentials to approximate the given numbers

(1) $\sqrt{35.9}$

(2) $\sqrt[3]{27.01}$

Problem 12. The diameter of a sphere is measured as 20 ± 0.1 centimeters. Find the absolute error and the relative error in the volume . Also estimate the volume of the sphere.

Problem 13. Find the linear approximation to $f(x) = 2x + \cos(3x)$ at $x_0 = \frac{\pi}{3}$. (Write answer in form $y = f(x_0) + f^{'}(x_0)(x - x_0)$.)

Problem 14. Identify the critical points and find the maximum value and minimum value on the given interval.

(1) $f(x) = \sin x$ on $\left[-\frac{\pi}{4},\ \frac{\pi}{6}\right]$

(2) $f(x) = x^3 - 3x + 1$ on $\left(-\frac{3}{2},\ 3\right)$

(3) $f(x) =\mid x - 1 \mid$ on $[0,\ 3]$

Problem 15. Where is $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ increasing, decreasing, concave up, concave down? Find all minimum and maximum points. Sketch the graph.

Problem 16. Find the inflection points of $f(x) = x^{\frac{1}{3}} + 2$.

Problem 17. Find (if any exist) the maximum and minimum values of $f(x) = \frac{1}{x(1-x)}$ on $(0,\ 1)$.

Problem 18. Sketch the graphs of the given functions. Include asymptotes, minimum points, maximum points and inflection points.

(1) $f(x) = 2x^3 - 3x^2 - 12x + 3$

(2) $f(x) = \frac{x^2-2x+4}{x-2}$