6.1 The Natural Logarithm Function

$D_x\left(\frac{x^3}{3}\right) = x^2$

$D_x\left(\frac{x^2}{2}\right) = x$

$D_x(x) = 1$

$D_x(?) = \frac{1}{x}$ ← what should go here in place of "?" to give a derivative of $1/x$? Can we complete the pattern?

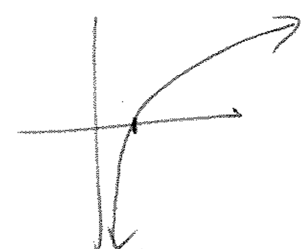
$D_x\left(\frac{-1}{x}\right) = \frac{1}{x^2}$

$D_x\left(\frac{-1}{2x^2}\right) = \frac{1}{x^3}$

$$\boxed{\underline{\text{Defn}} \quad \ln x = \int_1^x \frac{1}{t}\,dt \quad , \quad x > 0}$$

(area accumulation fn)

Remember graph of $y = \ln x$



domain of $y = \ln x$ is $x > 0$

From 1st Fundamental Thm of Calculus

$$\boxed{D_x\left(\int_1^x \frac{1}{t}\,dt\right) = D_x(\ln x) = \frac{1}{x} \quad , \quad x > 0}$$

Ex 1 Find $\frac{dy}{dx}$ if $y = \ln(x^2)$

6.1 (cont)

Ex 2 Find $\frac{dy}{dx}$ <u>and</u> state domain.

(a) $y = \ln\left(\sqrt[3]{2x}\right)$

(b) $y = \ln\left(3x^2 + 14x - 5\right)$

$$\boxed{D_x\left[\ln|x|\right] = \frac{1}{x} \qquad x \neq 0}$$

Proof ① If $x > 0$, then $|x| = x$.

$\Rightarrow D_x\left[\ln|x|\right] = D_x(\ln x) = \frac{1}{x}$ (by previous box pg ①)

② If $x < 0$, then $|x| = -x$.

$\Rightarrow D_x(\ln|x|) = D_x\left[\ln(-x)\right] = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x}$ //

6.1 (cont).

Ex 3 Evaluate integrals.

(a) $\displaystyle\int \frac{6}{3x-2}\,dx$

(b) $\displaystyle\int_2^5 \frac{3x}{7-2x^2}\,dx$ (Note: this integral is valid because $7-2x^2 > 0 \ \forall\, x \in [2,5]$.)

6.1 (cont)

Properties of Logarithms

① $\ln 1 = 0$

② $\ln(ab) = \ln a + \ln b$

③ $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

④ $\ln a^r = r \ln a$

Proof ① $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$ // (area under curve)

② $D_x(\ln(ax)) = \frac{1}{ax}(a) = \frac{1}{x}$ and $D_x(\ln x) = \frac{1}{x}$

$\Rightarrow D_x(\ln(ax)) = D_x(\ln x)$

$\Rightarrow \ln(ax) = \ln x + C \quad \forall x > 0$

if $x = 1$, then $\ln a = \ln 1 + C$

$\Rightarrow C = \ln a$

$\Rightarrow \ln(ax) = \ln x + \ln a$ //

We can prove ③ + ④ similarly.

Ex 4 (#30) Rewrite as single log.

$\ln(x^2-9) - 2\ln(x-3) - \ln(x+3)$

6.1 (cont)

Ex 5 (#34) Find $\frac{dy}{dx}$ by logarithmic differentiation.

$$y = \frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}$$

$$\Rightarrow \ln y = \ln\left[\frac{(x^2+3)^{2/3}(3x+2)^2}{\sqrt{x+1}}\right]$$

6.2 Inverse Fns & Their Derivatives

If $f(x)$ is a function, then $f^{-1}(x)$ is notation for the inverse of f.
(read as "f inverse"; $f^{-1} \neq \frac{1}{f}$)

If we have complete graph of $f(x)$, we can use horizontal line test to check for existence of $f^{-1}(x)$.

ex $f(x)=x^3 \Rightarrow f^{-1}(x)=\sqrt[3]{x}$ (i.e. way to "undo" cubing is to take the cube root)

If $f(x)$ & $f^{-1}(x)$ are inverses, then $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$

- inverses exist when we can get back to an x given a y, i.e. $x = f^{-1}(y) \Leftrightarrow f(x)=y$.
- inverses exist when a fn is <u>one-to-one</u> (which means $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$)

If we don't have graph, how can we algebraically test if a fn has an inverse?

Thm A If f is strictly monotonic on its domain, then f has an inverse.

- domain of f = range of f^{-1}
  range of f = domain of f^{-1}

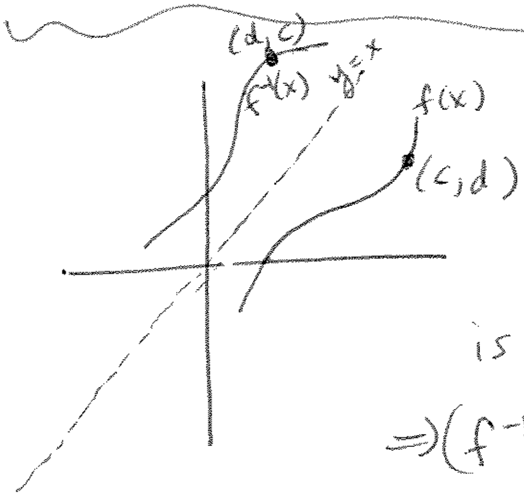
6.2 (cont)

Ex 1 Show that $f(x) = 3x^7 + 4x^3 + x - 3$ has an inverse.
(notice I didn't ask you to find the inverse. ☺)

Ex 2 Explore whether or not $f(x) = x^2 - 4$ has an inverse? If ~~not~~, can we restrict the domain so it does? If so, find $f^{-1}(x)$.

6.2 (cont)

Ex 3 Find $f^{-1}(x)$ for $y = \dfrac{2x-1}{3+5x}$, + check your work.



graph of $f^{-1}(x)$ is $f(x)$ reflected across $y=x$ line!

$\Rightarrow$ you can see slope at (d,c) is reciprocal of slope at (c,d)

$$\Rightarrow (f^{-1})'(d) = \frac{1}{f'(c)}$$

6.2 (cont)

Thm B: Inverse Function Thm

If f is differentiable, strictly monotonic on I, & $f'(x) \neq 0$ at some $x \in I$, then $f^{-1}(x)$ is differentiable at corresponding pt $y = f(x)$ in range of f &

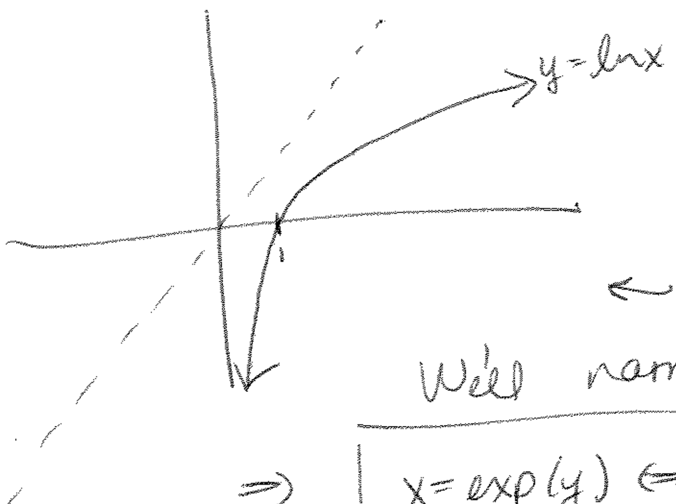
$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

(i.e. $\frac{dx}{dy} = \frac{1}{dy/dx}$)

Ex 4 For $f(x) = x^5 + 5x - 4$, find $(f^{-1})'(2)$ using Thm B.

6.3 The Natural Exponential Function

Remember graph of $y = \ln x$.



It is strictly monotonic, $\Rightarrow$ it has an inverse fn.

Can we draw the inverse fn's graph? $\leftarrow$

We'll name the inverse fn "exp".

$\Rightarrow$ $\boxed{x = \exp(y) \Leftrightarrow y = \ln x}$

domain of $\ln x$: $x > 0$
$=$ range of $\exp(x)$: $y > 0$

domain of $\exp(x)$: $x \in \mathbb{R}$
$=$ range of $\ln x$: $y \in \mathbb{R}$

Since "ln" + "exp" are inverse fns,

$$\ln(\exp(x)) = \exp(\ln(x)) = x.$$

$\boxed{\text{\underline{Defn} Let } e \in \mathbb{R} \text{ denote unique } \# \ni \ln e = 1.}$

(e is irrational; named after Euler; $e \simeq 2.718$; we know e exists because we can see it on graph above).

Important understanding $\Rightarrow$

$r \in \mathbb{R}$ $\exp(r) = \exp(r \ln e)$ (since $\ln e = 1$)
$= \exp(\ln e^r)$ (Log property)
$= e^r$ (since "exp" and "ln" are inverses)

very cool!

6.3 (cont)

$$\boxed{\textbf{Thm} \quad \text{Let } a,b \in \mathbb{R}. \text{ Then } e^a e^b = e^{a+b} \text{ \& } \frac{e^a}{e^b} = e^{a-b}.}$$

Proof $\dfrac{e^a}{e^b} = \exp\left(\ln\left(\dfrac{e^a}{e^b}\right)\right)$

$= \exp(\ln e^a - \ln e^b)$

$= \exp(a \ln e - b \ln e)$

$= \exp(a-b) = e^{a-b}$ //

Let $y = e^x \iff \ln y = x$.

$\Rightarrow \dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(x)$

$\dfrac{1}{y}\dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = y = e^x$

$\Rightarrow \boxed{D_x(e^x) = e^x}$ or $\boxed{D_x(e^u) = e^u D_x u \text{ chain Rule}}$

Ex 1 Find y'. $y = e^{x^2-3x}$

6.3 (cont)

Ex 2 Find y'. $y = e^{\sqrt{x}\ln x}$

Ex 3 For $f(x) = e^{x} - e^{-x}$, analyze the graph.
(ie. min, max pts, concavity, pts of inflectn, sketch graph)

6.3 (cont)

Since $D_x[e^x] = e^x$, then $\boxed{\int e^x\,dx = e^x + C}$

Ex 4 Evaluate integrals.

(a) $\int e^{-6x}\,dx$

(b) $\int e^{x+e^x}\,dx$

(c) $\int_1^2 \frac{e^{3/x}}{x^2}\,dx$

6.4 General Exponential & Logarithmic Functions

$a \in \mathbb{R}$, $a > 0$: $a^x = \exp(\ln a^x) = \exp(x \ln a) = e^{x \ln a}$

$$\Rightarrow \boxed{a^x = e^{x\ln a}, \ \forall a > 0, \ \forall x \in \mathbb{R}}$$

$\left(\Rightarrow \ln(a^x) = \ln(e^{x\ln a}) = x \ln a \quad (\text{Proof of log property})\right)$

Properties of Exponents

(i) $a^x a^y = a^{x+y}$

(ii) $\dfrac{a^x}{a^y} = a^{x-y}$

(iii) $(a^x)^y = a^{xy}$

(iv) $(ab)^x = a^x b^x$

(v) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Pf (v) $\left(\frac{a}{b}\right)^x = e^{\ln\left(\frac{a}{b}\right)^x} = e^{x\ln\left(\frac{a}{b}\right)}$

$= e^{x(\ln a - \ln b)} = e^{x\ln a - x \ln b}$

$= e^{\ln a^x - \ln b^x} = e^{\ln\left(\frac{a^x}{b^x}\right)} = \dfrac{a^x}{b^x}$ //

$$D_x[a^x] = D_x\left[e^{x\ln a}\right] = e^{x\ln a}(\ln a) = a^x(\ln a)$$

and (for $a \neq 1$) $\int a^x\,dx = \int e^{x\ln a}\,dx$ let $u = x\ln a$, $du = \ln a\,dx$

$\frac{1}{\ln a}\int e^u\,du = \frac{1}{\ln a}e^u + C$

$= \frac{1}{\ln a}e^{x\ln a} + C = \frac{1}{\ln a}a^x + C$

6.4 (cont)

$$\boxed{\begin{array}{l} D_x[a^x] = a^x \ln a \\ \int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C \\ \qquad\qquad\qquad\qquad a \neq 1 \end{array}}$$

★ Don't confuse exponential fns w/ power (or polynomial) fns!

Ex 1 Find y'.

$$y = (2x^3 + 9x)^4 + 4^{2x^3+9x}$$

Ex 2 Evaluate $\int \frac{2^{\sqrt{x}}}{3\sqrt{x}}\,dx$

6.4 (cont)

Remember log defn (from algebra)

$$y = \log_a x \iff a^y = x$$

$$\Rightarrow \ln(a^y) = \ln x$$

$$\iff y \ln a = \ln x$$

$$\iff y = \boxed{\frac{\ln x}{\ln a} = \log_a x}$$ change of base formula

$$\Rightarrow \boxed{D_x(\log_a x) = D_x\left(\frac{\ln x}{\ln a}\right) = \left(\frac{1}{\ln a}\right)\left(\frac{1}{x}\right)}$$

very cool $\Rightarrow$ We know $D_x(x^a) = ax^{a-1}$ if $a \in \mathbb{Q}$

but what if a is irrational?

$$D_x(x^a) = D_x(e^{a\ln x}) = e^{a\ln x}\left(\frac{a}{x}\right) = x^a\left(\frac{a}{x}\right) = ax^{a-1}$$

$$\boxed{D_x(x^a) = ax^{a-1} \quad \forall a \in \mathbb{R}}$$

Ex 3 Find y'. $\quad y = \sin^2 x + 2^{\sin x}$

6.4 (cont)

Ex 4 Find y'. $y = (\ln x^2)^{2x+3}$ (Hint: We must take ln of both sides first.)

Ex 5 Evaluate $\int_0^1 (10^{3x} + 10^{-3x})\, dx$.

6.4 (cont)

Ex 6 If $y = x^x$, find y'. :) (tricky)

6.5 Exponential Growth & Decay

Population Growth $\Rightarrow$ let y_0 = population at start

then $\frac{dy}{dt} = ky$ is reasonable assumption

(i.e. the rate of growth, or decay, is proportional to the population)

$$\Rightarrow dy = ky\,dt$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln|y| = kt + c \qquad \text{but } y>0 \Rightarrow |y| = y$$

$$\ln y = kt + c$$

$$y = e^{kt+c}$$

$$y = e^{kt}e^{c} \qquad e^{c} \in \mathbb{R} \text{ (i.e. a constant)}$$

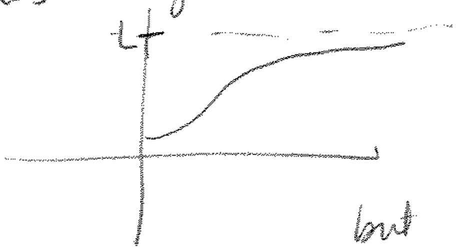
when $t=0$, $y = e^{c}$

so we write $y = y_0 e^{kt}$

$k>0 \Rightarrow$ growth 

$k<0 \Rightarrow$ decay 

(There are other factors that affect population, like resources & land. So population more likely follows logistic model.



where $\frac{dy}{dt} = ky(L-y)$

L = limiting population

but for small y, $\frac{dy}{dt} \approx kLy$)

6.5 (cont)

Ex 1 Population of U.S. was 3.9 million in 1790 + 17.8 million in 1960. If the rate of growth is assumed proportional to the population, what estimate would you give for the population in 2000? (compare your answer w/ actual population of 275 million.)

6.5 (cont)

Ex 2 If a radio-active substance loses 15% of its radioactivity in 2 days, what is its half-life?

Compound Interest

Put \$100 in a bank w/ 8% interest compounded quarterly.

$\left(\frac{0.08}{4} = 0.02 \text{ quarterly interest} \right)$

after qtr	value
0	100
1	$100(1+0.02)$
2	$[100(1+0.02)](1+0.02) = 100(1+0.02)^2$
3	$[100(1+0.02)(1+0.02)](1+0.02) = 100(1+0.02)^3$
$\vdots$	
n	?

6.5 (cont) Compound Interest

$$\Rightarrow \boxed{A(t) = A_0 \left(1+\frac{r}{n}\right)^{nt}}$$

A_0 = initial amount
$A(t)$ = value of account after t years
r = annual interest rate
n = # compoundings per year

What if we compound continuously?

$$\star \Rightarrow A(t) = \lim_{n\to\infty} A_0 \left(1+\frac{r}{n}\right)^{nt} = ?$$

Thm $\boxed{\lim_{h\to 0} (1+h)^{1/h} = e}$

Proof If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ + $f'(1) = 1$.

Use defn of derivative for $f(x)$.

$$1 = f'(1) = \lim_{h\to 0}\left[\frac{f(1+h)-f(1)}{h}\right] = \lim_{h\to 0}\left[\frac{\ln(1+h)-\ln(1)}{h}\right]$$

$$1 = \lim_{h\to 0}\left[\frac{\ln(1+h)}{h}\right] = \lim_{h\to 0}\left[\frac{1}{h}\ln(1+h)\right]$$

$$1 = \lim_{h\to 0}\left[\ln(1+h)^{1/h}\right]$$

But $\lim_{h\to 0}(1+h)^{1/h} = \lim_{h\to 0}\left[\exp\left(\ln(1+h)^{1/h}\right)\right]$

$$= \exp\left[\lim_{h\to 0}\left(\ln(1+h)^{1/h}\right)\right]$$

$$= \exp(1) = e$$

can only do this step because exp is continuous fn

i.e. $\lim_{h\to 0}(1+h)^{1/h} = e$ //

6.5 (cont)

Let's go back to ★ question. (continuous compounding)

$$A(t) = \lim_{n\to\infty} A_0 \left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = A_0 \lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

$$= A_0\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

as $n\to\infty$

$\frac{r}{n}\to 0$

let $h=\frac{r}{n}$

$$= A_0\left[\underbrace{\lim_{h\to 0}(1+h)^{1/h}}_{e}\right]^{rt}$$

$$\boxed{A(t) = A_0 e^{rt}}$$ continuous compounding

EX 3 If Methuselah's parents had put \$100 in the bank for him at birth and he left it there, what would Methuselah have had at his death (969 years later) if interest was 4% compounded annually?

6.6 First Order Linear Differential Eqns

We looked at one kind of D.E. to solve in Calc 1, where we could separate the variables + solve.
Now, we'll look at a different kind of D.E. that requires a different strategy to solve.

First order linear D.E.

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

- 1st order because 1st derivative
- linear because all operations on y are linear

general solution ⇒ family of all solns for D.E.

particular solution ⇒ solution when you are given an extra condition

strategy: multiply by integrating factor $e^{\int P(x)\,dx}$

Let's try it.

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

$$\frac{d}{dx}\left[ye^{\int P(x)dx}\right] = e^{\int P(x)dx}Q(x)$$

$$\int d\left[ye^{\int P(x)dx}\right] = \int e^{\int P(x)dx}Q(x)\,dx$$

$$ye^{\int P(x)dx} = \int e^{\int P(x)dx}Q(x)\,dx$$

$$\Rightarrow \boxed{y = e^{-\int P(x)dx}\int e^{\int P(x)dx}Q(x)\,dx}$$

6.6 (cont)

Ex 1 Solve $$\frac{dy}{dx} + \frac{5}{x}y = \frac{\cos(2x)}{x^5} \qquad x > 0$$

6.6 (cont)

Ex 2 Solve $y' = e^{2x} - 3y$ given $y = 1$ when $x = 0$

6.6 (cont)

Ex 3 A tank initially contains 200 gallons of brine, with 50 pounds of salt in solution. Brine containing 2 pounds of salt per gallon is entering the tank at the rate of 4 gallons per minute & is flowing out at the same rate. If the mixture in the tank is kept uniform by constant stirring, find the amount of salt in the tank at the end of 40 minutes.

y = amt of salt in tank

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

6.6 (cont)

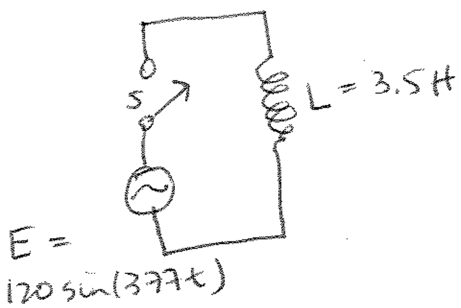
Kirchoff's law (for simple electrical circuit)

$$L\frac{dI}{dt} + RI = E(t)$$

R = resistance (in ohms Ω)
L = inductance (henrys)
E = voltage (volts)
I = current (amps)

Ex 4 Find I as a function of time for the circuit shown, assuming the switch is closed and $I=0$ at $t=0$.

[$\Rightarrow R=0$ (since there is no resistor)]

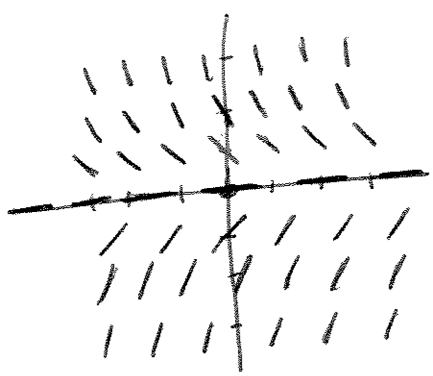


6.7 Approximations for Differential Eqns

For some diff. eqns, we cannot find explicit (analytic) solution, so we have to look for solution numerically (approximate soln).

<u>Slope Fields</u> $\Rightarrow$ a graphical representation of slopes, for $y' = f(x,y)$.

<u>ex</u> Plot slope field for $y' = -y$.

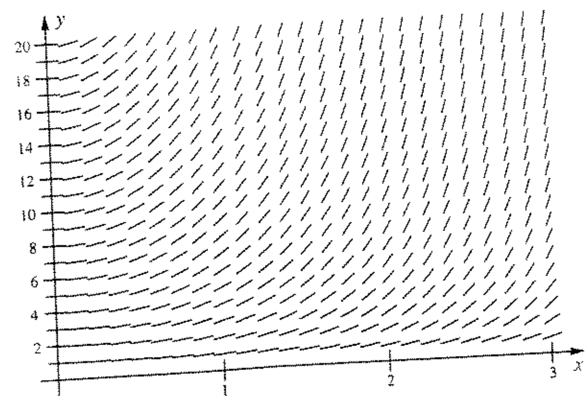


x	y	y'
0	0	0
0	1	-1
1	1	-1
0	2	-2
1	-1	1
-1	-1	1

Can you plot solution that satisfies $y(0) = 2$?

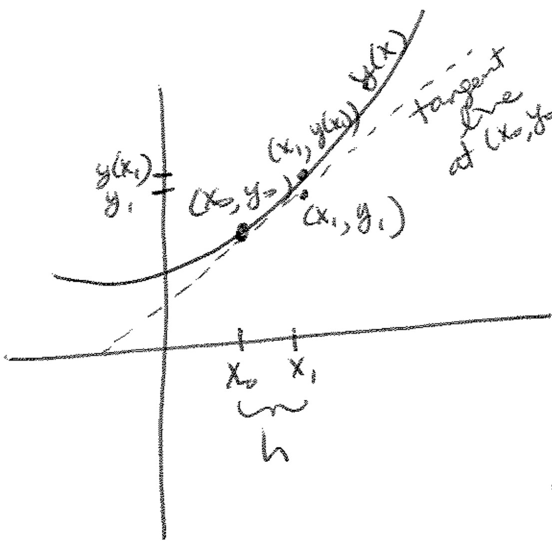
6.7 (cont)

Ex 1 Given the slope field for a differential eqn of form $y' = f(x,y)$, sketch the solution that satisfies $y(1)=3$. Find $\lim_{x\to\infty} y(x)$ + approximate $y(2)$.



6.7 (cont)

Euler's Method Let's say $y(x)$ is solution to $y' = f(x,y)$ $\ni$ $y(x_0) = y_0$.



We know at (x_0, y_0), slope is
$$m = f(x_0, y_0)$$

If we choose some small h, then we would expect y-value on tangent line to give approximate value for actual y-value.

tangent line: at (x_0, y_0)

$$y - y_0 = f(x_0, y_0)(x - x_0)$$
$$y = y_0 + f(x_0, y_0)(x - x_0)$$

at x_1: $y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + hf(x_0, y_0)$ (approximate y-value)

To get approximate y_2 value, we basically repeat process.

$$\Rightarrow y_2 = y_1 + hf(x_1, y_1)$$

So, we incrementally get y-values to approximate actual solution.

6.7 (cont)

(as h gets smaller, approx. better)

Euler's method

To approximate soln of $y' = f(x,y) \ni y(x_0) = y_0$, choose step size h (small) + repeat following steps, $n = 1, 2, 3, \ldots$

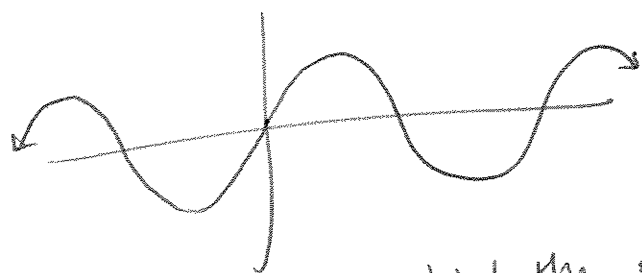
① $x_n = x_{n-1} + h$

② $y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$

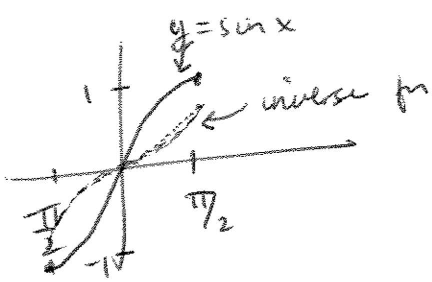
Ex 2 Use Euler's method w/ $h = 0.2$ to approximate soln to $y' = x^2$ given $y(0) = 0$ on $[0, 1]$.

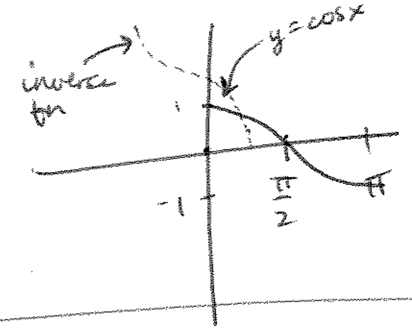
6.8 Inverse Trig Fns and Their Derivatives

For graph of $y = \sin x$, it doesn't pass the horizontal line test
$\Rightarrow$ it doesn't have an inverse function.



But, if we restrict the domain (to half a period), then we can talk about an inverse.





$\boxed{\begin{array}{lll} \underline{\text{Defn}} & & \\ x = \sin^{-1} y \iff y = \sin x & & x \in [-\pi/2, \pi/2] \\ x = \cos^{-1} y \iff y = \cos x & & x \in [0, \pi] \end{array}}$

$\left(\neq \frac{1}{\cos x}\right)$

notation $\quad \arccos(x) = \cos^{-1}(x)$

Ex 1 Evaluate (w/o calculator).

(a) $\cos^{-1}\left(\sqrt{3}/2\right)$

(b) $\sin^{-1}(1)$

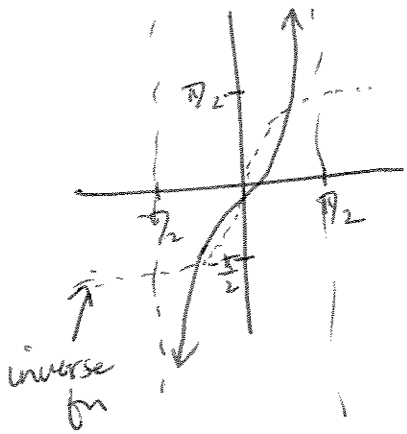
(c) $\cos^{-1}\left(\cos\left(-\pi/4\right)\right)$

(d) $\sin^{-1}\left(\sin\left(3\pi/2\right)\right)$

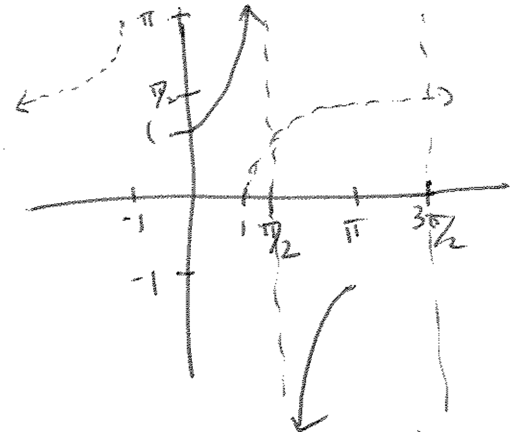
6.8 (cont)

$y = \tan x$

restrict domain to $x \in (-\pi/2, \pi/2)$



$y = \sec x$



restrict domain to $x \in [0, \pi]$, $x \neq \pi/2$

Defn

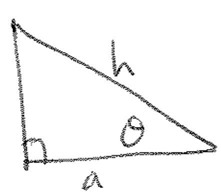
$$x = \tan^{-1} y \iff y = \tan x, \quad x \in (-\pi/2, \pi/2)$$

$$x = \sec^{-1} y \iff y = \sec x, \quad x \in [0, \pi/2) \cup (\pi/2, \pi]$$

Ex 2 Evaluate.

(a) $\tan^{-1}(-1)$

(b) $\sec^{-1}(2)$



Notice $\cos\theta = \frac{a}{h}$ and $\sec\theta = \frac{h}{a}$

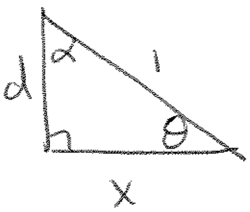
$$\Rightarrow \theta = \cos^{-1}\left(\frac{a}{h}\right) \quad + \quad \theta = \sec^{-1}\left(\frac{h}{a}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{a}{h}\right) = \sec^{-1}\left(\frac{h}{a}\right)$$

$$\iff \sec^{-1}\left(\frac{h}{a}\right) = \cos^{-1}\left(\frac{1}{h/a}\right)$$

$$\iff \boxed{\sec^{-1}(y) = \cos^{-1}\left(\frac{1}{y}\right)}$$

6.8 (cont)



Let $\theta = \cos^{-1} x$. Then the rt. Δ shown here depicts that info.

What is $\sin(\cos^{-1} x)$?

$$\sin(\cos^{-1} x) = \sin\theta = \frac{d}{1} = d$$

$$\& \quad x^2 + d^2 = 1 \Rightarrow d^2 = 1 - x^2 \Rightarrow d = \sqrt{1-x^2}$$

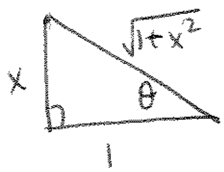
$$\Rightarrow \boxed{\sin(\cos^{-1} x) = \sqrt{1-x^2}}$$

(why does this have to be positive?)

Also, $\sin\alpha = x \Rightarrow \alpha = \sin^{-1} x$

$$\cos(\sin^{-1} x) = \cos\alpha = d = \sqrt{1-x^2}$$

i.e. $\boxed{\cos(\sin^{-1} x) = \sqrt{1-x^2}}$



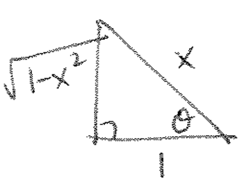
$$\tan\theta = x \Rightarrow \theta = \tan^{-1} x$$

$$\sec(\tan^{-1} x) = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{\sqrt{1+x^2}}}$$

$$\boxed{\sqrt{1+x^2} = \sec(\tan^{-1} x)}$$

$\tan(\sec^{-1} x) = ?$

let $\theta = \sec^{-1} x$



$$\Rightarrow \tan(\sec^{-1} x) = \tan\theta = \sqrt{1-x^2}$$

but we have to consider sign in this one.

(if $x \geq 1$, in QI; if $x \leq 1$, in QII)

$$\Rightarrow \boxed{\tan(\sec^{-1} x) = \begin{cases} \sqrt{1-x^2} & x \geq 1 \\ -\sqrt{1-x^2} & x \leq -1 \end{cases}}$$

6.8 (cont)

$$\boxed{\begin{array}{ll} D_x[\sin x] = \cos x & D_x[\cos x] = -\sin x \\ D_x[\tan x] = \sec^2 x & D_x[\cot x] = -\csc^2 x \\ D_x[\sec x] = \sec x \tan x & D_x[\csc x] = -\csc x \cot x \end{array}}$$

Ex 3 Calculate $\sin[2\cos^{-1}(\frac{1}{4})]$ (w/o calculator).

Derivatives of Inverse Trig Fns

Let $y = \cos^{-1} x$. We want to find y'.

$\Leftrightarrow x = \cos y$

$D_x[x] = D_x[\cos y]$

$1 = (-\sin y)\, y'$

$y' = \frac{-1}{\sin y} = \frac{-1}{\sin(\cos^{-1} x)} = \frac{-1}{\sqrt{1-x^2}}$

$D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad x \in (-1,1)$

$D_x(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad |x| > 1$

$D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ $\quad x \in (-1,1)$

$D_x(\tan^{-1} x) = \frac{1}{1+x^2}$

6.8 (cont)

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C$$

Ex 4 $D_x\left[\tan^{-1}(5x^2-3x+1)\right] = ?$

Ex 5 Evaluate integrals.

(a) $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

(b) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

6.9 Hyperbolic Fns + Their Inverses

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\qquad$ $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\qquad$ $\coth x = \dfrac{\cosh x}{\sinh x}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$ $\qquad$ $\operatorname{csch} x = \dfrac{1}{\sinh x}$

★ Related to trig fns:

Trig
1. $(\cos\theta, \sin\theta)$ pt on unit circle
2. $\sin\theta$ odd fn
3. $\cos\theta$ even fn
4. $\sin^2\theta + \cos^2\theta = 1$

Hyperbolic
1. $(\cosh\theta, \sinh\theta)$ pt on unit hyperbola
2. $\sinh\theta$ odd fn
3. $\cosh\theta$ even fn
4. $\cosh^2\theta - \sinh^2\theta = 1$

Prove $\cosh^2\theta - \sinh^2\theta = 1$

pf

6.9 (cont)

$$D_x(\cosh x) = D_x\left(\frac{e^x+e^{-x}}{2}\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$$

$$D_x(\sinh x) =$$

$$\boxed{\begin{array}{ll} D_x(\sinh x) = \cosh x & D_x(\cosh x) = \sinh x \\ D_x(\tanh x) = \operatorname{sech}^2 x & D_x(\coth x) = -\operatorname{csch}^2 x \\ D_x(\operatorname{sech} x) = -\operatorname{sech} x \tanh x & D_x(\operatorname{csch} x) = -\operatorname{csch} x \coth x \end{array}}$$

Ex 1 $D_x(\coth(4x)\sinh x) = ?$

Ex 2 $\int \tanh x \ln(\cosh x)\, dx$

6.9 (cont)

Ex 3 Verify identity.

$$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}.$$

(Hint: $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.)

6.9 (cont)

Inverse Hyperbolic Fns

Let $y = \sinh x \iff x = \sinh^{-1} y$ (if inverse exists).

Find $x = \sinh^{-1} y$.

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$2y = e^x - e^{-x}$$

$$e^x (2y) = \left(e^x - \frac{1}{e^x}\right) e^x$$

6.9 (cont)

$$\boxed{\begin{array}{ll} \sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right) & \cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right) \quad x \geq 1 \\ \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad x \in (-1,1) & \operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \quad x \in (0,1] \end{array}}$$

$$D_x(\sinh^{-1} x) = D_x\left(\ln\left(x+\sqrt{x^2+1}\right)\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$$\boxed{\begin{array}{ll} D_x(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}} & D_x(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \quad x > 1 \\ D_x(\tanh^{-1} x) = \frac{1}{1-x^2} \quad x \in (-1,1) & D_x(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}} \quad x \in (0,1) \end{array}}$$

Ex 4 Find y'.

$$y = x^2 \sinh^{-1}(x^5)$$

Chapter 7 Formula Sheet

Math1220
Kelly MacArthur

Log Properties

$\ln 1 = 0$

$\ln(ab) = \ln a + \ln b$

$\ln(\frac{a}{b}) = \ln a - \ln b$

$\ln(a^m) = m \ln a$

Integrals

$\int \frac{1}{x} dx = \ln|x| + C$

$\int e^x dx = e^x + C$

$\int a^x dx = \frac{1}{\ln a} a^x + C$

$\int x^a dx = \frac{x^{a+1}}{a+1} + C$

$a =$constant, $a \neq -1$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$

$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$

$\int \frac{2}{x\sqrt{x^2-1}} dx = \sec^{-1}|x| + C$

$\int \sinh(x) dx = \cosh x + C$

$\int \cosh(x) dx = \sinh x + C$

$\int \tanh(x) \sec h(x) dx = -\sec h(x) + C$

$\int \sec h^2(x) dx = \tanh(x) + C$

$\int \csc h^2(x) dx = -\coth(x) + C$

$\int \csc h(x) \coth(x) dx = -\csc h(x) + C$

Functions and Identities

$\sin(\cos^{-1} x) = \sqrt{1-x^2}$

$\cos(\sin^{-1} x) = \sqrt{1-x^2}$

$\sec(\tan^{-1} x) = \sqrt{1+x^2}$

$\tan(\sec^{-1} x) = \sqrt{x^2-1}$, if $x \geq 1$

$-\sqrt{x^2-1}$, if $x \leq -1$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), x \geq 1$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$

$\sec h^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), 0 < x \leq 1$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\coth x = \frac{\cosh x}{\sinh x}$

$\csc \mathrm{h}(x) = \frac{1}{\sinh x}$

$\sec \mathrm{h}(x) = \frac{1}{\cosh x}$

$\cosh^2 x - \sinh^2 x = 1$

Trig Identities

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

Derivatives

$D_x(e^x) = e^x$

$D_x(\ln x) = \frac{1}{x}$

$D_x(x^a) = ax^{a-1}$

$D_x(a^x) = (\ln a)a^x$

$D_x(\sin x) = \cos x$

$D_x(\cos x) = -\sin x$

$D_x(\tan x) = \sec^2 x$

$D_x(\cot x) = -\csc^2 x$

$D_x(\sec x) = \sec x \tan x$

$D_x(\csc x) = -\csc x \cot x$

$D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \in (-1,1)$

$D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, x \in (-1,1)$

$D_x(\tan^{-1} x) = \frac{1}{1+x^2}$

$D_x(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

More Derivatives

$D_x(\sinh x) = \cosh x$

$D_x(\cosh x) = \sinh x$

$D_x(\operatorname{csc h} x) = -\operatorname{csc h} x \coth x$

$D_x(\tanh x) = \operatorname{sec h}^2 x$

$D_x(\operatorname{sec h} x) = -\operatorname{sec h} x \tanh x$

$D_x(\coth x) = -\operatorname{csc h}^2 x$

$D_x(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$

$D_x(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$

$D_x(\tanh^{-1} x) = \frac{1}{1-x^2}, -1 < x < 1$

$D_x(\operatorname{sec h}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$

Miscellaneous

$\lim_{h \to 0}(1+h)^{1/h} = e$

To solve $\frac{dy}{dx} + P(x)y = Q(x)$, multiply both sides by $e^{\int P(x)dx}$ and solve.