9.1 Infinite Sequences

<u>Infinite Sequence</u> $\Rightarrow$ ordered arrangement of $\mathbb{R}$ #s

$a_1, a_2, a_3, a_4, \ldots$

$\{a_n\}_{n=1}^{\infty}$

$\{a_n\}$

notation

<u>iteration</u> (explicit) vs. <u>recursion</u> (implicit)

<u>ex</u> $a_n = 5n-3$

<u>ex</u> $a_1 = 2$

$a_n = a_{n-1} + 5 \quad n \geq 2$

or we can just write out terms

$2, 7, 12, 17, 22, \ldots$

<u>Defn</u> convergence

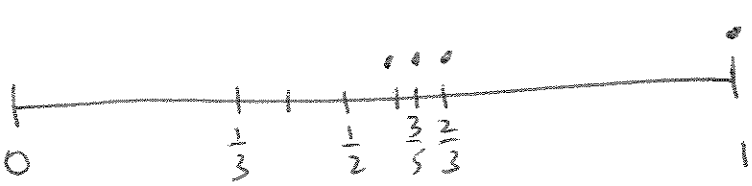
$\{a_n\}$ converges to L, written $\lim\limits_{n\to\infty} a_n = L$,

if $\forall$ positive ε $\exists$ corresponding positive $N \ni$

$n \geq N \Rightarrow |a_n - L| < \varepsilon$.

If a sequence fails to converge to a finite L, then it diverges.

<u>ex</u> $a_n = \dfrac{n}{2n-1} \Rightarrow 1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots \rightarrow \frac{1}{2}$



9.1 (cont)

Ex 1 Does $\{a_n\}$ converge? $a_n = \dfrac{5n^2-3n+1}{2n^2+7}$

If so, what is its limit?

Properties of limits of Sequences

① $\lim_{n\to\infty} k = k$

② $\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

③ $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ (assuming $\lim_{n\to\infty} b_n \neq 0$)

② + ③ are true only if limits are finite

④ $\lim_{n\to\infty} k a_n = k \lim_{n\to\infty} a_n$

⑤ $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

④ + ⑤ ⇒ limit is linear operator

9.1 (cont)

$$\boxed{\text{If } \lim_{x\to\infty} f(x) = L, \text{ then } \lim_{n\to\infty} f(n) = L.}$$

x is continuous variable
n is discrete variable

$\Rightarrow$ we can use L'Hopital's Rule!

Ex2 Determine if $\{a_n\}$ converges & if so, find $\lim_{n\to\infty} a_n$.

(a) $a_n = \dfrac{\ln(1/n)}{\sqrt{2n}}$

(b) $a_n = \dfrac{n^{100}}{e^n}$

9.1 (cont)

Squeeze Theorem

If $\{a_n\}$ + $\{c_n\}$ both converge to L and $a_n \le b_n \le c_n$ for $n \ge K$ (some fixed integer), then $\{b_n\}$ also converges to L.

Ex3 Determine if $\{a_n\}$ converges & if so, find $\lim_{n\to\infty} a_n$. $a_n = e^{-n} \sin n$

9.1 (cont)

$\boxed{\text{Thm If } \lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0}$

Ex 4 Show that if $r \in (-1,1)$, then $\lim_{n\to\infty} r^n = 0$.

9.1 (cont)

Monotonic Sequence Thm

If U is an upper bound for a nondecreasing sequence $\{a_n\}$, then the sequence converges to a limit $A \ni A \leq U$. Also, if L is a lower bound for a nonincreasing sequence $\{b_n\}$, then $\{b_n\}$ converges to a limit B $\ni B \geq L$.

Ex5 Write 1st 4 terms for $a_n = \frac{n}{n+1}\left(2 - \frac{1}{n^2}\right)$. Show that $\{a_n\}$ converges.

9.2 Infinite Series

Zeno's paradox ⇒ step from 0 to $\frac{1}{2}$, then keep taking steps halfway between where I'm at and 1 ⇒ I'll never get to 1!

$$\Rightarrow \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots+\frac{1}{2^n} \stackrel{?}{=} 1$$

Let S_i be the <u>partial sum</u> of first i terms in a sequence.

$$S_1 = \frac{1}{2} = 1-\frac{1}{2}$$
$$S_2 = \frac{1}{2}+\frac{1}{4} = 1-\frac{1}{4}$$
$$S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1-\frac{1}{8}$$
$$\vdots$$
$$S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} = 1-\frac{1}{2^n}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{2^n}\right) = 1.$$

<u>Infinite Series</u> ⇒ $a_1+a_2+a_3+\cdots+a_n+\cdots = \sum_{i=1}^{\infty} a_i$ (sometimes also written $\sum a_i$)

<u>Partial Sum</u> ⇒ $\sum_{i=1}^{n} a_i = S_n$

<u>Defn</u> $\sum a_i$ converges & has sum S if sequence of partial sums converges to S, i.e. $\lim_{n\to\infty} S_n = S$.
If $\{S_n\}$ diverges, then the series diverges & has no sum.

9.2 (cont)

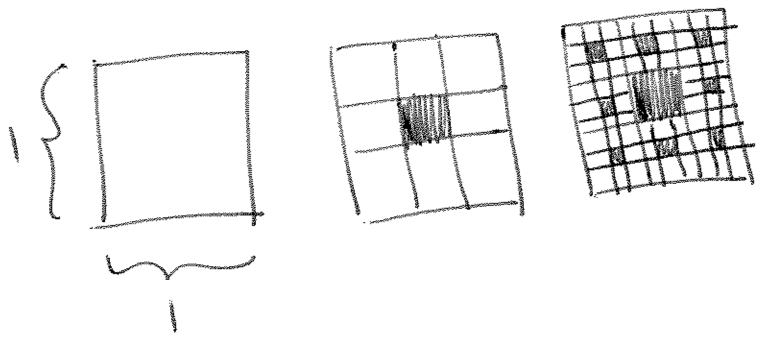
$$\boxed{\begin{array}{l}\underline{\text{Geometric Series}}\\ (a\neq 0) \quad \sum_{i=1}^{\infty} ar^{i-1} = a+ar+ar^2+ar^3+\dots\end{array}}$$

<u>Ex 1</u> Show that a geometric series converges for at least some r + find its sum.

$$S_n = \sum_{i=1}^{n} ar^{i-1} = a+ar+ar^2+\dots+ar^n$$

9.2 (cont)

Ex 2 If the pattern shown is continued indefinitely, what fraction of the original square will eventually be painted?



9.2 (cont)

Thm
$n^{\underline{th}}$ term test for divergence

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

$\Leftrightarrow$ If $\lim_{n\to\infty} a_n \neq 0$ (or if $\lim_{n\to\infty} a_n$ dne), then series diverges!

Pf Let S_n be partial sum and $S = \lim_{n\to\infty} S_n$.

Then $S_n - S_{n-1} = (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1})$

$= a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$ //

Ex 3 Does $\sum_{i=1}^{\infty} \frac{3i-7}{4i+3}$ converge or diverge?

9.2 (cont)

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$

$= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ but does S_n converge?

$S_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$

$= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}\right.$

$\left.+\frac{1}{15}+\frac{1}{16}\right)+\cdots+\frac{1}{n}$

$> 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} S_n > \lim_{n\to\infty} \underbrace{1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{n}}_{\text{keeps growing as } n \text{ increases}}$

$\Rightarrow$ harmonic series <u>diverges</u>

$\Rightarrow \lim_{n\to\infty} a_n = 0$ DOES <u>NOT</u> imply $\sum_i a_i$ converges!

9.2 (cont)

Ex 4 Does $\sum_{i=1}^{\infty} \frac{3}{i(i+1)}$ converge or diverge?

Linearity of a Convergent Series

If $\sum_{i=1}^{\infty} a_i$ & $\sum_{i=1}^{\infty} b_i$ both converge, $c \in \mathbb{R}$,
then $\sum_{i=1}^{\infty} c a_i = c \sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} (a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$
also converge.

Ex 5 Does $\sum_{k=1}^{\infty} \left[5\left(\frac{1}{2}\right)^k - 3\left(\frac{1}{7}\right)^{k+1}\right]$ diverge or converge?

9.2 (cont)

Thm
If $\sum_{k=1}^{\infty} a_k$ diverges and $c \neq 0$, then $\sum_{k=1}^{\infty} c a_k$ diverges.

Grouping Terms in an ∞ Series

The terms in a convergent series can be grouped in any way & the new series will still converge w/ same sum.

Why don't we just use computers to tell if a series converges?

Consider $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ (Harmonic series)

We know this diverges. Partial sums S_n grow so slowly that a computer would not detect its divergence.

e.g. $S_n = 100$ for $n = 10^{43}$ & $S_{272,000,000} \approx 20$

Eventually, the computer will continue to return same value for S_n & we would think it converges. So we need a bunch of tests to determine convergence/divergence. Let the fun begin!

9.3 Positive Series: Integral Test

Bounded Sum Test

A series $\sum a_i$ of nonnegative terms converges $\Leftrightarrow$ its partial sums are bounded above.

pf Let $S_n = a_1 + a_2 + \cdots + a_n$

$a_k \geq 0 \Rightarrow S_{n+1} \geq S_n \Rightarrow \{S_n\}$ nondecreasing.

$\Rightarrow$ By Monotonic Sequence Thm, $\{S_n\}$ converges if $\exists\, U \ni S_n \leq U \;\forall n$. Otherwise, $\{S_n\}$ diverges. //

Ex 1 Does $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{(k+1)!}$ converge?

9.3 (cont)

Integral Test

If $f(x)$ is continuous, positive + nonincreasing on $[N, \infty)$ and $a_k = f(k)$ $\forall k \in \mathbb{Z}^+$, then

$$\sum_{k=N}^{\infty} a_k \text{ converges} \iff \int_N^{\infty} f(x)\,dx \text{ converges}$$

(i.e. the series $\sum_{i=1}^{\infty} f(i)$ + the improper integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together)

Ex 2 Does $\sum_{k=1}^{\infty} \frac{5k^2}{1+k^3}$ converge or diverge?

$f(x) = \frac{5x^2}{1+x^3}$ always positive $\forall x > 0$

9.3 (cont)

p-series test

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-series. It converges if $p > 1$ + diverges if $p \le 1$.

Pf $f(x) = \frac{1}{x^p}$ is positive, continuous, nonincreasing fn on $[1, \infty)$ if $p \ge 0$.

Use Integral Test ⇒ (for ①, ② + ③)

① $p > 1$ $\quad \int_1^{\infty} \frac{1}{x^p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} = 0 - \frac{1}{1-p} = \frac{1}{p-1}$ (if $p>1$) converges

② $p = 1$ $\quad \int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty} \to \infty$ diverges.

③ $0 \le p < 1$ $\quad \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} \to \infty$ diverges

④ $p < 0$ $\quad$ Notice $\lim_{k\to\infty} \frac{1}{k^p} \neq 0$ (if $p<0$) ⇒ series diverges, by n^{th} term test. //

Ex3 Does $\sum_{k=3}^{\infty} \frac{1}{k^3}$ converge or diverge?

9.3 (cont)

Ex 4 Estimate error made by approximating the series by the sum of the first 5 terms.

$$S_n = \sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$$

$$E = \sum_{k=6}^{\infty} \frac{1}{k\sqrt{k}}$$

9.4 Positive Series: Other Tests

Geometric Series: $\sum_{n=1}^{\infty} r^n$ converges if $r \in (-1, 1)$

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

Ordinary Comparison Test

If $0 \le a_n < b_n \ \forall n \ge N$

(i) If $\sum_n b_n$ converges, so does $\sum a_n$

(ii) If $\sum a_n$ diverges, so does $\sum b_n$.

Ex 1 Does $\sum_{n=1}^{\infty} \frac{3n+4}{4n^2-2n-5}$ converge or diverge?

9.4 (cont)

Limit Comparison Test

Assume $a_n \geq 0$, $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ + $\sum b_n$ converge or diverge together. If $L = 0$ + $\sum b_n$ converges, then $\sum a_n$ converges.

★ used much more than ordinary comparison test.

Pf Let $\varepsilon = L/2$. In our defn of convergence

$\exists N \ni n \geq N \Rightarrow \left| \frac{a_n}{b_n} - L \right| < \varepsilon = L/2$

① $0<L<\infty$

$$\Leftrightarrow \quad -\frac{L}{2} < \frac{a_n}{b_n} - L < \frac{L}{2}$$

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$$

$$\frac{L}{2} b_n < a_n < \frac{3L}{2} b_n$$

by ordinary comparison test, if $\sum b_n$ converges so does $\sum a_n$ & if $\sum a_n$ converges, so does $\sum b_n \Rightarrow \sum a_n$ + $\sum b_n$ converge together.

If $\sum a_n$ diverges, so does $\sum b_n$ + if $\sum b_n$ diverges, so does $\sum a_n \Rightarrow$ they diverge together.

9.4 (cont)

② $L=0$ We have $a_n \geq 0$, $b_n > 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Again, by defn of limit, we know $\exists$

$\varepsilon > 0$, $\exists$ $\left|\frac{a_n}{b_n}\right| < \varepsilon$ $\forall\, n \geq N$.

$\Rightarrow \frac{a_n}{b_n} < \varepsilon$ since $a_n \geq 0$ $b_n > 0$

$\Leftrightarrow a_n < \varepsilon b_n \leq b_n$

by ordinary comp. test, if $\sum b_n$ converges, so does $\sum a_n$ //

Ex 2 ~~Does~~ this series converge or diverge?

$$\frac{1}{1^2+1} + \frac{2}{2^2+1} + \frac{3}{3^2+1} + \cdots$$

9.4 (cont)

Ex 3 ~~Does~~ $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+1}$ diverge or converge?

Ratio Test

If Σa_n is a series of positive terms + $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$,

then (i) if $\rho < 1$, series converges.

(ii) if $\rho > 1$ or if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, series diverges.

(iii) if $\rho = 1$, then test is inconclusive.

9.4 (cont)

Ex 4 Does this series converge or diverge?

$$3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots$$

Ex 5 Diverge or converge? $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5+n}$