Key Definitions: Sections 2.1-2.4

- The identity matrix I_n is

- A diagonal matrix is

- A zero matrix is

- The transpose of a matrix A is

- An elementary matrix is

- A partitioned or block matrix A is

Major Theorems: Sections 2.1-2.4

Section 2.1

Theorem 1 Properties of Matrix Addition and Scalar Multiplication *Let A, B, and C be matrices of the same size, $m \times n$, and let r and s be scalars.*

(a) $A + B =$

(b) $(A + B) + C =$

(c) $A + 0 =$

(d) $r(A + B) =$

(e) $(r + s)A =$

(f) $r(sA) =$

Theorem 2 Properties of Matrix Multiplication *Let $A, B,$ and C be matrices and r be a scalar such that the sums and products below are defined. Then,*

(a) $A(BC) =$

(b) $A(B + C) =$

(c) $(B + C)A =$

(d) $r(AB) =$

(e) $I_m A =$

Theorem 3 Transpose Properties *Let A and B be matrices whose sizes are appropriate for the following sums and products.*

(a) $(A^T)^T =$

(b) $(A + B)^T =$

(c) For any scalar r, $(rA)^T =$

(d) $(AB)^T =$

Note: *The transpose of a product of matrices equals the product of their transposes in reverse order.*

Section 2.2

Theorem 4 2×2 **Inverses** *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible, and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, *then* A *is not invertible (is singular).*

The quantity $ad - bc$ *is called the* **determinant** *of* A.

Thus, a 2×2 *matrix is invertible if and only if* $\det A \neq 0$.

Theorem 5 Matrix Equation Solutions and Inverses *If* A *is an invertible* $n \times n$ *matrix, then for each* $\mathbf{b} \in \mathbb{R}^n$, *the equation* $A\mathbf{x} = \mathbf{b}$ *has the unique solution,* $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 6 Properties of Inverses

(a) If A *is an invertible matrix, then* A^{-1} *is invertible and*

$$(A^{-1})^{-1} = \underline{\hspace{3cm}}$$

(b) If A *and* B *are* $n \times n$ *invertible matrices, then* AB *is also invertible. The inverse of* AB *is the product of the inverses of* A *and* B *in reverse order. That is,*

$$(AB)^{-1} = \underline{\hspace{3cm}}$$

(c) If A *is an invertible matrix, then* A^T *is also invertible. The inverse of* A^T *is the transpose of* A^{-1}. *That is,*

$$(A^T)^{-1} = \underline{\hspace{3cm}}$$

Theorem 7 *An* $n \times n$ *matrix* A *is invertible if and only if* A *is row equivalent to* I_n, *and in this case, any sequence of elementary row operations that reduces* A *to* I_n *also transforms* I_n *to* A^{-1}.

Section 2.3

Theorem 8 The Invertible Matrix Theorem *Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.*

(a) A is an <u>invertible</u> matrix.

(b) A is row equivalent to the $n \times n$ ____________________ matrix.

(c) A has ____________________ postions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the ____________________ solution.

(e) The columns of A form a linearly ____________________ set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is ____________________.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has ____________________ solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A ____________________ $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ ____________________ $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA =$ ____________________.

(k) There is an $n \times n$ matrix D such that $AD =$ ____________________

(l) A^T is an ____________________ matrix.

Theorem 9 *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then, T is invertible if and only if A is an invertible matrix, and the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function such that*

$$A^{-1}(A\mathbf{x}) = S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

$$A(A^{-1}\mathbf{x}) = T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

where $S = T^{-1}$ is the inverse of T.

Supplemental Practice Problems:

1. Compute the inverse of $A = \begin{bmatrix} 1 & 0 & 5 \\ 4 & 2 & 20 \\ 0 & 4 & -5 \end{bmatrix}$ using the inverse algorithm, $[A\ I] \sim [I\ A^{-1}]$.

2. Find the inverse of the following matrices: (you can look up answers on matrix calculator)

   (a) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -4 & -3 \\ -1 & 1 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 5 & 5 & 2 \\ 4 & 5 & 2 \\ -2 & 1 & 0 \end{bmatrix}$

3. Consider the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ -2 & 5 & 2 \\ 0 & 3 & 1 \end{bmatrix}$

   (a) Show that $A^{-1} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ -1 & -1 & 3 \\ 3 & 3 & -8 \end{bmatrix}$. (multiply AA^{-1} to see it gives I)

   (b) Using matrix A or A^{-1}, determine the number of pivots of A and whether the columns of A are linearly independent or dependent. Since A^{-1} exists, # pivots = 3 + cols of A are lin. II.

   (c) Using matrix A or A^{-1}, find all solutions, if any, to the matrix equation $A\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$.

   $\vec{x} = A^{-1}\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$

4. Suppose that an $n \times n$ matrix has a column which is a multiple of another column. Either give an example of an invertible matrix of this type or explain why such a matrix is not invertible.

   inverse matrix cannot exist because columns are lin. dep

5. Consider the following matrices

$$A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & 5 \\ 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = C$$

   Calculate (if possible) each of the following matrix products:

   (a) AB $= \begin{bmatrix} 12 & 3 \\ 4 & 19 \end{bmatrix}$

   (b) BA

   $(3\times2)(2\times3)$ $= \begin{bmatrix} 8 & -4 & 2 \\ -11 & 8 & 13 \\ -3 & 3 & 15 \end{bmatrix}$

   (c) AC $(2\times3)(2\times2)$ DNE # cols of $A \neq$ # rows of C

   (d) CA

   $(2\times2)(2\times3)$ $= \begin{bmatrix} 19 & -12 & 3 \\ -11 & 7 & -1 \end{bmatrix}$

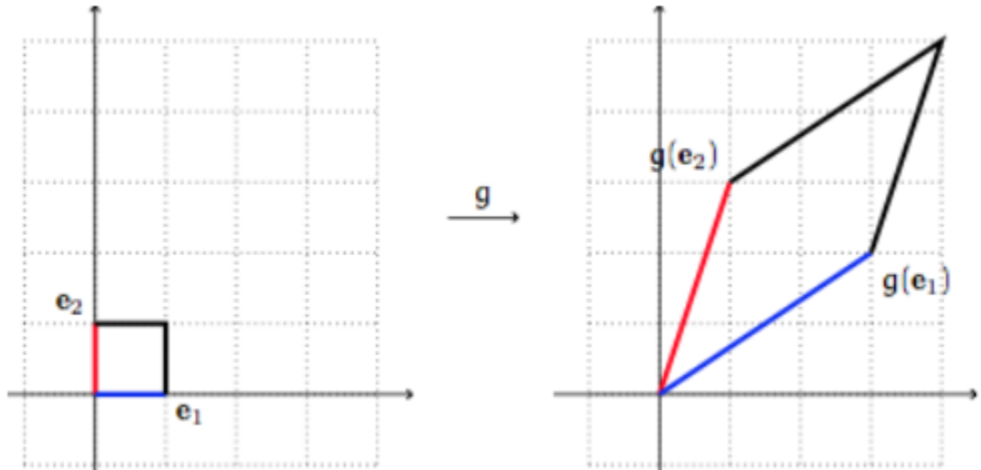
From Midterm I Review Chp 2

6. Let R be the rectangle with vertices (-2, -1), (-2, 2), (3, 2), (3, -1). Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(x_1, x_2) = (4x_1 - 3x_2, -x_1 + x_2)$.

   (a) Find the standard matrix A of the linear transformation T, and sketch the image of the rectangle R under T.

   (b) Find the standard matrix A^{-1} corresponding to the inverse of T and sketch the image of the rectangle R under T^{-1}.

7. Let R be the rectangle with vertices (-2, -1), (-2, 2), (3, 2), (3, -1). Consider the linear transformation $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which maps the unit square to the parallelogram pictured below.



$T(\vec{e}_1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\Rightarrow B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$

   (a) Find the standard matrix B associated to S and sketch the image of the rectangle R under S.

   (b) Find the matrix B^{-1} associated to the inverse of S and sketch the image of the rectangle R under S^{-1}. $B^{-1} = \frac{1}{9-2} \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3/7 & -1/7 \\ -2/7 & 3/7 \end{bmatrix}$

8. Consider the matrices $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ c & -1 \end{bmatrix}$ where b and c are unknowns. Find values of b and c such that $AB = BA$.

9. Given $\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$. Find formulas for X, Y and Z in terms of A, B and C. Justify your calculations. That is, in some cases, you may need to make assumptions about the size of a matrix in order to produce a formula.

✓10. Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ where B and C are square blocks. Show that A is invertible if and only if both B and C are invertible.

($\Rightarrow$) Assume A^{-1} exists. let $A^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$

$C \neq 0$

and we know $AA^{-1} = I$.

$B \neq 0$

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

① $BE = I$ ② $CH = I$ ③ $BF = 0 \Rightarrow F = 0$

$\Rightarrow E = B^{-1}$ $\Rightarrow H = C^{-1}$ ④ $CG = 0$

$G = 0$

$\Rightarrow B^{-1} = E,\ C^{-1} = H$ //

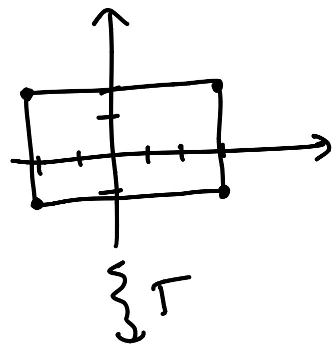
($\Leftarrow$) Assume B^{-1}, C^{-1} exists. Prove A^{-1} exists.

Let $D = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$.

$$AD = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

$$\Rightarrow D = A^{-1}$$ //

#6) $(-2,-1)$ $(-2,2)$ $(3,2)$ $(3,-1)$

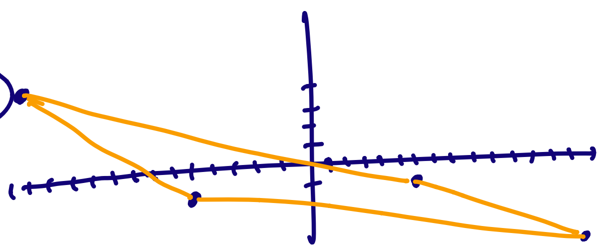


$T(x_1,x_2) = (4x_1 - 3x_2, -x_1 + x_2)$

(a) $T(-2,-1) = (4(-2) - 3(-1),\ 2 + -1)$

$= (-5, 1)$

$T(-2,2) = (-8-6,\ 2+2) = (-14, 4)$



$T(3,2) = (12-6,\ -3+2) = (6,-1)$

$T(3,-1) = (12+3,\ -3-1) = (15,-4)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix} = x_1\begin{bmatrix} 4 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(6) $A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{4-3}\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

$A^{-1}\begin{bmatrix} 15 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 15 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$A^{-1}\begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ etc. we'll undo

T by multiplying all the pts in the transformed quadrilateral by A^{-1}, and get back to original pts.

#8) $\begin{bmatrix} 3+5c & 3b-5 \\ 1+2c & b-2 \end{bmatrix} = \begin{bmatrix} 3+b & 5+2b \\ 3c-1 & 5c-2 \end{bmatrix}$

$\Leftrightarrow$ ① $3+5c = 3+b$

$5c = b$

② $3b-5 = 5+2b$

$\boxed{b=10}$

$\Leftarrow$

$5c = 10$

$\boxed{c=2}$

③ $1+2c = 3c-1$

$-c = -2$

$\boxed{c=2}$

④ $b-2 = 5c-2$

check $10-2 = 5(2)-2)$ ✓

$\Rightarrow B = \begin{bmatrix} 1 & 10 \\ 2 & -1 \end{bmatrix}$

#9)

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{bmatrix} AX & AY & AZ + B \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$\Rightarrow$ $AX = I$ $\Rightarrow$ $\boxed{X = A^{-1}}$

$AY = 0$ $\Rightarrow$ $\boxed{Y = 0}$ (since $A \neq 0$ and A^{-1} exists)

$AZ + B = 0$

$AZ = -B$

$\boxed{Z = -A^{-1}B}$

we had to assume A is square and A^{-1} exists.