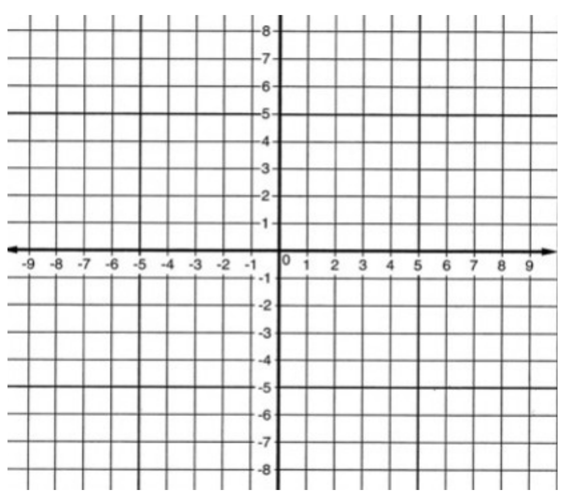
3.1 Maxima/Minima Values

Ex 1: Find the min and max points for the function $f(x)=\dfrac{1}{1+x^2}$ on the interval [-3, 1].	We're guaranteed max and min points if (1) the curve is continuous and (2) we have a closed interval. Max and min points in this case will occur: 1. as endpoints of the closed interval OR 2. as stationary points (where f'(x) = 0) OR 3. as singular points (where f'(x) does not exist) Max/min points is usually synonymous with extreme points. If f(c) is an extreme value (aka critical value or min/max value), then c is a critical x-value.
Ex 2: Under what conditions are we guaranteed min and max points?	Note: Remember that for curves that live in 2-d, a point has two coordinates!!!

Ex 3: Find all critical points for the curve given by $f(x)=x^5-\frac{25}{3}x^3+20x-1$ on the interval [-3, 2]. Identify the min and max values.

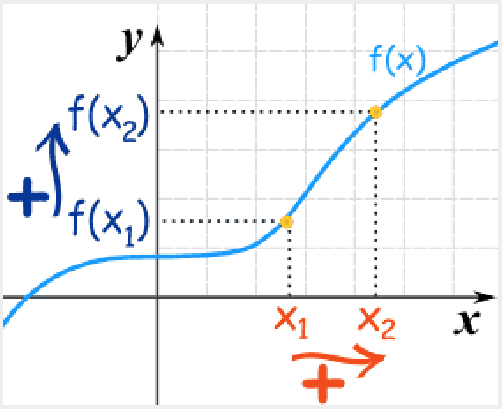
Ex 4: Sketch a graph of a function that meets the following conditions:
1. f(x) is continuous
2. f(x) is not necessarily differentiable.
3. domain of f(x) is [0, 6]
4. f(x) has a max value of 4 (at x = 3)
5. f(x) has a min value of 2 (at x = 1)
6. f(x) has no stationary points



3.2-3.3 Monotonicity and Concavity & Local Extrema

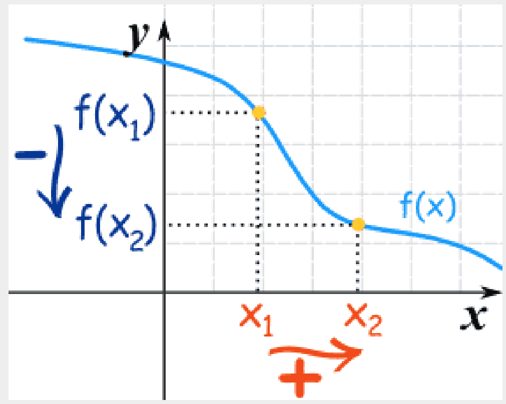
- $f'(x)>0$ on interval I $\Rightarrow$ $f(x)$ is increasing

increasing means
$x_1<x_2 \Rightarrow f(x_1)<f(x_2)$



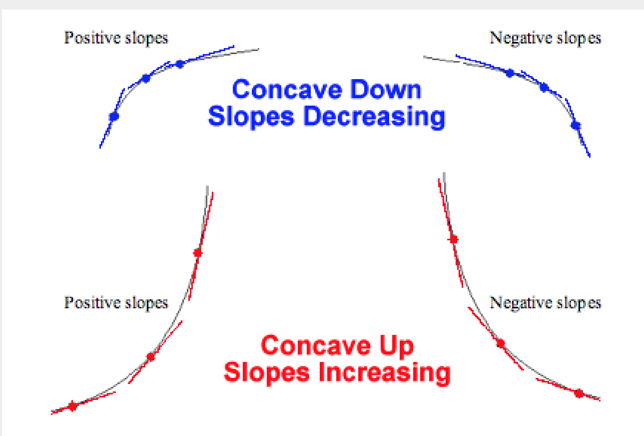
- $f'(x)<0$ on interval I $\Rightarrow$ $f(x)$ is decreasing

decreasing means
$x_1<x_2 \Rightarrow f(x_1)>f(x_2)$



concave up means
$f'(x)$ is increasing on interval I

concave down means
$f'(x)$ is decreasing on interval I



inflection point:
where the concavity changes (from up to down or down to up)

To find min/max x-values:
1. find where the first derivative is either zero or undefined.
2. create a first derivative sign line.
3. interpret the sign line for possible min/max points.

To find inflection point x-values:
1. find where the second derivative is either zero or undefined.
2. create a second derivative sign line.
3. interpret the sign line for possible inflection points.

Second Derivative Test:
If $f'(c)=0$ then (1) $f''(c)<0 \Rightarrow (c, f(c))$ is a max .
and (2) $f''(c)>0 \Rightarrow (c, f(c))$ is a min.

3.2-3.3 (continued)

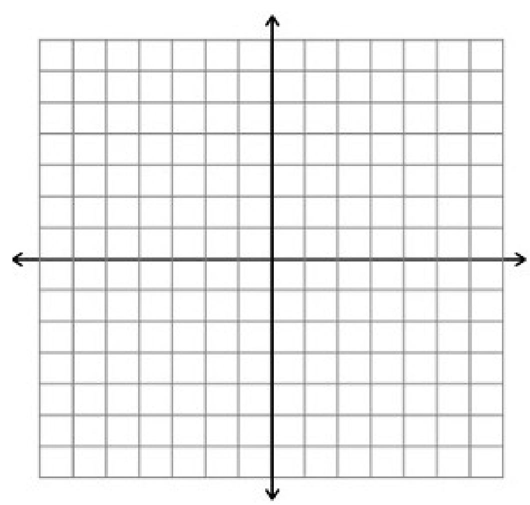
Ex 1: For $f(x)=\dfrac{x^2}{x^2+1}$, find all min/max points, inflection points, where f(x) is increasing and decreasing and sketch the graph.

Step 1: Find the first derivative and create the first derivative sign line.

Step 2: Find the second derivative and create the second derivative sign line.

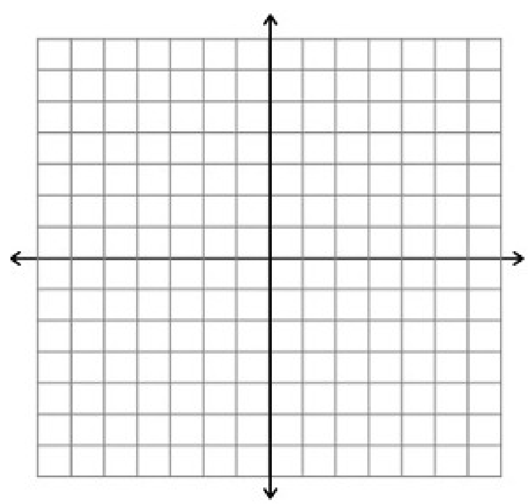
Step 3: Find all min/max and inflection points. (Remember you need two coordinates for every point.)

Step 4: Sketch the graph with all of that information.



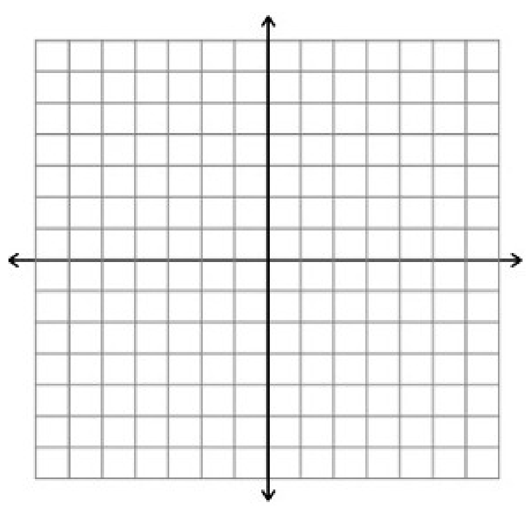
3.2-3.3 (continued)

Ex 2: For $f(x)=x^2-\frac{2}{x}$, find all min/max points, inflection points, where f(x) is increasing and decreasing, where it's concave up and concave down, and sketch the graph.



3.2-3.3 (continued)

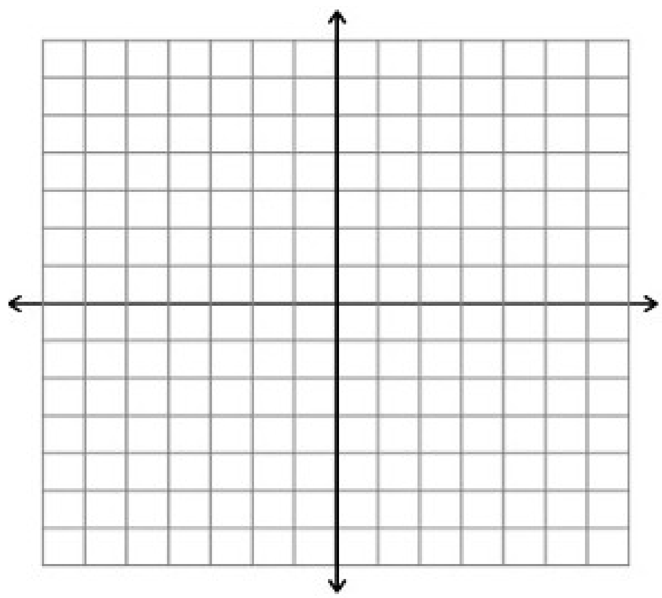
Ex 3: For $f(x)=x\sqrt{x-2}$, find all min/max points, inflection points, where f(x) is increasing and decreasing and sketch the graph.



3.2-3.3 (continued)

Ex 4: Sketch the graph of a continuous function that satisfies all of the following conditions:

1. f(0) = f(3) = 3
2. f(2) = 4
3. f(4) = 2
4. f(6) = 0
5. $f'(x) > 0$ on (0, 2)
6. $f'(x) < 0$ on (2, 4) and on (4, 5)
7. $f'(2) = f'(4) = 0$
8. $f'(x) = -1$ on (5, 6)
9. $f''(x) < 0$ on $(0,3) \cup (4,5)$
10. $f''(x) > 0$ on $(3,4)$



3.4 Min/Max Story (aka Optimization) Problems

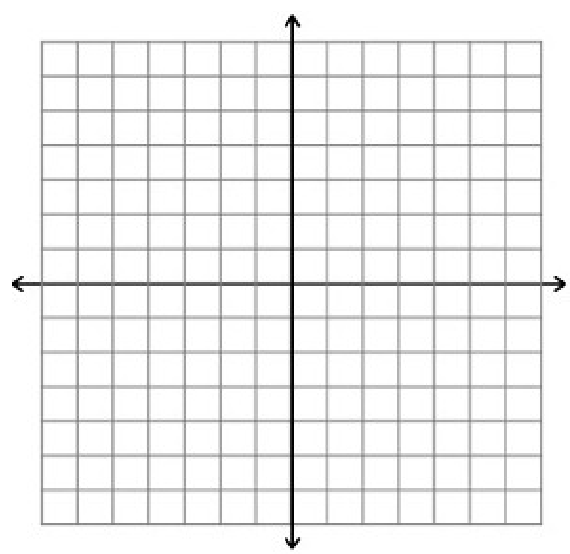
Ex 1: For what number does the principal square root exceed 8 times the number by the largest amount?	Steps to solve min/max story problems: 1. Draw a picture and/or list all the information given. 2. Write down what needs to be optimized, as a function. 3. If you have more than one input variable for the function you're optimizing, find an equation to eliminate one of those input variables. 4. Differentiate your function. 5. Set the derivative to zero or find where it's undefined, i.e. look for singular and stationary points for the function. 6. Check to ensure you have found the min or max (whatever the problem asked for). 7. Answer the stated questions.

Ex 2: Find the volume of the largest open box that can be made from a piece of cardboard that is 24 by 9 inches. Find the dimensions of the box that yields the maximum volume. (You'll form the box by cutting squares from each corner and fold up.)

Ex 3: A flower bed will be in the shape of a sector of a circle of radius r and vertex angle θ. Find r and θ if its area is a constant A and the perimeter is a minimum.

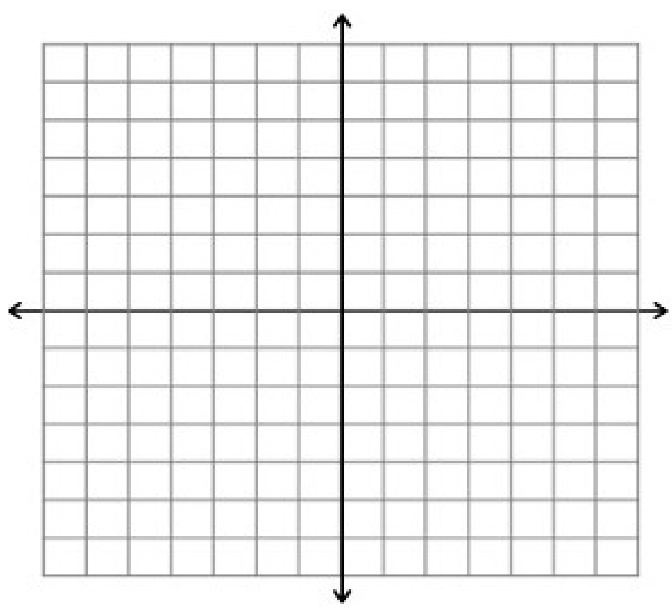
Ex 4: A farmer has 80 feet of fencing. They need to enclose three identical pens along one side of the barn. What dimensions for the total enclosure make the area of the pens as large as possible?
Ex 5: Show that the rectangle with maximum perimeter that can be inscribed in a circle is a square.

3.5 Graphing Functions

Ex 1: Analyze the function and sketch its graph. $f(x)=\frac{x^2+x-6}{x-1}$	<u>To analyze a function's graph</u>: 1. Find all VA, HA and SA and its domain. (VA = vertical asymptotes, HA = horizontal asymptotes, SA = slant asymptotes) 2. Find x-intercepts. 3. Find the derivative and fill in the sign-line; find min/max points. 4. Find the second derivative and fill in that sign-line; find inflection points. 5. Start the graph, by plotting • x-intercepts • all asymptotes • min/max points • inflection points • one or two more points, as needed. 6. Fill in the rest of the graph with knowledge of slope, concavity, etc. 

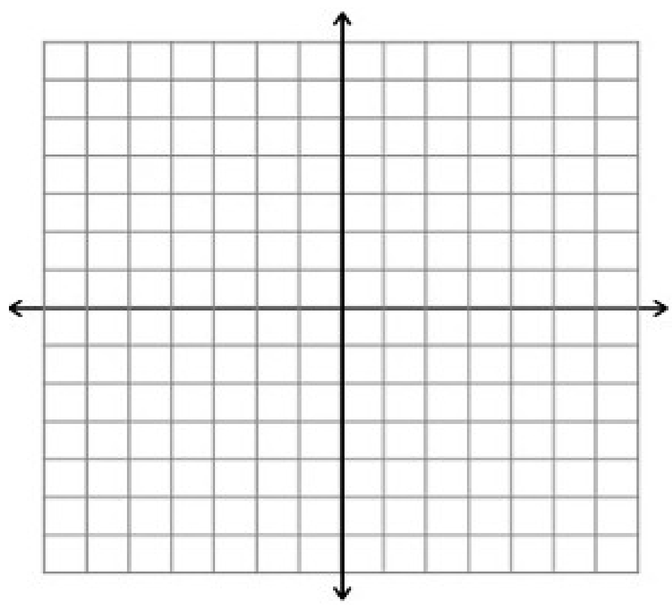
3.5 (continued)

Ex 2: For $f(x) = \dfrac{\sin x}{1 + \cos x}$ for $x \in [0\,,\, 2\pi]$, find all asymptotes, min/max points, inflection points, where f(x) is increasing and decreasing, where it's concave up and concave down, and sketch the graph.



3.5 (continued)

Ex 3: Analyze the function and sketch its graph. $f(x)=(1+x^5)^{-1}$



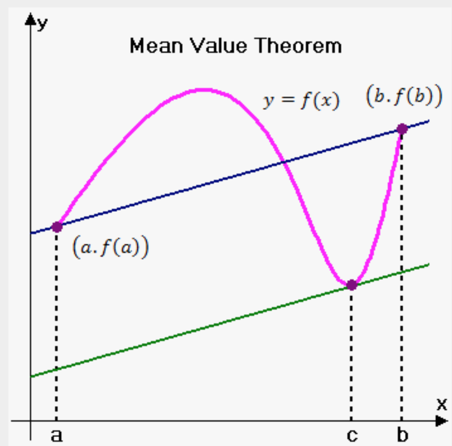
3.6 Mean Value Theorem for Derivatives

<table>
<tr>
<td>Ex 1: Find all possible c-values given by MVTD, if any exist.
(a) $f(x) = x + \frac{1}{x}$ on [2, 4]

(b) $f(x) = x + \frac{1}{x}$ on [-1, 2]</td>
<td><u>MVTD</u>

If
(1) f(x) is continuous on the interval [a, b]
AND
(2) f(x) is differentiable on (a, b),
then there is at least one c-value such that
$f'(c) = \frac{f(b) - f(a)}{b - a}$ and $c \in (a, b)$

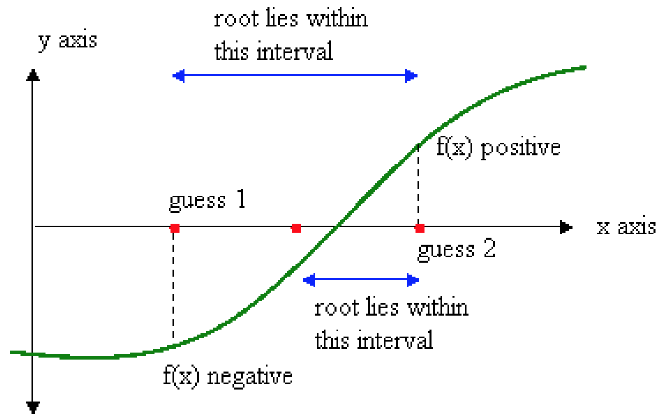
In other words, for a "nice" function, there is at least one point on the curve in the designated interval where the slope is the average slope.


</td>
</tr>
<tr>
<td colspan="2">(Ex 2: Find all possible c-values given by MVTD, if any exist, for $f(x) = \frac{2x - 7}{x - 5}$ on [6, 8].</td>
</tr>
</table>

3.6 (continued)

Ex 3: Suppose F'(x) = 5 and F(0) = 4. Find F(x).	If $F'(x)=G'(x)$, then $F(x)=G(x)+C$ for some arbitrary constant C.
Ex 4: Show that if f(x) is a quadratic function $f(x)=ax^2+bx+c$, $a\neq 0$, then the c given by MVTD is always the midpoint on any given interval $[\alpha, \beta]$.	

3.7 Bisection and Newton's Method



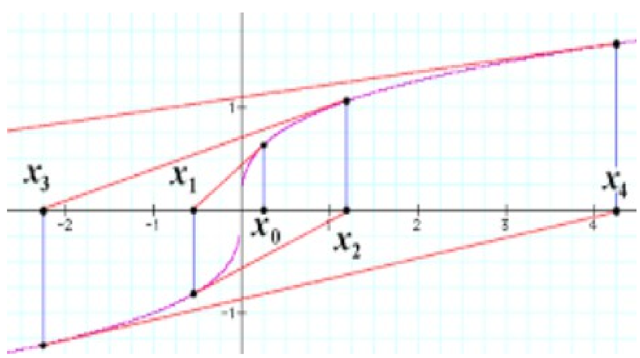
Ex 1: Use the Bisection method to approximate the real root of $x-2+2\cos x=0$ on the interval [1, 2].

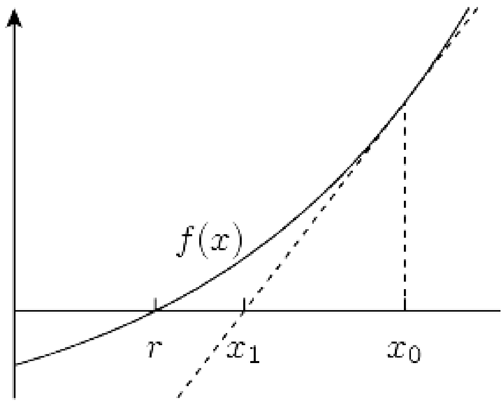
check: Is there a zero in that interval?

n	a_n	b_n	$m_n=\frac{a_n+b_n}{2}$	$f(a_n)$	$f(b_n)$	$f(m_n)$
1						
2						
3						
4						
5						
6						
7						

What is our final approximate answer?

What is the error? $h_n=\left|\frac{b_n-a_n}{2}\right|$





Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ used to solve $f(x) = 0$.

Note: You have to find a reasonable "first guess" of x_0 .

Ex 2: Use Newton's method to approximate the smallest root of $2\cos x - \sin x = 0$.

Find a first guess x_0 and explain your reasoning:

Find specific formula (for this given function) for x_{n+1} .

n	x_n
0	
1	
2	
3	
4	
5	

3.8 Antiderivatives

Ex 1: Evaluate the following antiderivatives.	Antiderivative Rules:
(a) $\int \left(\frac{\sqrt{2\text{x}}}{x} + \frac{3}{x^5} \right) dx$	1. $\int 1\, dx = x + C$ 2. $\int x^r\, dx = \frac{x^{r+1}}{r+1} + C$ (Power Rule) for $r \in \mathbb{Q}$, $r \neq -1$ 3. $\int \sin x\, dx = -\cos x + C$ 4. $\int \cos x\, dx = \sin x + C$ 5. $\int (g(x))^r g'(x)\, dx = \frac{(g(x))^{r+1}}{r+1} + C$ (generalized power rule, aka u-sub)
(b) $\int \frac{x(x+1)^2}{\sqrt{x}} dx$	Antiderivative (aka Indefinite Integral) operator is a linear operator. That is, it satisfies both of these properties. (a) $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$ (i.e. antiderivatives distribute through addition) AND (b) $\int k\, f(x)\, dx = k \int f(x)\, dx$ for any constant k (i.e. antiderivatives commute with multiplication by a constant).

Ex 2: Evaluate these integrals. (a) $\int x^2 \sqrt[3]{x^3+8}\ dx$	(b) $\int \sin x\ \cos x\ \sqrt{1+\sin^2 x}\ dx$

3.9 Introduction to Differential Equations

Ex 1: Solve these separable differential equations.

(a) $\frac{dy}{dx} = y^3(x^3 - x)$ if the function goes through the point (0, 4).

(b) $\frac{dy}{dx} = -y^2 x(x^2+2)^3$ if $y(0) = 1$.