Standard form of quadratic function

Our goal is to write a quadratic function given in the following form:

$$f(x) = ax^2 + bx + c$$

in the standard form:

$$f(x) = a(x-h)^2 + k$$

Our task is then to decide what h and k should be. We will start with the case when $a = 1$
Remember that squaring a monomial gives us

$$(x+p)^2 = x^2 + 2px + p^2$$

For example:

$$(x+3)^2 = x^2 + 6x + 9$$
$$(x-5)^2 = x^2 - 10x + 25$$

Consider then, for instance, the following function:

$$f(x) = x^2 - 10x + 28$$

This function has some similarities with the last example above, except for the constant term, 28. However, it is nc very important that we have exactly a perfect square, just that we find a square there. We can do that:

$$f(x) = x^2 - 10x + 28 = x^2 - 10x + 25 + 3 = (x-5)^2 + 3$$

Let's look at another example:

$$f(x) = x^2 + 8x - 12$$

Firstly, I need to decide what p is in this particular instance. To do that, I look at the x term ($8x$), and write it as $2px$ $(2\cdot 4x)$. Then I will need p^2 $(4^2=16)$. It is not completely obvious how to write a constant term, -12, in the above function so that we 16. In order to avoid this, we can simply add and subtract 16.

$$f(x) = x^2 + 8x - 12 = x^2 + 8x + 16 - 16 - 12 = (x+4)^2 - 28$$

In general, then, we can do the following (remember: b and c are constants, real numbers):

$$f(x) = x^2 + bx + c = x^2 + 2\frac{b}{2}x + c = x^2 + 2\frac{b}{2}x + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = (x + \frac{b}{2})^2 - \frac{b^2}{4} + c$$

Here is another example:

$$f(x) = x^2 - 3x + 7 = x^2 + 2\frac{3}{2}x + 7 = x^2 + 2\frac{3}{2}x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 7 =$$
$$= (x + \frac{3}{2})^2 - \frac{3^2}{4} + 7 = (x + \frac{3}{2})^2 - \frac{9}{4} + 7 = (x + \frac{3}{2})^2 + \frac{21}{4}$$

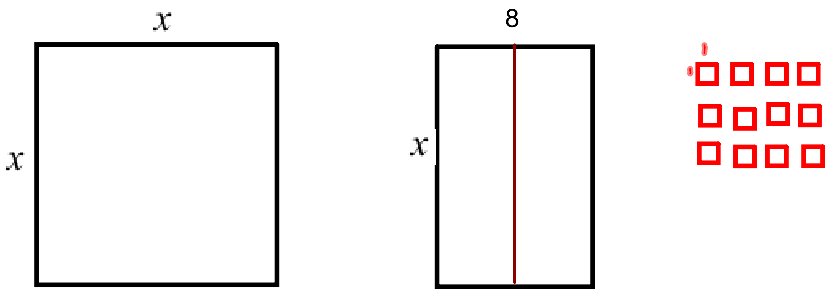
We said above that the standard form is: $f(x) = a(x-h)^2 + k$

So we still have to figure out what h and k are:

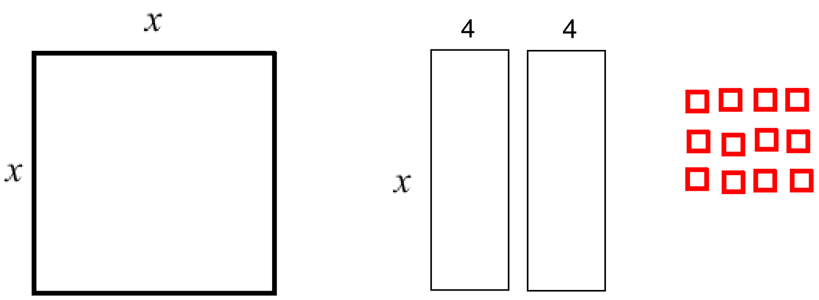
$$f(x) = (x + \frac{3}{2})^2 + \frac{21}{4} = \left(x - \left(-\frac{3}{2}\right)\right)^2 + \frac{21}{4}$$

Geometric interpretation of completing the square

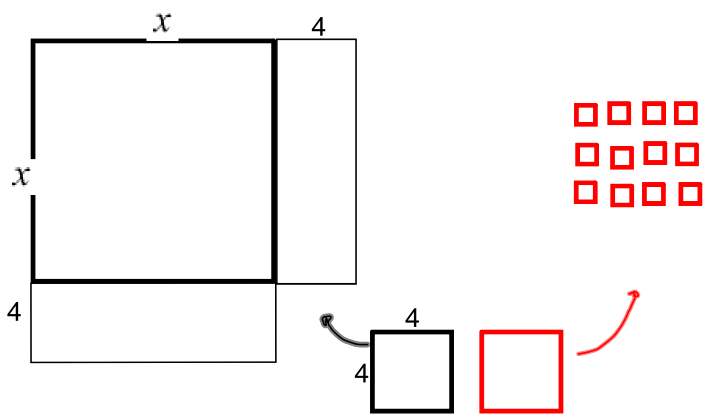
$$f(x) = x^2 + 8x - 12$$



Area: x^2 $8x$ 12 but since we have -12, we'll put the squares in red



Area: x^2 $4x$ $4x$ 12



In order for me to have a complete square here I need a 4x4 square, or rather 16 sqaure units, so I will add those in, but then I need to subtract them as well! We now have

$$(x+4)^2 \qquad -12-16$$

$$f(x) = x^2 + 8x - 12 = (x+4)^2 - 28$$

When leading coefficient is not 1

In this instance we will first factor out the leading coefficient and then perform the same procedure. We will show an example first, before showing a general case:

$$f(x) = 3x^2 - 12x + 11 = 3(x^2 - 4x) + 11 = 3(x^2 - 4x + 4 - 4) + 11 =$$

$$= 3\left((x-2)^2 - 4\right) + 11 = 3(x-2)^2 - 12 + 11 = 3(x-2)^2 - 1$$

In general:

$$f(x) = ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c =$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c =$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

You are now only seconds away from quadratic formula. Can you get the roots of this function?