Major Theorems: Section 2.1

Section 2.1

Theorem 1 Properties of Matrix Addition and Scalar Multiplication *Let A, B, and C be matrices of the same size, $m \times n$, and let r and s be scalars.*

(a) $A + B =$ $B + A$

(b) $(A + B) + C =$ $A + (B + C)$

(c) $A + 0 =$ A

(d) $r(A + B) =$ $rA + rB$

(e) $(r + s)A =$ $rA + sA$

(f) $r(sA) =$ $(rs)A$

Theorem 2 Properties of Matrix Multiplication *Let $A, B,$ and C be matrices and r be a scalar such that the sums and products below are defined. Then,*

(a) $A(BC) =$ $(AB)C$

(b) $A(B + C) =$ $AB + AC$

(c) $(B + C)A =$ $BA + CA$

(d) $r(AB) =$ $A(rB) = (rA)B$

(e) $I_m A =$ A

Theorem 3 Transpose Properties *Let A and B be matrices whose sizes are appropriate for the following sums and products.*

(a) $(A^T)^T =$ A

(b) $(A + B)^T =$ $A^T + B^T$

(c) *For any scalar r,* $(rA)^T =$ rA^T

(d) $(AB)^T =$ $B^T A^T$

Note: *The transpose of a product of matrices equals the product of their transposes in reverse order.*

Supplemental Practice Problems:

1. Consider the following matrices

$$\begin{bmatrix} 8 & -5 & 2 \\ -11 & 8 & 13 \\ -3 & 3 & 15 \end{bmatrix} \qquad \underset{2\times 3}{A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{bmatrix}}, \quad \underset{3\times 2}{B = \begin{bmatrix} 1 & -2 \\ 2 & 5 \\ 3 & 3 \end{bmatrix}}, \quad \underset{2\times 2}{\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = C}$$

Calculate (if possible) each of the following matrix products:

(a) AB $\begin{bmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 12 & 3 \\ 4 & 19 \end{bmatrix}}$

(b) BA

(c) AC can't do because # cols of A ≠ # rows of C

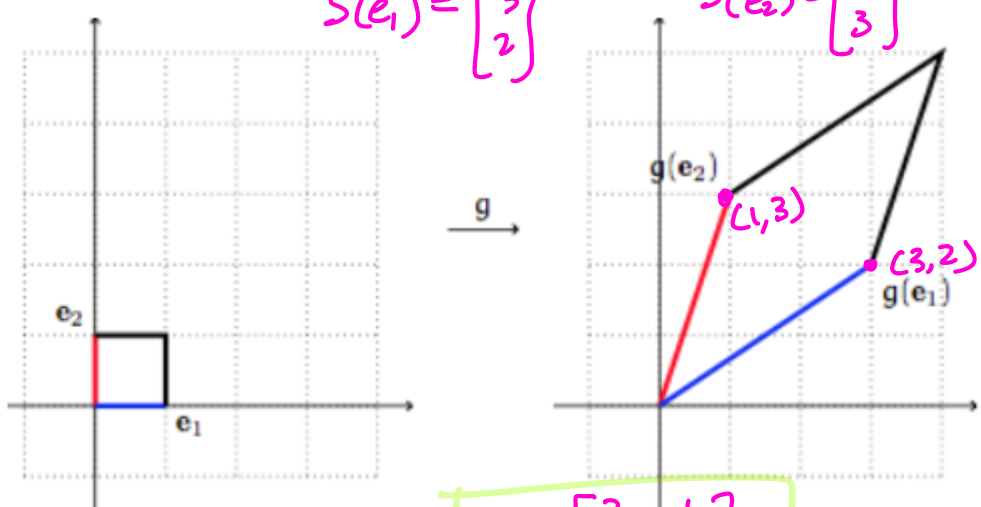
(d) CA $= \boxed{\begin{bmatrix} 19 & -12 & 3 \\ -11 & 7 & -1 \end{bmatrix}}$

2. Let R be the rectangle with vertices (-2, -1), (-2, 2), (3, 2), (3, -1). Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(x_1, x_2) = (4x_1 - 3x_2, -x_1 + x_2)$.

   Find the standard matrix A of the linear transformation T, and sketch the image of the rectangle R under T.

3. Let R be the rectangle with vertices (-2, -1), (-2, 2), (3, 2), (3, -1). Consider the linear transformation $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which maps the unit square to the parallelogram pictured below.

$S(\vec{e}_1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\qquad$ $S(\vec{e}_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



Find the standard matrix B associated to S. $\boxed{B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}}$

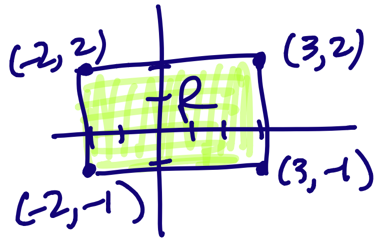
4. Consider the matrices $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}$ where b and c are unknowns. Find values of b and c such that $AB = BA$.

$$AB = \begin{bmatrix} 3+5c & 3b-5 \\ 1+2c & b-2 \end{bmatrix} \qquad BA = \begin{bmatrix} 3+b & 5+2b \\ 3c-1 & 5c-2 \end{bmatrix}$$

we want $\begin{bmatrix} 3+5c & 3b-5 \\ 1+2c & b-2 \end{bmatrix} = \begin{bmatrix} 3+b & 5+2b \\ 3c-1 & 5c-2 \end{bmatrix}$

(continued below)

2) $T(x_1,x_2) = (4x_1 - 3x_2, -x_1 + x_2)$



or put another way

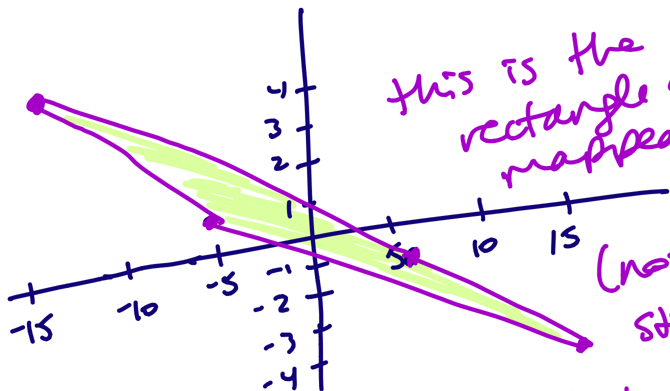
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \boxed{A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}}$$

(x_1, x_2) pt in 2d	where it mapped to $T(x_1, x_2)$
$(-2,-1)$	$(4(-2) - 3(-1), -(-2) + -1) = (-5, 1)$
$(-2, 2)$	$(-8-6, 2+2) = (-14, 4)$
$(3, 2)$	$(12-6, -3+2) = (6, -1)$
$(3, -1)$	$(12+3, -3-1) = (15, -4)$



this is the rectangle above mapped by T (notice it's still a quadrilateral)

4) $\begin{bmatrix} 3+5c & 3b-5 \\ 1+2c & b-2 \end{bmatrix} = \begin{bmatrix} 3+b & 5+2b \\ 3c-1 & 5c-2 \end{bmatrix}$

$\Leftrightarrow$ ① $3 + 5c = 3 + b$ and ② $3b - 5 = 5 + 2b$

$5c = b$ $\qquad \boxed{b = 10}$

$\longrightarrow$ $10 = 5c$

$\boxed{c = 2}$

and ③ $1 + 2c = 3c - 1$

check if $c = 2$ works: $1 + 4 = 6 - 1$ ✓

and ④ $b - 2 = 5c - 2$

check: $b = 10$, $c = 2$

$8 = 8$ ✓