9.1 Infinite Sequences

Ex 1: Write the first four terms and determine if the sequence $\{a_n\}$ converges or diverges given $a_n = (2n)^{1/2n}$	A sequence $\{a_n\}$ converges if $\lim_{n \to \infty} a_n =$ finite number . Otherwise, $\{a_n\}$ diverges. (In other words, the limit could be some sort of infinity or the limit could not exist and the sequence would diverge.)

Ex 2: Determine if the sequence $\{a_n\}$ converges or diverges.

(a) $a_n = 2 + (0.99)^n$	(b) $a_n = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdot \cdot \cdot \left(1 - \frac{1}{n^2}\right)$, for $n \geq 2$

Ex 3: Find an explicit formula for a_n . Determine if $\{a_n\}$ converges or diverges. (Hint: first decide which n-value you will start with and then make a table of values.)

(a) $1-\frac{1}{2}\ ,\ \frac{1}{2}-\frac{1}{3}\ ,\ \frac{1}{3}-\frac{1}{4}\ ,\ \frac{1}{4}-\frac{1}{5}\ ,\ \ldots$	(b) $-1\ ,\ \frac{2}{3}\ ,\ -\frac{3}{5}\ ,\ \frac{4}{7}\ ,\ -\frac{5}{9}\ ,\ \ldots$

Ex 4: Find an explicit formula for a_n . Determine if $\{a_n\}$ converges or diverges.

$$a_1=1\ ,\ a_{n+1}=1+\frac{1}{2}a_n\ ,\ n=2,3,4,\ldots$$

9.2 Infinite Series

Ex 1: Determine if these series converge or diverge. (a) $\sum_{n=1}^{\infty} \frac{5^{n+1}}{8^{n-1}}$	Assuming $\sum_{n=1}^{\infty} a_n$ is a positive series (meaning that each of the a_n terms are positive), you can use these tests to determine convergence or divergence of the series. (1) <u>nth term test for DIVERGENCE</u>: If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. (2) <u>Geometric Series</u>: For series of the form $\sum_{n=k}^{\infty} ar^n$ where *k* and *a* are constants, $\sum_{n=k}^{\infty} ar^n = \frac{\text{first term}}{1-r}$ if $\lvert r \rvert < 1$. Otherwise $\sum_{n=k}^{\infty} ar^n$ diverges if $\lvert r \rvert \geq 1$. (And, first term = the term in the series when you plug in the first value of n, i.e. ar^k .)
(b) $\sum_{n=3}^{\infty} \frac{2\text{n}+1}{2\text{n}-3}$	Note: The infinite sum operator $\sum_{n=1}^{\infty}$ is a linear operator only on convergent positive series!!!!! That is, (a) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ and (b) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$ (where c is a constant) IF both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent positive series. In other words, we can distribute the infinite summation ONLY when we already know the series are each convergent.

Ex 2: Write 0.1256565656... as an infinite series and then as a simplified fraction, using the series.

Ex 3: Do these series converge or diverge? Explain your answer.

(a) $\sum_{n=1}^{\infty}\left(2\left(\frac{3}{5}\right)^n+500\left(\frac{1}{2}\right)^n\right)$	(b) $\sum_{n=1}^{\infty} n^2\sin\left(\frac{1}{n^2}\right)$

9.3 Positive Series Tests

Ex 1: Do these series converge or diverge?	Positive Series Tests:
(a) $\sum_{n=1}^{\infty} n^3 e^{-3n^4}$	Assuming $\sum_{n=1}^{\infty} a_n$ is a positive series. (1) <u>nth term test for DIVERGENCE</u>: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. (2) <u>Geometric Series</u>: For series of the form $\sum_{n=k}^{\infty} ar^n$ where k and a are constants, $\sum_{n=k}^{\infty} ar^n = \frac{\text{first term}}{1-r}$ if $\lvert r\rvert < 1$. Otherwise $\sum_{n=k}^{\infty} ar^n$ diverges if $\lvert r\rvert \geq 1$. (3) <u>p-series</u>: $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges, if } p>1 \\ \text{diverges, if } p \leq 1 \end{cases}$
(b) $\sum_{n=2}^{\infty} \left(\frac{1}{n^{5/3}} + \frac{1}{3^n} \right)$	(7) <u>Integral Test</u>: If f is a (a) continuous, (b) positive, and (c) non-increasing function on $[k, \infty)$, then $\sum_{n=k}^{\infty} a_n$ converges iff $\int_k^{\infty} f(x)\,dx$ where $a_n = f(n)$. (8) <u>Argument by Partial Sums</u>: If $S_p = \sum_{n=1}^{p} a_n$ and $\lim_{p\to\infty} S_p = S < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges to S. Otherwise, if $\lim_{p\to\infty} S_p$ either DNE or goes to some sort of infinity, then $\sum_{n=1}^{\infty} a_n$ diverges. Note: This is the order of tests I prefer. The other tests will be filled in later.

9.3 (continued)

Ex 2: Determine if these series converge or diverge. And, if they converge, find their sum.

(a) $\sum_{n=4}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n-3}\right)$

(b) $\sum_{n=5}^{\infty}\left(4\left(\frac{5}{2}\right)^{-n}+5\left(\frac{1}{3}\right)^{2n}\right)$

Ex 3: Show that $\sum_{n=3}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges.

Ex 4: Rewrite this as a p-series and determine if this series converges or diverges.

$$\sum_{n=9}^{\infty} \frac{4}{(n+3)^2}$$

9.4 Positive Series: More Tests

Ex 1: Determine if these series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{3n+1}}{2n^2-1}$

(b) $\sum_{n=1}^{\infty} \frac{n^{10}+1}{3^n}$

Positive Series Tests:

Assuming $\sum_{n=1}^{\infty} a_n$ is a positive series.

(4) LCT (Limit Comparison Test):

If $a_n \geq 0$, $b_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and if $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

If L = 0 AND $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Otherwise, there is no conclusion.

(Note: $\sum_{n=1}^{\infty} a_n$ refers to the series that we are given, $\sum_{n=1}^{\infty} b_n$ is a series we CHOOSE to compare our given series to. With this test, you always have to choose $\sum_{n=1}^{\infty} b_n$ on your own. Typically, we choose p-series since we know everything about them.)

(5) RT (Ratio Test):

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \quad \begin{cases} \text{if } \rho < 1 \text{, the series converges} \\ \text{if } \rho > 1 \text{, the series diverges} \\ \text{if } \rho = 1 \text{, there's no conclusion} \end{cases}$$

(If there is no conclusion, it means you have to try a different test until you get a conclusion.)

(6) OCT (Ordinary Comparison Test):

If $0 \leq a_n \leq b_n$ for $n \geq N$ (for some finite N value), then:

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

$\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ diverges

Note: This is the order of tests I prefer. The other tests are given on a previous page, from the last section of notes.

9.4 (continued)

Ex 2: Determine if these series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n^{1000}}$	(b) $\sum_{n=1}^{\infty} \frac{1}{5+\cos^2 n}$
(c) $\frac{\ln 2}{4}+\frac{\ln 3}{9}+\frac{\ln 4}{16}+\ldots$	(d) $1+\frac{2}{9\sqrt[4]{3}}+\frac{3}{25\sqrt[4]{5}}+\frac{4}{49\sqrt[4]{7}}+\ldots$

9.4 (continued)

Ex 3: Determine if these series converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$	(b) $\sum_{n=1}^{\infty} \left(1-\frac{2}{n}\right)^{5n}$
(c) $\sum_{n=1}^{\infty} \frac{10^n+n^{10}}{(2n)!}$	(d) $\sum_{n=1}^{\infty} n\left(\frac{1}{5}\right)^{n}$

9.5 Alternating Series

Ex 1: Do these series converge absolutely, converge conditionally or diverge? (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n-1}}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$	Important Note: Now we need to test for convergence differently. Remember that ALL the tests given so far are only for positive series. So you can use any of those tests to test for absolute convergence. You do this by doing any positive series test (the one that makes the most sense) on the series of terms in absolute value. If a series does not converge absolutely (i.e. the series of terms in absolute value diverges by one of the positive series tests), then (a) if it's an all-positive series, then you're done, it diverges. (b) if it's an alternating series, try the AST to see if it converges conditionally. In other words, if you're given the series $\sum_{n=1}^{\infty} c_n$, then test for absolute convergence on the corresponding series $\sum_{n=1}^{\infty} \lvert c_n \rvert$. <u>AST (Alternating Series Test)</u>: If we have an alternating series, $\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n > 0$ for all n and if $\{a_n\}$ is decreasing and $\lim_{n \to \infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges (at least conditionally). Another Note: If you have an alternating series and you do the AST first and find conditional convergence, you STILL have to test for absolute convergence before you can make your final conclusion.

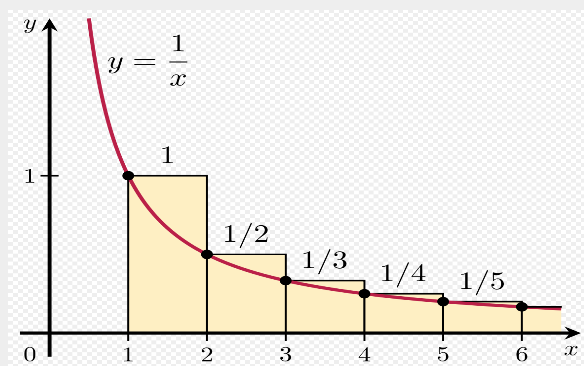
9.5 (continued)

ART (Absolute Ratio Test):
This test is exactly the RT except you put absolute values on the term you're taking the limit of.

Ex 2: Determine if these series converge absolutely, converge conditionally, or diverge.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n^{1.01}+10}$	(b) $\sum_{n=1}^{\infty} \frac{(-5)^n n^3}{8^n}$
(c) $1-\frac{5^2}{2!}+\frac{5^4}{4!}-\frac{5^6}{6!}+....$	(d) $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$

9.3-9.5 Notes on Error Bounds

Ex 1: How large should p be so that S_p approximates $S=\sum_{n=1}^{\infty}\frac{1}{n^{9/8}}$ with error no bigger than 0.001?	For any convergent series, we can write it as $S=\sum_{n=1}^{\infty}a_n=S_p+E_p=\sum_{n=1}^{p}a_n+\sum_{n=p+1}^{\infty}a_n$ where $S_p=\sum_{n=1}^{p}a_n$ and $E_p=\sum_{n=p+1}^{\infty}a_n$, and use S_p (the pth partial sum) to approximate the real sum of the infinite series. But there is an error to pay for that approximation. These are ways to bound that error, E_p. (1) For an all positive series, $\sum_{n=1}^{\infty}a_n$, $E_p=\sum_{n=p+1}^{\infty}a_n<\int_p^{\infty}f(x)dx$ where $f(n)=a_n \forall n=1,2,3,...$ and $\int_p^{\infty}f(x)dx$ converges. 

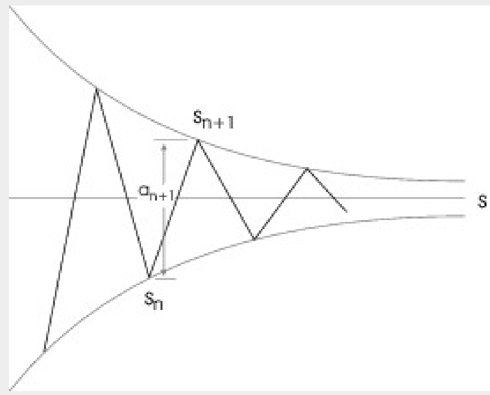
<u>9.3-9.5 (Error bounds continued)</u>

Ex 2: (a) Estimate the error made by using S_{10} as an approximation to $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2n}{n^2-1}$

(2) For an alternating series, $\sum_{n=1}^{\infty} (-1)^n a_n$,

$$|E_p| = \left| \sum_{n=p+1}^{\infty} (-1)^n a_n \right| \leq a_{p+1} \ .$$

Notice that the way this alternating series is written, we are assuming all the a_n terms are positive.



(b) What if we want to ensure an error of 0.001, then what must p be? (That is, if we want $E_p \leq 0.001$, what must p be?)

(c) And what might that tell you about the rate of convergence for this series?

9.6 Power Series

<table>
<tr>
<td rowspan="2">Ex 1: Find the convergence set for the series $\sum_{n=2}^{\infty} \frac{(2x-1)^n}{(n+1)!}$</td>
<td><u>Power Series</u>:
It is a function of x. Now the sum has two variables, one is the summation variable (usually n) and the other is a different variable (usually x).
$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$
A power series is basically an infinite-degree polynomial in x.

This series converges in one of the following ways:
(a) converges at x = 0.
OR
(b) converges on (-R, R) (or [-R, R) or (-R, R] or [-R, R]) (Notice that the center point for all of these intervals is x = 0.)
OR
(c) all real numbers (i.e. $(-\infty, \infty)$).</td>
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<td>To find the convergence set, you always do these steps:
(1) Use ART and force convergence to see what x-values make that true (i.e. do ART and force the limit result to be < 1 and then solve for the x-interval using algebra).

And, then
(2) Check the endpoints. By this I mean you need to separately, plug in each of the x-values that are the endpoints of the interval of convergence that you got in part (1). Then, use appropriate tests on those infinite series of numbers to see if those series converge or diverge.

What does convergence mean?
(i) For an infinite series of numbers to converge, it means their sum adds to something finite.

(ii) For a power series to converge, it means that the infinite-degree polynomial exactly matches the function of x on that interval of convergence.</td>
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</table>

9.6 (continued)

Ex 2: For each of these power series, find the convergence set and the radius of convergence, R.

(a) $x+4x^2+9x^3+16x^4+\ldots$

(b) $(x+3)-2(x+3)^2+3(x+3)^3-4(x+3)^4+\ldots$

9.6 (continued)

Ex 3: For each of these power series, find the convergence set and the radius of convergence, R.

(a) $\frac{x}{3}+\frac{x^2}{8}+\frac{x^3}{15}+\frac{x^4}{24}+...$
(b) $\sum_{n=1}^{\infty}\frac{(-1)^n(2\text{x}-3)^n}{4^n\sqrt{n}}$

9.7 Operations on Power Series

Ex 1: Find the power series for f(x), along with its radius of convergence. (a) $f(x)=\dfrac{2x^3}{1-x^4}$	If $f(x)=\sum_{n=0}^{\infty} a_n x^n$ converges on the interval I, then $f'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\int_0^x f(t)dt=\sum_{n=0}^{\infty} \dfrac{a_n x^{n+1}}{n+1}$ also converge on the interior of I.
(b) $f(x)=e^{3x}+3x^2-5$	<u>Power Series to have on your note card</u>: (along with their convergence sets) (1) $\dfrac{1}{1-x}=\sum_{n=0}^{\infty} x^n \quad \forall x \in (-1,1)$ (2) $\ln(1+x)=\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^n}{n} \quad \forall x \in (-1,1)$ (3) $\arctan x=\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^{2n-1}}{2n-1} \quad \forall x \in [-1,1]$ (4) $e^x=\sum_{n=0}^{\infty} \dfrac{x^n}{n!} \quad \forall x \in \mathbb{R}$

Ex 2: Find the power series for f(x), along with its radius of convergence.

(a) $f(x)=\int_0^x \frac{\arctan t}{t}\,dt$	(b) $f(x)=\frac{x}{x^2-3x+2}$ (Hint: Use PFD.)

9.7 (continued)

Ex 3: Find the sum of $\sum_{n=1}^{\infty} n\,x^n$, i.e. find the function that this power series represents, and on what interval.

9.8 Taylor and Maclaurin Series

<table>
<tr>
<td rowspan="2">Ex 1: Find the Taylor Series for the following function, given the center value and state the radius of convergence. $f(x)=e^x$, $a=2$</td>
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<u>Taylor's Theorem:</u>

Assume $f(x)$ is a function with derivatives of all orders in some interval $(a-R, a+R)$
The Taylor Series for $f(x)$ is given by

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

on $(a-R, a+R)$ where R is the radius of convergence,

iff

$\lim_{n\to\infty} R_n(x)=0$, i.e. the remainder goes to zero, where the remainder is given by

$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c\in(a-R, a+R)$.

Remember that a Maclaurin series is just a Taylor Series with *a* = 0, i.e. the center value is 0.

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<u>Taylor's Formula with Remainder</u>:

Assume $f(x)$ is a function with at least (n+1) derivatives existing for each *x* in an open interval, I, containing *a*. Then, for each *x* in that interval, I,

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x)$$

where the remainder (or error) $R_n(x)$ is given by $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c\in[min(a,x), max(a,x)]$.

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Ex 2: Find the Taylor Series for the following functions, given the center value and state the radius of convergence. (a) $f(x) = \sec x$, $a = \frac{\pi}{4}$, out to the third degree term	Power (Maclaurin) Series to have on your note card: (5) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$ (6) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \forall x \in \mathbb{R}$ (7) $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R}$ (8) $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \forall x \in \mathbb{R}$
(b) $f(x) = \frac{1}{1-x}$, $a = 3$	

Ex 3: Find the Taylor Series for the following functions, given the center value and state the radius of convergence.

$f(x) = x(\sin(2x) + \sin(3x))$, $a = 0$

Ex 4: Given $f(x) = \begin{cases} 0 , & x < 0 \\ x^4 , & x \geq 0 \end{cases}$, explain why we cannot use Maclaurin series, where $a = 0$, to represent this function. Can we use a different Taylor Series (centered somewhere other than 0)? Why?

9.9 Taylor Approximation to a Function

Some new algebra practice first for error bounds.

Ex 1: Find a good bound for the max value of these given expressions. (a) $\left\lvert \frac{c^2+\sin c}{10\ln c} \right\rvert$ on the interval [2,4]	Note: An error bound is just an upper bound on the worst case scenario of how off your estimate can be. For example of an upper bound, I could say that my age is less than 200 years. This is an upper bound because surely I'm not more than 200 years old. However, it's a terrible upper bound because it doesn't give you a decent estimate of my actual age and it doesn't even make sense since most people don't live to be 200 years old. If I said that an upper bound for my age is 70, then I'm guaranteeing that I'm not older than 70 years, but it doesn't tell you how close I am to 70 years old. However, it's a MUCH better upper bound on my age than 200.
(b) $\lvert \tan c + \sec c \rvert$ on the interval $[0, \frac{\pi}{3}]$	

Ex 2: Find a Taylor polynomial of order 3 based at *a* for the following functions. (a) $f(x)=\sqrt{x}$, $a=2$	Ex 3: Use the Taylor polynomials in Ex 2 to (i) find the formula for $R_3(x)$, at any x-value, and (ii) find a good upper bound for $\lvert R_3(x)\rvert$ for the given x-values. (a) x = 1 and x = 3
(b) $f(x)=2^x$, $a=1$	(b) x = 0 and x = 2

<u>9.9 (continued)</u>

Ex 4: Find (i) a formula for $R_6(x)$ for any x-value and (ii) find a good upper bound for $|R_6(x)|$ for the given x-value.

(a) $f(x)=\frac{1}{x^2}$, $a=1$, at $x=\frac{1}{2}$	(b) $f(x)=\frac{1}{x^2}$, $a=1$, at $x=5$
(c) $f(x)=\frac{2}{x-3}$, $a=1$, at $x=0$	(d) $f(x)=\frac{2}{x-3}$, $a=1$, at $x=2$