9.1 Practice (Infinite Sequences)

Ex 1 Write first four terms + determine if sequence converges or diverges.

$$a_n = (2n)^{1/2n}$$

A sequence $\{a_n\}$ converges if $\lim_{n\to\infty} a_n =$ finite #

otherwise, $\{a_n\}$ diverges

Ex 2 Find explicit formula for a_n. Determine if it converges or diverges.

$$1-\frac{1}{2},\ \frac{1}{2}-\frac{1}{3},\ \frac{1}{3}-\frac{1}{4},\ \frac{1}{4}-\frac{1}{5},\ \ldots$$

Ex 3 Determine if sequence $\{a_n\}$ converges or diverges

(a) $a_n = 2 + (0.99)^n$

(b) $a_n = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right), n \geq 2$

Ex 4 Write explicit formula + determine if sequence converges or diverges

(a) $-1, \frac{2}{3}, \frac{-3}{5}, \frac{4}{7}, \frac{-5}{9}, \ldots,$

(b) $a_1 = 1$
$a_{n+1} = 1 + \frac{1}{2} a_n$

9.2 Practice (Infinite Series)

Ex1 Do these series converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{8^{n-1}}$

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{2k+1}{2k-3}$

$\boxed{\displaystyle\sum_{n=1}^{\infty} a_n}$

① n^{th} term test for Divergence

if $\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges

② Geometric series

$\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ (a constant) if $|r| < 1$

otherwise, if $|r| \geq 1$, this diverges

Notes:

$\displaystyle\sum_{k=1}^{\infty}$ is a linear operator on convergent positive series, i.e.

① $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

and

② $\displaystyle\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$

if $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ are both convergent positive series

Ex 2 Write as infinite series & then fraction.

0.12565656...

Ex 3 Do these series converge or diverge?

(a) $\sum_{n=4}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n-3} \right)$

(b) $\sum_{n=1}^{\infty} \left[2\left(\frac{3}{5}\right)^n + 500\left(\frac{1}{2}\right)^n \right]$

Ex 4 Show $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges.

9.3 Practice (Positive Series Tests)

Ex 1 Do these series converge or diverge?

(a) $\sum_{n=1}^{\infty} n^3 e^{-3n^4}$

(b) $\sum_{n=3}^{\infty} \left(\frac{1}{n^{4/3}} + \frac{1}{3^n} \right)$

① n^{th} term test for Divergence

if $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ diverges

② Geometric series

$\sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}$ if $|r|<1$

diverges if $|r| \geq 1$

③ P-series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converges, } p>1 \\ \text{diverges, } p\leq 1 \end{cases}$

★

⑦ Integral Test

f continuous, positive, nonincreasing fn on $[k, \infty)$

$\sum_{n=k}^{\infty} a_n$ converges $a_n = f(n)$

$\Leftrightarrow \int_k^{\infty} f(x)\,dx$ converges

⑧ Argument of Partial sums

if $S_n = \sum_{k=1}^{n} a_k$

and $\lim_{n\to\infty} S_n = S < \infty$,

then $\sum_{k=1}^{\infty} a_k$ converges

if $\lim_{n\to\infty} S_n$ DNE or goes to ∞,

$\sum_{k=1}^{\infty} a_k$ diverges

★ note: This is the order of tests I prefer. The other tests are filled in on page 42.

Ex 2 How large should n be so that S_n approximates S w/ error no bigger than 0.001?

$$\sum_{j=1}^{\infty} \frac{1}{j^{9/8}}$$

$$\left(E_n = \sum_{k=n+1}^{\infty} \frac{1}{j^{9/8}} \right)$$

$$\boxed{\begin{array}{l} E_n < \int_n^{\infty} f(x)\,dx \\ \text{when } f(k) = a_k \ \forall k = 1, 2, \ldots \\ \text{and} \\ \int_n^{\infty} f(x)\,dx \text{ converges} \end{array}}$$

Ex 3 Do these series converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} n^2 \sin\left(\frac{1}{n^2}\right)$

(b) $\displaystyle\sum_{n=10}^{\infty} \frac{5}{(n+1)^2}$

9.4 Practice (Positive Series: More Tests)

Positive series $\sum_{k=1}^{\infty} a_k$

Ex 1 Do these series converge or diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{3n+1}}{2n^2-1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}+1}{5^n}$

④ LCT (Limit Comparison Test)

$a_n \geq 0$, $b_n > 0$

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$ if $0 < L < \infty$, $\sum a_n$ and $\sum b_n$ converge or diverge together

(if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

⑤ RT (ratio Test)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \begin{cases} \rho < 1 & \text{converges} \\ \rho > 1 & \text{diverges} \\ \rho = 1 & \text{inconclusive} \end{cases}$$

⑥ OCT (ordinary Comparison Test)

$0 \leq a_n \leq b_n$ for $n \geq N$.

$\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

$\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

also see summary note on page 473 of book

Ex 2 Do these series converge or diverge?

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n^{1000}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{5+\cos^2 n}$

(c) $\frac{\ln 2}{2^2} + \frac{\ln 3}{3^2} + \frac{\ln 4}{4^2} + \cdots$

(d) $1 + \frac{2}{3^2 \sqrt[4]{3}} + \frac{3}{5^2 \sqrt[4]{5}} + \frac{4}{7^2 \sqrt[4]{7}} + \cdots$

Ex 3 Do these series converge or diverge?

(a) $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$

(b) $\sum_{n=3}^{\infty} \left(1-\frac{2}{n}\right)^{5n}$

(c) $\sum_{j=1}^{\infty} \frac{10^j + j^{10}}{(2j)!}$

(d) $\sum_{n=2}^{\infty} n\left(\frac{1}{5}\right)^n$