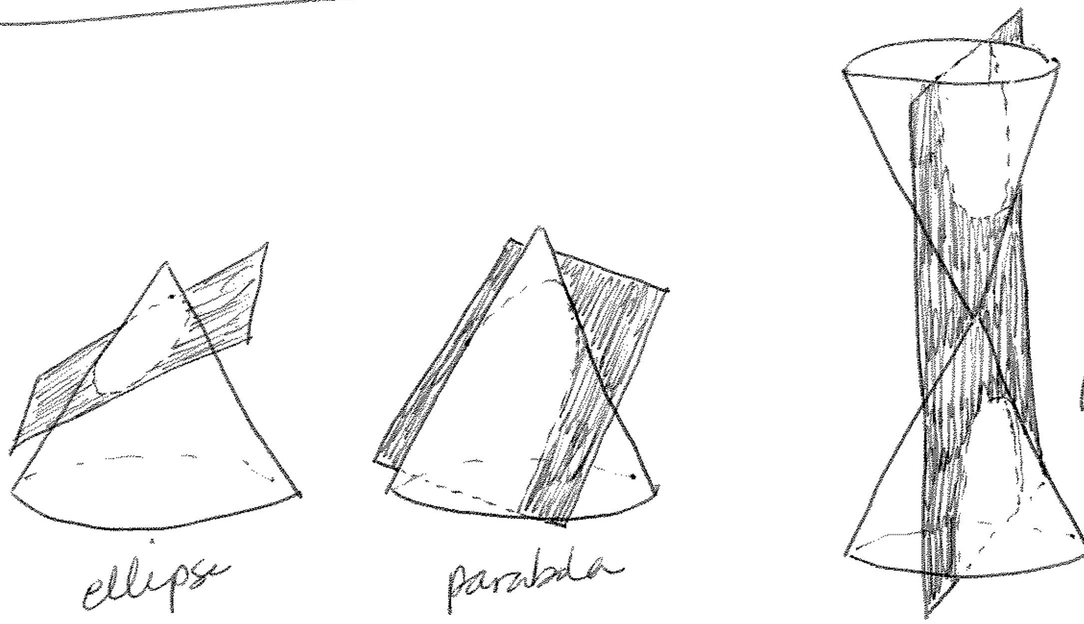
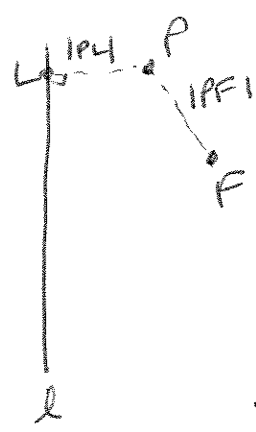
10.1-10.2 Review of Conics

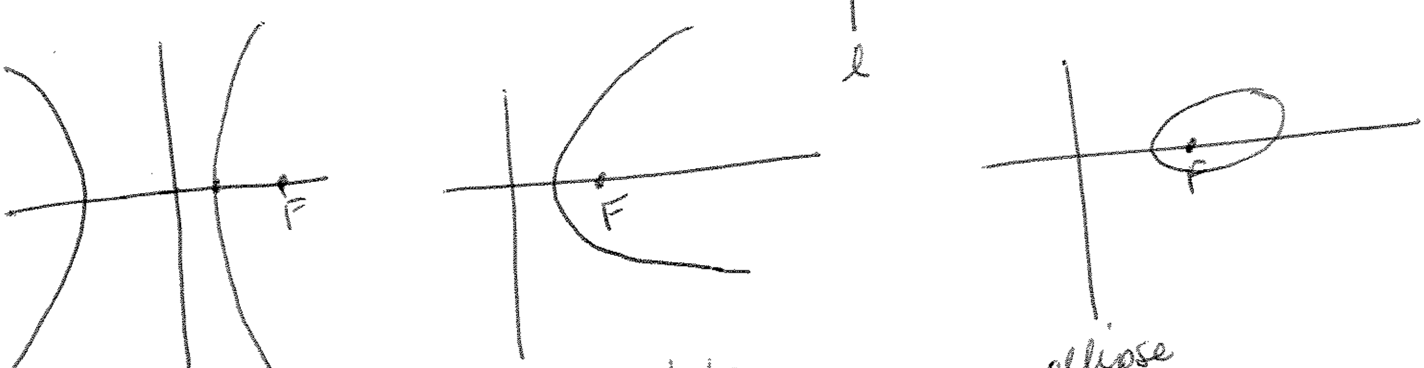


ellipse   parabola   hyperbola

Let l be a fixed line (directrix) and F be a fixed point (focus) not on the line. For the set of pts P where the ratio of the distance from F to the distance to l is a positive constant e (called eccentricity), we call it a conic.

$$\iff \quad |PF| = e|PL|$$

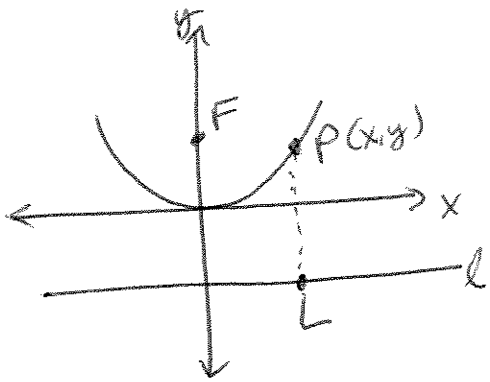




hyperbola $e>1$ parabola $e=1$ ellipse $0<e<1$

10.1-10.2 (cont)

Parabola



$F:(0,p)$ $\quad$ $L:(x,-p)$

$l:\ y=-p$

$P:(x,y)$

$|PF| = \sqrt{(y-p)^2+(x-0)^2}$

$|PL| = \sqrt{(x-x)^2+(y-(-p))^2}$

$$|PF|=|PL| \iff \sqrt{(y-p)^2+x^2} = \sqrt{(y+p)^2}$$

$$(y-p)^2+x^2=(y+p)^2$$

$$\cancel{y^2}-2py+\cancel{p^2}+x^2=\cancel{y^2}+2py+\cancel{p^2}$$

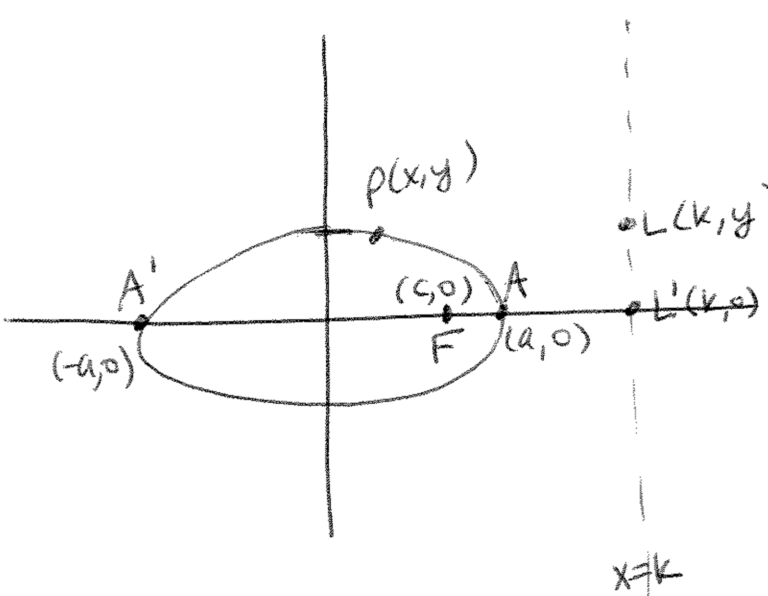
$$x^2-2py=2py$$

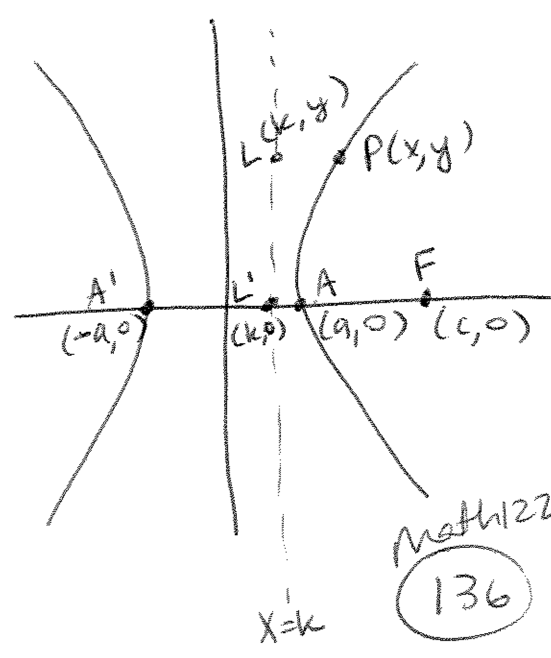
$$x^2=4py \Rightarrow \boxed{y=\frac{1}{4p}x^2}$$

(our familiar eqn of a parabola)

Ellipse + Hyperbola

These are called central conics because for each of them, there are 2 vertices, A + A' (on the major axis). They are symmetric w.r.t. their centers.





10.1 - 10.2 (cont)

Note that $|PF| = e|PL|$ for all pts P on ellipse or hyperbola.

Specifically,

$|AF| = e|AL'| \Rightarrow a - c = e(k - a)$ ①

and $|A'F| = e|A'L'| \Rightarrow a + c = e(a + k)$ ②

Solve ① + ② simultaneously for c and k.

$$\begin{aligned} a-c &= ek - ea \\ + \quad a+c &= ek + ea \\ \hline 2a &= 2ek \\ \boxed{\frac{a}{e} = k} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} a-c &= e\left(\frac{a}{e} - a\right) \\ a-c &= a - ea \\ -c &= -ea \\ \boxed{c = ea} \end{aligned}$$

$\Rightarrow |PF| = e|PL|$ becomes

$$\sqrt{(x-c)^2 + (y-0)^2} = e\sqrt{(x-k)^2 + (y-y)^2}$$

$$\sqrt{(x-c)^2 + y^2} = e\sqrt{(x-k)^2}$$

$$\left(\sqrt{(x-ea)^2 + y^2}\right)^2 = \left(e\left|x - \frac{a}{e}\right|\right)^2$$

$$(x-ea)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$x^2 - \cancel{2eax} + e^2a^2 + y^2 = e^2x^2 - \cancel{2aex} + a^2$$

$$(1-e^2)x^2 + y^2 = a^2 - e^2a^2$$

10.1–10.2 (cont)

$$(1-e^2)x^2+y^2=a^2(1-e^2)$$

$$x^2+\frac{y^2}{1-e^2}=a^2$$

$$\star\quad \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

ellipse eqn + hyperbola

Ellipse

we know $0<e<1$.

$\Rightarrow 1-e^2>0$

Let $b=a\sqrt{1-e^2}$

$\Rightarrow b^2=a^2(1-e^2)$

$$\star \Leftrightarrow \boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

standard formula for ellipse

Hyperbola

we know $e>1$.

$\Rightarrow 1-e^2<0$

$\Rightarrow e^2-1>0$

Let $b=a\sqrt{e^2-1}$

$\Rightarrow b^2=a^2(e^2-1)$

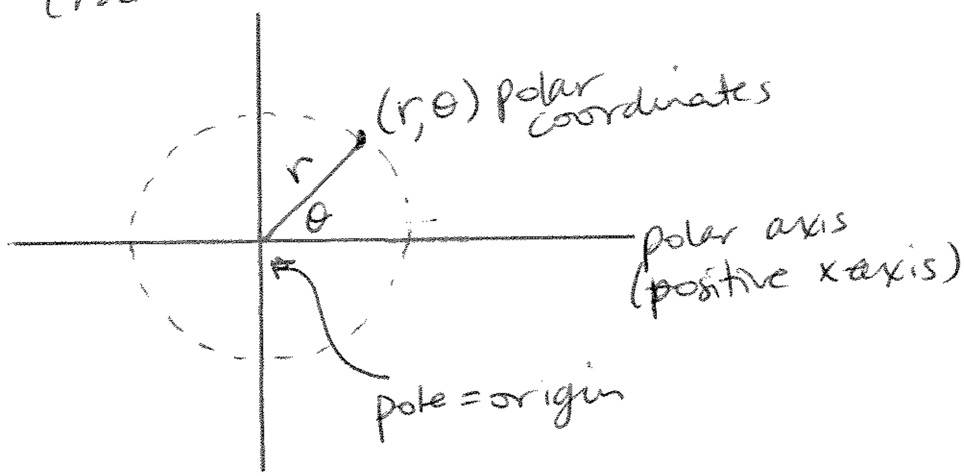
$\Leftrightarrow b^2=-a^2(1-e^2)$

$$\star \Leftrightarrow \boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

standard formula for hyperbola

10.5 The Polar Coordinate System

A different way to designate points in a plane (rather than rectangular coordinate system (x,y)).



$x^2+y^2=r^2$ (by Pythagorean Thm)

and $\tan\theta = \dfrac{y}{x}$ and $(x,y) = (r\cos\theta, r\sin\theta)$ (from Trig)

Rectangular/Polar Coord. Conversion

$r=\sqrt{x^2+y^2}$ $\tan\theta = \dfrac{y}{x}$	$x=r\cos\theta$ $y=r\sin\theta$
(from rect. to polar)	(from polar to rect.)

EX1 Find Rectangular (aka Cartesian) coords for $(4, \pi/6)$.

10.5 (cont)

Ex 2 Find polar coordinates for $(-\sqrt{2}, \sqrt{2})$.

(★ There are ∞ly many ways to write the same pt since we have as many ways to designate an angle.)

Ex 3 Find 4 other ways to write $(-3, 2\pi/3)$

Plot angle first, then radius!

10.5 (cont)

Note:
$(0, a)$ where $a \in \mathbb{R}$ is the origin in polar coords.

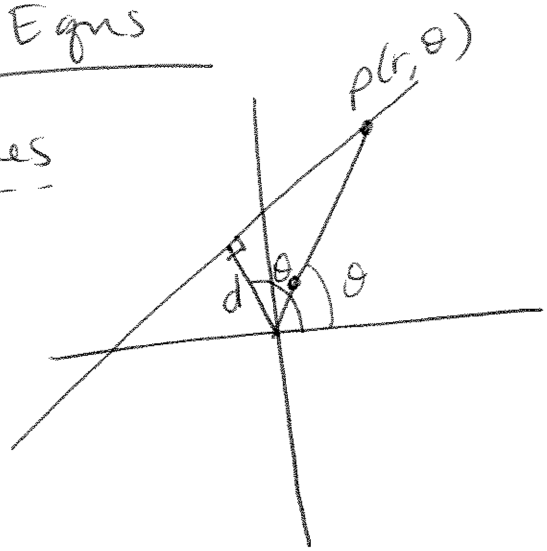
Ex 4 Plot $r = 6\sin\theta$

Ex 5 Show that $r = 6\sin\theta$ is a circle in Cartesian coord system.

10.5 (cont)

Polar Eqns

① Lines



P = pt on line

d = ⊥ distance from origin to line

θ_0 = angle from polar axis to line d

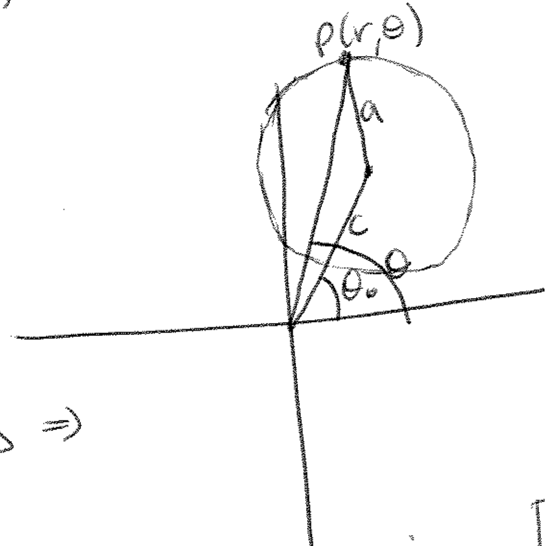
extract Δ ⇒

$$\Rightarrow \cos(\theta_0 - \theta) = \frac{d}{r} \iff \boxed{r = \frac{d}{\cos(\theta_0 - \theta)}}$$

generic eqn of line (θ_0, d constants)

(Remember, $\cos(\theta - \theta_0) = \cos(\theta_0 - \theta)$ since cosine is even.)

② Circles



P = generic pt on circle

a = radius of circle

c = distance from origin to center of circle

θ_0 = angle to line c

extract Δ ⇒

by Law of Cosines $\boxed{a^2 = r^2 + c^2 - 2rc\cos(\theta - \theta_0)}$

generic circle eqn

10.5 (cont)

If $c=a$, then the circle passes thru the origin. $\Rightarrow a^2 = r^2 + a^2 - 2ra\cos(\theta-\theta_0)$

$\Leftrightarrow r^2 = 2ra\cos(\theta-\theta_0)$

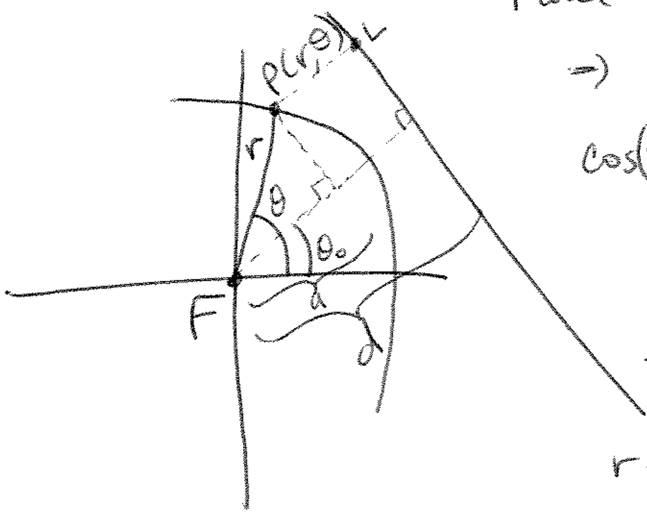
$\Rightarrow \boxed{r = 2a\cos(\theta-\theta_0)}$ circle eqn (if it goes thru origin)

If $\theta_0 = 0$, we have $\underline{r = 2a\cos\theta}$ (center of circle on x-axis)

If $\theta_0 = \pi/2$, we have $\underline{r = 2a\sin\theta}$ (center of circle on y-axis)

③ Conics (Parabolas, Hyperbolas + Ellipses)

Place focus (F) at origin.



$\Rightarrow |PF| = e|PL| \Leftrightarrow r = e|PL|$

$\cos(\theta-\theta_0) = \frac{a}{r} \Rightarrow a = r\cos(\theta-\theta_0)$

$|PL| = d - a = d - r\cos(\theta-\theta_0)$

$\Rightarrow r = e|PL|$ becomes

$r = e(d - r\cos(\theta-\theta_0))$

$r = ed - r(e\cos(\theta-\theta_0))$

$r(1 + e\cos(\theta-\theta_0)) = ed$

generic conic in polar coords $\boxed{r = \dfrac{ed}{1+e\cos(\theta-\theta_0)}}$ θ_0, e, d constants

10.5 (cont)

Ex 6 Name the curve. If it's a conic, give its eccentricity. Sketch graph.

$e=1$ parabola
$0<e<1$ ellipse
$e>1$ hyperbola

(a) $r = \dfrac{4}{2+2\cos(\theta - \pi/3)}$

(b) $r = -4\cos(\theta - \pi/4)$

(c) $\theta = \dfrac{2\pi}{3}$

10.6 Graphs of Polar Equations

Symmetry

① about x-axis ⇒ replacing (r,θ) w/ $(r,-\theta)$ produces equivalent equation

② about y-axis ⇒ replacing (r,θ) w/ $(-r,-\theta)$ produces equivalent equation

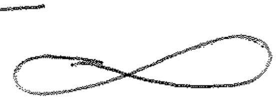
③ about origin ⇒ replacing (r,θ) w/ $(-r,\theta)$ produces equivalent equation

Limacon $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$

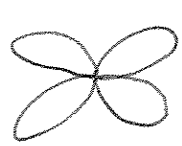
$a>b$ $a=b$ $a<b$

(**Cardioid** $r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$)

Lemniscates $r^2 = \pm a\cos(2\theta)$ or $r^2 = \pm a\sin(2\theta)$



Rose $r = a\cos(n\theta)$ $r = a\sin(n\theta)$ (n leaves, n odd; 2n " , n even)



10.6 (cont)

Spiral $r = a\theta$

Ex 1 sketch graph of given polar eqns.

(a) $r = 4 \sin \theta$

(b) $r^2 = -16 \cos(2\theta)$

10.6 (cont)

(c) $r = 4 - 3\sin\theta$

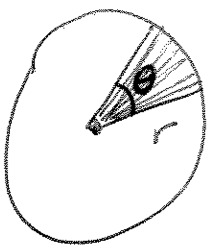
(d) $r = 2\theta$

10.6 (cont)

(e) $r = \sqrt{2} - \sqrt{2} \sin \theta$

(f) $r = 4 \cos 2\theta$

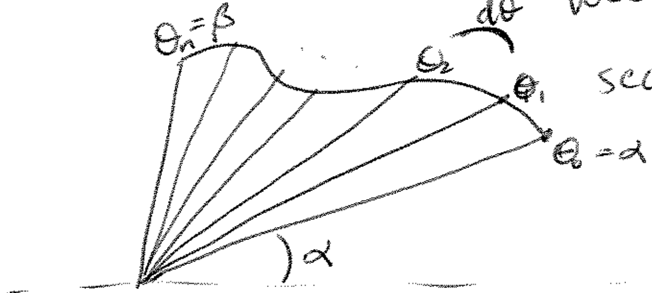
10.7 Calculus in Polar Coordinates

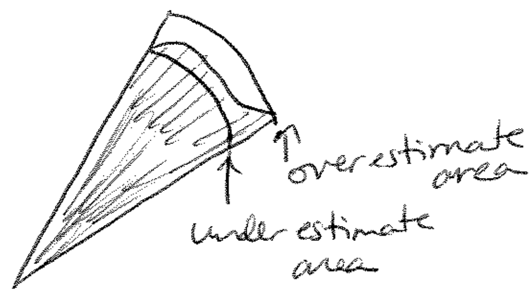


Area of shaded pie piece (sector)

$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}\theta r^2$$

To find area under a curve in a plane, we'll add up areas of sectors.





$$\Rightarrow A = \sum_{i=1}^{n} \frac{1}{2}[r_i]^2 \Delta\theta_i = \frac{1}{2}\sum_{i=1}^{n} r_i^2 d\theta_i$$

take limit as $n \to \infty$ $\Rightarrow A = \frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$ but $r = f(\theta)$ (a function of θ)

$$\Rightarrow A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 \, d\theta$$

<u>Ex 1</u> Find area inside $r = 3 + 3\sin\theta$

(more space on next page)

10.7 (cont)

Ex1 (cont)

Ex2 Find area inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

10.7 (cont)

Tangent line Slope

$m = \frac{dy}{dx}$ in rectangular coords.

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$y = r\sin\theta = f(\theta)\sin\theta$

$x = r\cos\theta = f(\theta)\cos\theta$

$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$

$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \text{slope in polar coords}}$$

Ex3 Find the slope of the tangent line to $r = 2 - 3\sin\theta$ at $\theta = \pi/6$

10.7 (cont)

Ex 4 For $r = 2-3\sin\theta$, what is θ when the tangent line is horizontal? (same fn as in Ex 3)