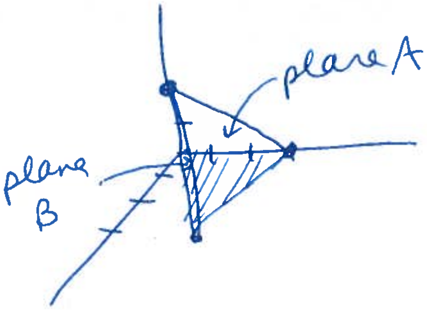
13.7 #17

tetrahedron $(0,0,0)$ $(3,2,0)$ $(0,3,0)$ $(0,0,2)$



plane A: $(3,2,0)$ $(0,3,0)$ $(0,0,2)$

$\vec{u} = \langle 3,-1,0\rangle$ $\quad \vec{v} = \langle 3,2,-2\rangle$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 3 & 2 & -2 \end{vmatrix} = \langle 2, 6, 9\rangle$$

$2x+6y+9z = D \qquad 9(2) = D = 18$

$2x+6y+9z = 18 \quad \Rightarrow \quad z = 2 - \frac{2}{9}x - \frac{2}{3}y$

$y = 3 - \frac{1}{3}x - \frac{3}{2}z$

$x = 9 - 3y - \frac{9}{2}z$

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plane B: $(3,2,0)$ $(0,0,0)$ $(0,0,2)$

$\vec{u} = \langle 3,2,0\rangle, \quad \vec{v} = \langle 0,0,2\rangle$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \langle 4,-6,0\rangle = 2\langle 2,-3,0\rangle$$

$2x - 3y = D \qquad 0 - 0 = D$

$\Rightarrow \quad 2x - 3y = 0$

$2x = 3y \quad \Rightarrow \quad y = \frac{2}{3}x$

$x = \frac{3}{2}y$

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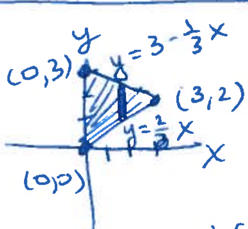
① **roof in z-direction:**



$0 \le z \le 2 - \frac{2}{9}x - \frac{2}{3}y$

$\frac{2}{3}x \le y \le 3 - \frac{1}{3}x$

$0 \le x \le 3$

$$V = \int_0^3 \int_{\frac{2}{3}x}^{3-\frac{1}{3}x} \int_0^{2-\frac{2}{9}x-\frac{2}{3}y} dz\, dy\, dx$$

②

$$V = \int_0^3 \int_{\frac{2}{3}x}^{3-\frac{1}{3}x} \left(2 - \tfrac{2}{9}x - \tfrac{2}{3}y\right) dy\, dx$$

$$= \int_0^3 \left(\left(2-\tfrac{2}{9}x\right)y - \tfrac{1}{3}y^2\right)\Big|_{\frac{2}{3}x}^{3-\frac{1}{3}x} dx$$

$$= \int_0^3 \left(\left(2-\tfrac{2}{9}\right)\left(3-\tfrac{1}{3}x-\tfrac{2}{3}x\right) - \tfrac{1}{3}\left[\left(3-\tfrac{1}{3}x\right)^2 - \left(\tfrac{2}{3}x\right)^2\right]\right) dx$$

$$= \int_0^3 \left(\left(6 - \tfrac{2}{3}x - \tfrac{4}{3}x - \tfrac{2}{3} + \tfrac{2}{27}x + \tfrac{4}{27}x\right) - \tfrac{1}{3}\left[9 - 2x + \tfrac{1}{9}x^2 - \tfrac{4}{9}x^2\right]\right) dx$$

$$= \int_0^3 \left(\tfrac{7}{3} - \tfrac{10}{9}x + \tfrac{1}{9}x^2\right) dx = \left(\tfrac{7}{3}x - \tfrac{10}{9}\left(\tfrac{x^2}{2}\right) + \tfrac{1}{9}\left(\tfrac{x^3}{3}\right)\right)\Big|_0^3$$

$$= \tfrac{7}{3}(3) - \tfrac{5}{9}(3^2) + \tfrac{1}{27}(3^3) - 0$$

$$= 7 - 5 + 1 = 3$$

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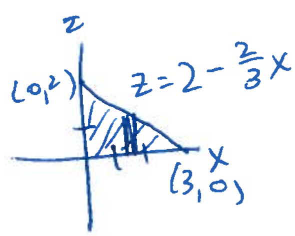
② roof in y-direction:



$$\tfrac{2}{3}x \le y \le 3 - \tfrac{1}{3}x - \tfrac{3}{2}z$$

$$0 \le z \le 2 - \tfrac{2}{3}x$$

$$0 \le x \le 3$$

$$V = \int_0^3 \int_0^{2-\frac{2}{3}x} \int_{\frac{2}{3}x}^{3-\frac{1}{3}x-\frac{3}{2}z} dy\, dz\, dx$$

$$= \int_0^3 \int_0^{2-\frac{2}{3}x} \left(3 - \tfrac{1}{3}x - \tfrac{3}{2}z - \tfrac{2}{3}x\right) dz\, dx$$

$$= \int_0^3 \int_0^{2-\frac{2}{3}x} \left(3 - \tfrac{3}{2}z - x\right) dz\, dx$$

$$= \int_0^3 \left(3z - xz - \tfrac{3}{4}z^2\right)\Big|_0^{2-\frac{2}{3}x} dx$$

$$= \int_0^3 \left(3\left(2-\tfrac{2}{3}x\right) - x\left(2-\tfrac{2}{3}x\right) - \tfrac{3}{4}\left(2-\tfrac{2}{3}x\right)^2\right) dx$$

$$= \int_0^3 \left(6 - 2x - 2x + \tfrac{2}{3}x^2 - \tfrac{3}{4}\left(4 - \tfrac{8}{3}x + \tfrac{4}{9}x^2\right)\right) dx$$

$$= \int_0^3 \left(3 - 2x + \tfrac{1}{3}x^2\right) dx = \left(3x - x^2 + \tfrac{1}{9}x^3\right)\Big|_0^3$$

$$= 3(3) - 3^2 + \tfrac{1}{9}(3^3) - 0 = 9 - 9 + 3 = 3$$

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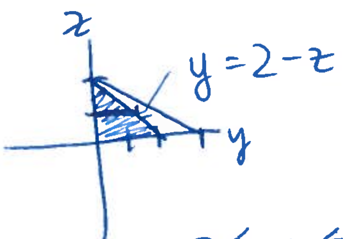
③ roof in x-direction:

(i) $0 \le x \le \frac{3}{2}y$



$0 \le y \le 2 - z$

$0 \le z \le 2$

$$V_i = \int_0^2 \int_0^{2-z} \left(\tfrac{3}{2}y\right) dy\, dz$$

$$= \int_0^2 \left(\tfrac{3}{4}y^2\right)\Big|_0^{2-z} dz$$

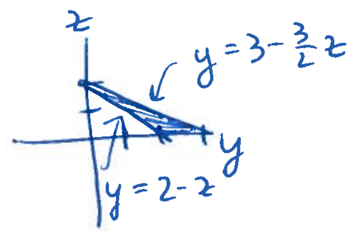
$$= \int_0^2 \left(\tfrac{3}{4}(2-z)^2\right) dz$$

$$= \int_0^2 \left(3 - 3z + \tfrac{3}{4}z^2\right) dz$$

$$= \left(3z - \frac{3z^2}{2} + \tfrac{1}{4}z^3\right)\Big|_0^2$$

$$= 6 - 6 + 2 = 2$$

(ii) $0 \le x \le 9 - 3y - \frac{9}{2}z$



$2 - z \le y \le 3 - \frac{3}{2}z$

$0 \le z \le 2$

$$V_{ii} = \int_0^2 \int_{2-z}^{3-\frac{3}{2}z} \left(9 - 3y - \tfrac{9}{2}z\right) dy\, dz$$

$$= \int_0^2 \left(\left(9 - \tfrac{9}{2}z\right)y - \frac{3y^2}{2}\right)\Big|_{2-z}^{3-\frac{3}{2}z} dz$$

$$= \int_0^2 \left(\left(9 - \tfrac{9}{2}z\right)\left(3 - \tfrac{3}{2}z - (2-z)\right) - \tfrac{3}{2}\left[\left(3 - \tfrac{3}{2}z\right)^2 - (2-z)^2\right]\right) dz$$

$$= \int_0^2 \left(9 - \tfrac{9}{2}z - \tfrac{9}{2}z + \tfrac{9}{4}z^2 - \tfrac{3}{2}\left(9 - 9z + \tfrac{9}{4}z^2 - 4 + 4z - z^2\right)\right) dz$$

$$= \int_0^2 \left(\tfrac{3}{2} - \tfrac{3}{2}z + \tfrac{3}{8}z^2\right) dz$$

$$= \left(\tfrac{3}{2}z - \tfrac{3}{4}z^2 + \tfrac{1}{8}z^3\right)\Big|_0^2$$

$$= 3 - 3 + 1 = 1$$

$V = 2 + 1 = 3$ ✓