Practice for section 2.5 Zeros of Polynomial functions

1. Determine all roots (real and complex) and write in factored form each of these polynomials:
   a. $3x^3 - 4x^2 + 8x + 8$

Notice that the usual suspects, $-1, 0, 1$, do not work. Our possibilities for the roots are

$\frac{b}{c}$; $b \mid 8$; $c \mid 3$, so:

$\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}, \frac{8}{3}, -\frac{8}{3}$

$-\frac{1}{3}$: $3\left(-\frac{1}{3}\right)^3 - 4\cdot\left(-\frac{1}{3}\right)^2 + 8\left(-\frac{1}{3}\right) + 8 = -\frac{1}{9} - \frac{4}{9} - \frac{8}{3} + 8 > 0$

$\frac{1}{3}$: $3\cdot\frac{1}{3^3} - 4\cdot\frac{1}{3^2} + 8\frac{1}{3} + 8 = -3\frac{1}{3^2} + 8\frac{1}{3} + 8 = \frac{7}{3} + 8 > 0$

$-\frac{2}{3}$: $3\cdot\left(-\frac{2}{3}\right)^3 - 4\left(-\frac{2}{3}\right)^2 + 8\left(-\frac{2}{3}\right) + 8 = -\frac{8}{9} - \frac{16}{9} - \frac{16\cdot 3}{3\cdot 3} + 8 = \frac{-24-48}{9} + 8 =$

$= -\frac{72}{9} + 8 = -8 + 8 = \boxed{0}$

$3x^3 - 4x^2 + 8x + 8 \div \left(x + \frac{2}{3}\right) = 3x^2 - 6x + 12$

$$
\begin{array}{r}
3x^3 - 4x^2 + 8x + 8 \\
-\ (3x^3 + 2x^2) \\
\hline
-6x^2 + 8x \\
-\ (-6x^2 - 4x) \\
\hline
12x + 8 \\
-\ (12x + 8) \\
\hline
0
\end{array}
$$

$3x^3 - 4x^2 + 8x + 8 = \left(x + \frac{2}{3}\right)\left(3x^2 - 6x + 12\right)$

$= 3\left(x + \frac{2}{3}\right)\left(x^2 - 2x + 4\right)$

We find roots of x^2-2x+4:

$$x_{1/2} = \frac{2 \pm \sqrt{4-4\cdot 4}}{2} = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} =$$

$$= \frac{2 \pm i\,2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

So roots are: $-\frac{2}{3}$, $1+i\sqrt{3}$, $1-i\sqrt{3}$

$$3x^3-4x^2+8x+8 = 3\left(x+\frac{2}{3}\right)\left(x-1-i\sqrt{3}\right)\left(x-1+i\sqrt{3}\right)$$

b. $12z^3 - 4z^2 - 27z + 9$

This can be factored:

$$12z^3-4z^2-27z+9 = 4z^2(3z-1)-9(3z-1) =$$

$$= (3z-1)(4z^2-9) = (3z-1)(2z-3)(2z+3)$$

So roots are $\frac{1}{3}$, $\frac{3}{2}$, $-\frac{3}{2}$

c. $5x^4 + 9x^3 - 7x^2 - 9x + 2$

Let's check 1: $5+9-7-9+2 = 7+9-7-9=0$

-1: $5\cdot 1+9(-1)-7(1)-9(-1)+2 =$

$= 7-9-7+9=0$

So $(x-1)$ and $(x+1)$ are both factors of our polynomial

$$5x^4+9x^3-7x^2-9x+2 = (x-1)(x+1)\,g(x)$$

$$5x^4+9x^3-7x^2-9x+2 = (x^2-1)\,g(x)$$

$5x^4+9x^3-7x^2-9x+2 \div x^2-1 = 5x^2+9x-2$

$$\begin{array}{l} 5x^4+9x^3-7x^2-9x+2 \\ -\underline{5x^4 \quad \mp 5x^2} \\ \quad 9x^3-2x^2 \\ \quad -\underline{9x^3 \quad \mp 9x} \\ \quad 0 \quad -2x^2+2 \\ \quad\quad \underline{\mp 2x^2 \pm 2} \\ \quad\quad\quad 0 \end{array}$$

$$5x^4+9x^3-7x^2-9x+2 = (x-1)(x+1)(5x^2+9x-2) =$$
$$= (x-1)(x+1)(5x^2+10x-x-2) =$$
$$= (x-1)(x+1)(5x(x+2)-(x+2)) =$$
$$= (x-1)(x+1)(x+2)(5x-1)$$

Roots are $-2, -1, 1, \frac{1}{5}$

2. Write a polynomial function which has -2i, 3, and -1 as roots.

$$f(x) = (x+1)(x-3)(x+2i)(x-2i) =$$
$$= (x+1)(x-3)(x^2+4)$$

This is only one of many possible answers.
$k f(x)$ is also a fine solution, for any $k \in \mathbb{R} \setminus \{0\}$