Extending the Theory of Composites to Other Areas of Science

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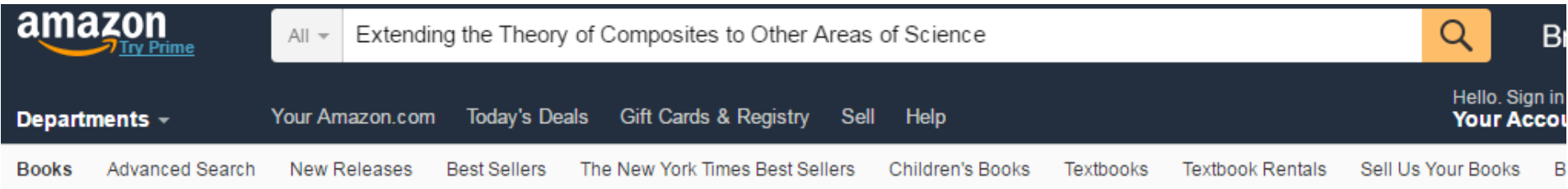
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With tremendous help from Nelson Beebe.

Lecture at Durham, UK (Recorded)
Graeme W. Milton
9am July 19th, 2016

Its available !
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Extending the Theory of Composites to Other Areas of Science

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In Chapter 1 we review many of the linear equations of physics, and write them in a canonical form appropriate to the theory of composites. We show how conservation laws, which have played a key role throughout the history of science, can be generalized to equalities which we call "boundary field equalities and inequalities". Chapter 2 reviews the abstract theory of composites, both for the effective tensor and for the associated "Y -tensor". Chapter 3 shows that the problem of finding the Dirichlet-to-Neumann map which governs the response of inhomogeneous bodies, for acoustics, elastodynamics,

▾ Read more

Rewriting some of the linear equations of Physics.

Constitutive Law:

$$\begin{pmatrix} \epsilon(\mathbf{x}) \\ \mathbf{d}(\mathbf{x}) \\ \mathbf{b}(\mathbf{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{S}(\mathbf{x}) & \mathcal{D}(\mathbf{x}) & \mathcal{Q}(\mathbf{x}) \\ \mathcal{D}^T(\mathbf{x}) & \varepsilon(\mathbf{x}) & \beta(\mathbf{x}) \\ \mathcal{Q}^T(\mathbf{x}) & \beta^T(\mathbf{x}) & \mu(\mathbf{x}) \end{pmatrix}}_{} \begin{pmatrix} \sigma(\mathbf{x}) \\ \mathbf{e}(\mathbf{x}) \\ \mathbf{h}(\mathbf{x}) \end{pmatrix},$$

Differential Constraints:

$$\begin{aligned} \epsilon &= [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2, & \nabla \cdot \sigma = 0; \\ \nabla \cdot \mathbf{d} &= 0, & \mathbf{e} = -\nabla V; \\ \nabla \cdot \mathbf{b} &= 0, & \mathbf{h} = -\nabla \psi. \end{aligned}$$

Key Identity: $\begin{pmatrix} \epsilon(\mathbf{x}) \\ \mathbf{d}(\mathbf{x}) \\ \mathbf{b}(\mathbf{x}) \end{pmatrix} \cdot \begin{pmatrix} \sigma(\mathbf{x}) \\ \mathbf{e}(\mathbf{x}) \\ \mathbf{h}(\mathbf{x}) \end{pmatrix} = \nabla \mathbf{u}(\mathbf{x}) : \sigma(\mathbf{x}) - \nabla V(\mathbf{x}) \cdot \mathbf{d}(\mathbf{x}) - \nabla \psi(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x}) = \nabla \cdot \mathbf{Q}(\mathbf{x}),$

$$\mathbf{Q}(\mathbf{x}) = \sigma(\mathbf{x})\mathbf{u}(\mathbf{x}) - V(\mathbf{x})\mathbf{d}(\mathbf{x}) - \psi(\mathbf{x})\mathbf{b}(\mathbf{x}),$$

$$\int_\Omega \begin{pmatrix} \epsilon(\mathbf{x}) \\ \mathbf{d}(\mathbf{x}) \\ \mathbf{b}(\mathbf{x}) \end{pmatrix} \cdot \begin{pmatrix} \sigma(\mathbf{x}) \\ \mathbf{e}(\mathbf{x}) \\ \mathbf{h}(\mathbf{x}) \end{pmatrix} = \int_{\partial\Omega} \mathbf{n} \cdot [\sigma(\mathbf{x})\mathbf{u}(\mathbf{x}) - V(\mathbf{x})\mathbf{d}(\mathbf{x}) - \psi(\mathbf{x})\mathbf{b}(\mathbf{x})].$$

Time Harmonic Equations:

Acoustics:
$$\underbrace{\begin{pmatrix} -i\mathbf{v} \\ -i\nabla \cdot \mathbf{v} \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -(\omega\rho)^{-1} & 0 \\ 0 & \omega/\kappa \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla P \\ P \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Elastodynamics:
$$\underbrace{\begin{pmatrix} -\boldsymbol{\sigma}(\mathbf{x})/\omega \\ i\mathbf{p}(\mathbf{x}) \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -\boldsymbol{C}/\omega & 0 \\ 0 & \omega\rho \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla\mathbf{u} \\ \mathbf{u} \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Maxwell:
$$\underbrace{\begin{pmatrix} -i\mathbf{h} \\ i\nabla \times \mathbf{h} \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} -[\omega\mu(\mathbf{x})]^{-1} & 0 \\ 0 & \omega\varepsilon(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}} \underbrace{\begin{pmatrix} \nabla \times \mathbf{e} \\ \mathbf{e} \end{pmatrix}}_{\mathcal{F}},$$

Schrödinger equation,
$$\begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla \cdot \mathbf{q}(\mathbf{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{A} & 0 \\ 0 & E - V(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} \nabla\psi(\mathbf{x}) \\ \psi(\mathbf{x}) \end{pmatrix},$$

Thermoacoustics: $(s = 0)$

$$
\begin{pmatrix} i\boldsymbol{\sigma} \\ i\nabla\cdot\boldsymbol{\sigma} \\ \mathbf{q} \\ \nabla\cdot\mathbf{q} \\ -iP \end{pmatrix} = \underbrace{\begin{pmatrix} i\boldsymbol{\mathcal{D}}(\mathbf{x}) + \frac{\mathbf{I}\otimes\mathbf{I}}{\omega\beta_T} & 0 & 0 & \frac{-i\alpha_0 T_0\mathbf{I}}{\beta_T} & \frac{\mathbf{I}}{\omega\beta_T} \\ 0 & -\omega\rho_0 & 0 & 0 & 0 \\ 0 & 0 & ik(\mathbf{x})T_0 & 0 & 0 \\ \frac{i\alpha_0 T_0\mathbf{I}\cdot}{\beta_T} & 0 & 0 & \omega\frac{\alpha_0^2 T_0^2}{\beta_T} - \omega\rho_0 C_p T_0 & \frac{i\alpha_0 T_0}{\beta_T} \\ \frac{\mathbf{I}\cdot}{\omega\beta_T} & 0 & 0 & \frac{-i\alpha_0 T_0}{\beta_T} & \frac{1}{\omega\beta_T} \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} \nabla\mathbf{v} \\ \mathbf{v} \\ \nabla\theta/T_0 \\ \theta/T_0 \\ s \end{pmatrix}
$$

Key Identity:

$$
\begin{pmatrix} i\boldsymbol{\sigma} \\ i\nabla\cdot\boldsymbol{\sigma} \\ \mathbf{q} \\ \nabla\cdot\mathbf{q} \\ -iP \end{pmatrix} \cdot \begin{pmatrix} \nabla\mathbf{v} \\ \mathbf{v} \\ \nabla\theta/T_0 \\ \theta/T_0 \\ s \end{pmatrix} = \nabla\cdot\left[i\boldsymbol{\sigma}\mathbf{v} + \mathbf{q}\theta/T_0\right].
$$

Acoustics in the time domain:

$$\begin{pmatrix} \frac{\partial \mathbf{v}}{\partial t} \\ \nabla \cdot \mathbf{v} \end{pmatrix} = \underbrace{\begin{pmatrix} -\rho(\mathbf{x})^{-1} & 0 \\ 0 & \kappa(\mathbf{x})^{-1} \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} \nabla P \\ -\frac{\partial P}{\partial t} \end{pmatrix} .$$

Key Identity:

$$\begin{pmatrix} \frac{\partial \mathbf{v}}{\partial t} \\ \nabla \cdot \mathbf{v} \end{pmatrix} \cdot \begin{pmatrix} \nabla P \\ -\frac{\partial P}{\partial t} \end{pmatrix} = (\nabla P) \cdot \begin{pmatrix} \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix} - \begin{pmatrix} \frac{\partial P}{\partial t} \end{pmatrix} (\nabla \cdot \mathbf{v}) + P \frac{\partial \nabla \cdot \mathbf{v}}{\partial t} - P \frac{\partial \nabla \cdot \mathbf{v}}{\partial t}$$

$$= \begin{pmatrix} \nabla \\ -\frac{\partial}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} P\frac{\partial \mathbf{v}}{\partial t} \\ P\nabla \cdot \mathbf{v} \end{pmatrix}$$

$$= \overline{\nabla} \cdot \mathbf{Q},$$

Elastodynamics in the time domain:

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathcal{C}(\mathbf{x}) & 0 \\ 0 & \rho(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} -\frac{1}{2}\left[\nabla \mathbf{v} + \nabla \mathbf{v}^T\right] \\ \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix},$$

Key identity:

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2}\left[\nabla \mathbf{v} + \nabla \mathbf{v}^T\right] \\ \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix} = -\left(\frac{\partial \boldsymbol{\sigma}}{\partial t}\right) : \nabla \mathbf{v} + (\nabla \cdot \boldsymbol{\sigma}) \cdot \left(\frac{\partial \mathbf{v}}{\partial t}\right) + \boldsymbol{\sigma} : \frac{\partial(\nabla \mathbf{v})}{\partial t}$$

$$- \boldsymbol{\sigma} : \frac{\partial(\nabla \mathbf{v})}{\partial t}$$

$$= \begin{pmatrix} \nabla \\ -\frac{\partial}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\sigma} \frac{\partial \mathbf{v}}{\partial t} \\ \boldsymbol{\sigma} : \nabla \mathbf{v} \end{pmatrix} = \overline{\nabla} \cdot \mathbf{Q},$$

Elastodynamics in a moving frame: Galilean transformation

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}'}{\partial t'} \\ \nabla' \cdot \boldsymbol{\sigma}' \end{pmatrix} = \begin{pmatrix} \mathcal{I} & \mathbf{wI} \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{w}(\nabla \cdot \boldsymbol{\sigma})^T \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix},$$

$$\begin{pmatrix} -\nabla' \mathbf{v} \\ \frac{\partial \mathbf{v}'}{\partial t'} \end{pmatrix} = \begin{pmatrix} \mathcal{I} & 0 \\ \mathbf{Iw}^T & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix} = \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w}^T \nabla \mathbf{v} \end{pmatrix},$$

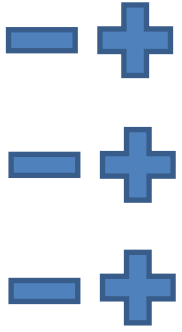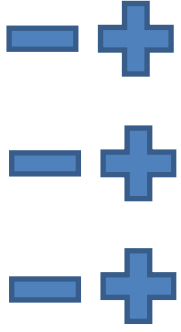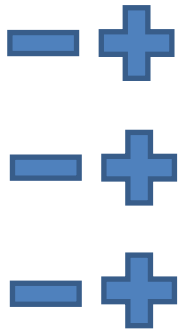## Matrix in the constitutive law now has Willis type couplings:

$$\begin{aligned} \mathbf{Z}'(\overline{\mathbf{x}}') &= \begin{pmatrix} \mathcal{I} & \mathbf{wI} \\ 0 & \mathbf{I} \end{pmatrix} \mathbf{Z}(\mathbf{x}) \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{Iw}^T & \mathbf{I} \end{pmatrix} \\ &= \begin{pmatrix} -\mathcal{C}(\mathbf{x}) + \mathbf{w}\rho(\mathbf{x})\mathbf{w}^T & \mathbf{w}\rho(\mathbf{x}) \\ \rho(\mathbf{x})\mathbf{w}^T & \rho(\mathbf{x}) \end{pmatrix}. \end{aligned}$$
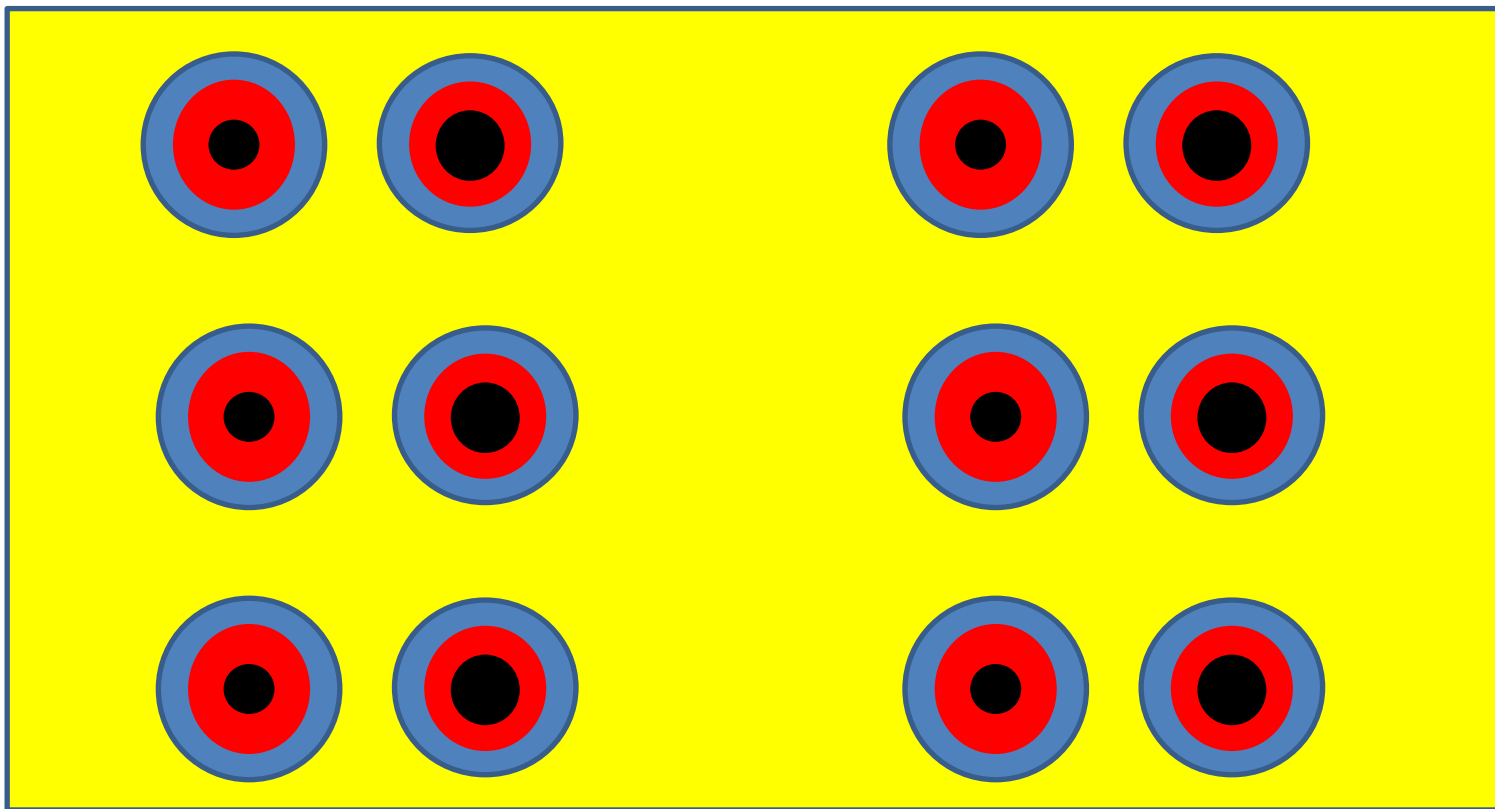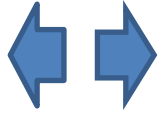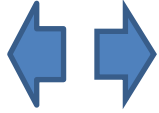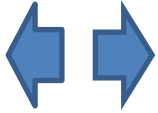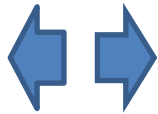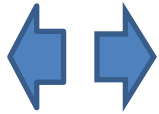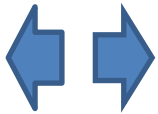
Yellow=Compliant, Blue=Stiff
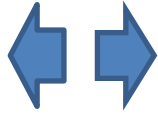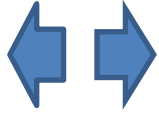Red=Rubber, Black=Lead

Time harmonic acceleration with no strain gives stress: Example of a Willis material

Electric dipole array
generates
polarization field

Force dipole array
generates
stress field
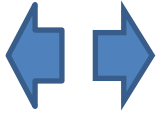
Piezoelectricity in the time domain:

$$
\begin{pmatrix} \frac{\partial \sigma}{\partial t} \\ \nabla \cdot \sigma \\ \frac{\partial d}{\partial t} \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathcal{C}(\mathbf{x}) & 0 & -\mathbf{a}(\mathbf{x}) \\ 0 & \rho(\mathbf{x}) & 0 \\ -\mathbf{a}^T(\mathbf{x}) & 0 & \varepsilon(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} -\frac{1}{2}\left[\nabla \mathbf{v} + \nabla \mathbf{v}^T\right] \\ \frac{\partial \mathbf{v}}{\partial t} \\ \frac{\partial \mathbf{e}}{\partial t} \end{pmatrix},
$$

Key identity:

$$
\begin{pmatrix} \frac{\partial \sigma}{\partial t} \\ \nabla \cdot \sigma \\ \frac{\partial d}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2}\left[\nabla \mathbf{v} + \nabla \mathbf{v}^T\right] \\ \frac{\partial \mathbf{v}}{\partial t} \\ \frac{\partial \mathbf{e}}{\partial t} \end{pmatrix} = \overline{\nabla} \cdot \mathbf{Q},
$$

$$
\mathbf{Q} = \begin{pmatrix} \sigma\frac{\partial \mathbf{v}}{\partial t} - \frac{\partial V}{\partial t}\frac{\partial \mathbf{d}}{\partial t} \\ \sigma \nabla \cdot \mathbf{v} \end{pmatrix},
$$

Biot equations in the time domain $(s = 0)$

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \\ -\nabla P \\ -\frac{\partial P}{\partial t} \\ M\zeta_{,t} \end{pmatrix} = \begin{pmatrix} -\mathcal{C} & 0 & 0 & \mathbf{M} & 0 \\ 0 & \rho & \rho_f & 0 & 0 \\ 0 & \rho_f & \widehat{m}_{ij}* & 0 & 0 \\ \mathbf{M} & 0 & 0 & M & M \\ 0 & 0 & 0 & M & M \end{pmatrix} \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} \\ \frac{\partial \mathbf{w}_t}{\partial t} \\ -\nabla \cdot \mathbf{w}_t \\ s \end{pmatrix}.$$

Key identity:

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \\ -\nabla P \\ -\frac{\partial P}{\partial t} \\ M\zeta_{,t} \end{pmatrix} \cdot \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} \\ -\frac{\partial \mathbf{w}_t}{\partial t} \\ -\nabla \cdot \mathbf{w}_t \\ s \end{pmatrix} = \begin{pmatrix} \nabla \\ -\frac{\partial}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} \sigma\frac{\partial \mathbf{v}}{\partial t} - P\frac{\partial \mathbf{w}_t}{\partial t} \\ \sigma\nabla \cdot \mathbf{v} - P\nabla \cdot \mathbf{w}_t \end{pmatrix}$$

Thermal Conduction and Diffusion:

$$\begin{pmatrix} \mathbf{q}_x \\ q_t \\ \nabla \cdot \mathbf{q}_x + \frac{\partial q_t}{\partial t} \end{pmatrix} = \underbrace{\begin{pmatrix} i\mathbf{k}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & -\frac{i\alpha(\mathbf{x})}{2} \\ 0 & \frac{i\alpha(\mathbf{x})}{2} & 0 \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} \nabla T \\ \frac{\partial T}{\partial t} \\ T \end{pmatrix} .$$

Key Identity:

$$\begin{pmatrix} \mathbf{q}_x \\ q_t \\ \nabla \cdot \mathbf{q}_x + \frac{\partial q_t}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} \nabla T \\ \frac{\partial T}{\partial t} \\ T \end{pmatrix} = \mathbf{q}_x \cdot \nabla T + q_t \frac{\partial T}{\partial t} + T \nabla \cdot \mathbf{q}_x + T \frac{\partial q_t}{\partial t}$$

$$= \begin{pmatrix} \nabla \\ -\frac{\partial}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{q}_x T \\ -T q_t \end{pmatrix}$$

Thermoelasticity in the time domain:

$$
\begin{pmatrix}
\frac{i\partial\boldsymbol{\sigma}}{\partial t} \\
i\nabla\cdot\boldsymbol{\sigma} \\
i\mathbf{q} \\
i\rho S\theta_0 \\
i\left(\nabla\cdot\mathbf{q}+\frac{\partial\rho S\theta_0}{\partial t}\right)
\end{pmatrix}
=
\underbrace{
\begin{pmatrix}
-i\mathcal{C}\frac{\partial}{\partial t} & 0 & 0 & i\boldsymbol{\beta}\theta_0 & 0 \\
0 & i\rho\frac{\partial}{\partial t} & 0 & 0 & 0 \\
0 & 0 & i\theta_0\boldsymbol{\kappa}* & 0 & 0 \\
-i\boldsymbol{\beta}\theta_0 & 0 & 0 & 0 & -i\theta_0\rho c \\
0 & 0 & 0 & i\theta_0\rho c & -i\theta_0\rho c\frac{\partial}{\partial t}
\end{pmatrix}
}_{\mathbf{Z(x)}}
\begin{pmatrix}
-\nabla\mathbf{u} \\
\frac{\partial\mathbf{u}}{\partial t} \\
-\nabla\theta/\theta_0 \\
-\frac{\partial\theta}{\partial t}/\theta_0 \\
-\theta/\theta_0
\end{pmatrix}.
$$

Key Identity:

$$
\begin{pmatrix}
\frac{i\partial\boldsymbol{\sigma}}{\partial t} \\
i\nabla\cdot\boldsymbol{\sigma} \\
i\mathbf{q} \\
i\rho S\theta_0 \\
i\left(\nabla\cdot\mathbf{q}+\frac{\partial\rho S\theta_0}{\partial t}\right)
\end{pmatrix}
\cdot
\begin{pmatrix}
-\nabla\mathbf{u} \\
\frac{\partial\mathbf{u}}{\partial t} \\
-\nabla\theta/\theta_0 \\
-\frac{1}{\theta_0}\frac{\partial\theta}{\partial t} \\
-\theta/\theta_0
\end{pmatrix}
=
\begin{pmatrix}
\nabla \\
-\frac{\partial}{\partial t}
\end{pmatrix}
\cdot
\begin{pmatrix}
i\boldsymbol{\sigma}\frac{\partial\mathbf{u}}{\partial t}-i\mathbf{q}\theta/\theta_0 \\
i\boldsymbol{\sigma}\nabla\cdot\mathbf{u}+i\rho S\theta
\end{pmatrix}.
$$

Maxwell's equations in the time domain:

$$\begin{pmatrix} -\mathbf{h} \\ \mathbf{d} \end{pmatrix} = \underbrace{\begin{pmatrix} -[\mu(\mathbf{x})]^{-1} & 0 \\ 0 & \varepsilon(\mathbf{x}) \end{pmatrix}}_{\mathbf{z}} \begin{pmatrix} \mathbf{b} \\ \mathbf{e} \end{pmatrix},$$

Differential constraints

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{e} \end{pmatrix} = \Theta \begin{pmatrix} \mathbf{\Phi} \\ V \end{pmatrix}, \quad \Theta^{\dagger} \begin{pmatrix} -\mathbf{h} \\ \mathbf{d} \end{pmatrix} = 0, \quad \Theta = \begin{pmatrix} \nabla \times & 0 \\ -\frac{\partial}{\partial t} & -\nabla \end{pmatrix}, \quad \Theta^{\dagger} = \begin{pmatrix} \nabla \times & \frac{\partial}{\partial t} \\ 0 & \nabla \cdot \end{pmatrix}$$

Key Identity:

$$\int_{\underline{\Omega}} \begin{pmatrix} -\mathbf{h} \\ \mathbf{d} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{b} \\ \mathbf{e} \end{pmatrix} = \int_{\underline{\Omega}} -\mathbf{h} \cdot (\nabla \times \mathbf{\Phi}) - \mathbf{d} \cdot \nabla V - \mathbf{d} \cdot \frac{\partial \mathbf{\Phi}}{\partial t} - \frac{\partial \mathbf{d}}{\partial t} \cdot \mathbf{\Phi} + (\nabla \times \mathbf{h}) \cdot \mathbf{\Phi}$$

$$= \int_{\partial \underline{\Omega}} \mathbf{n}_x \cdot (\mathbf{h} \times \mathbf{\Phi}) - \mathbf{n}_x \cdot (V\mathbf{d}) - n_t \mathbf{d} \cdot \mathbf{\Phi},$$

Schrödinger's equation in the time domain

$$\begin{pmatrix} \mathbf{q}_x \\ q_t \\ \nabla \cdot \mathbf{q}_x + \frac{\partial q_t}{\partial t} \end{pmatrix} = \begin{pmatrix} -\mathbf{A} & 0 & 0 \\ 0 & 0 & -\frac{i\hbar}{2} \\ 0 & \frac{i\hbar}{2} & -V \end{pmatrix} \begin{pmatrix} \nabla \psi \\ \frac{\partial \psi}{\partial t} \\ \psi \end{pmatrix} . \qquad \mathbf{A} = \hbar^2 \mathbf{I}/2m$$

$$\underbrace{\phantom{\begin{pmatrix} -\mathbf{A} & 0 & 0 \\ 0 & 0 & -\frac{i\hbar}{2} \\ 0 & \frac{i\hbar}{2} & -V \end{pmatrix}}}_{\mathbf{Z}}$$

Key Identity:

$$\int_\Omega \begin{pmatrix} \mathbf{q}_x \\ q_t \\ \nabla \cdot \mathbf{q}_x + \frac{\partial q_t}{\partial t} \end{pmatrix} \cdot \begin{pmatrix} \nabla \psi \\ \frac{\partial \psi}{\partial t} \\ \psi \end{pmatrix} = \int_\Omega \mathbf{q}_x \cdot \nabla \psi + q_t \frac{\partial \psi}{\partial t} + \psi \nabla \cdot \mathbf{q}_x + \psi \frac{\partial q_t}{\partial t}$$

$$= \int_\Omega \nabla \cdot (\mathbf{q}_x \psi) + \frac{\partial}{\partial t}(\psi q_t)$$

$$= \int_\Omega \overline{\nabla} \cdot \mathbf{Q}, \qquad \mathbf{Q} = (\mathbf{q}_x^T \psi, -q_t \psi)^T .$$

Schrödinger's equation in a magnetic field $(\hbar = 1)$

$$\begin{pmatrix} q_x \\ q_t \\ \nabla \cdot q_x + \frac{\partial q_t}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{-I}{2m} & 0 & \frac{ie\Phi}{2m} \\ 0 & 0 & -\frac{i}{2} \\ \frac{-ie\Phi}{2m} & +\frac{i}{2} & -eV \end{pmatrix} \begin{pmatrix} \nabla\psi \\ \frac{\partial\psi}{\partial t} \\ \psi \end{pmatrix}$$

Key identity still holds, and the above equation reduces to:

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}[i\nabla + e\Phi]^2\psi + eV\psi,$$

Minimization principles for Schrödinger's equation with complex energies

$$E\psi(\mathbf{x}) = -\nabla \cdot \mathbf{A}\nabla\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) - h(\mathbf{x})\theta_0 \qquad \mathbf{A} = \hbar^2\mathbf{I}/2m$$

Minimize over ψ'

$$W(\psi',p) = \sum_s \int_{\Omega^N} \underbrace{[p(\mathbf{x})]^2 + (E'')^2[\psi'(\mathbf{x})]^2 + 2\theta_0 p(\mathbf{x})h(\mathbf{x})}_{I(p,\psi')} \, d\mathbf{r}$$

where

$$p(\mathbf{x}) = p(\mathbf{x}, \psi') = \nabla \cdot \mathbf{A}\nabla\psi' + (E' - V(\mathbf{x}))\psi',$$

subject to suitable boundary conditions on ψ'

The Desymmetrization of Schrödinger's equation

$$\begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla \cdot \mathbf{q}(\mathbf{x}) \\ \nabla \cdot \mathbf{v}(\mathbf{x}) + S_0 \end{pmatrix} = \begin{pmatrix} -\mathbf{A} & 0 & 0 \\ 0 & E - V(\mathbf{x}) & h(\mathbf{x}) \\ 0 & \overline{h}(\mathbf{x}) & d(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \nabla \psi \\ \psi \\ \theta_0 \end{pmatrix} \qquad \mathbf{A} = \hbar^2 \mathbf{I}/2m$$

Replace with:

$$\mathbf{J}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{E}(\mathbf{x}),$$ Let $\mathbf{\Lambda}$ denote appropriate symmetrization operator:

$$\mathbf{\Lambda}\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla \cdot \mathbf{q}(\mathbf{x}) \\ \nabla \cdot \mathbf{v}(\mathbf{x}) \end{pmatrix} \qquad \mathbf{L}(\mathbf{x}) = \begin{pmatrix} -\mathbf{A} & 0 & 0 \\ 0 & a(\mathbf{x}_1, \mathbf{x}_2) & g(\mathbf{x}_1, \mathbf{x}_2) \\ 0 & \overline{g}(\mathbf{x}_1, \mathbf{x}_2) & d(\mathbf{x}_1, \mathbf{x}_2) \end{pmatrix} \qquad \mathbf{E}(\mathbf{x}) = \begin{pmatrix} \nabla \psi \\ \psi \\ \theta_0 \end{pmatrix}$$

Advantage: Can solve iteratively using FFT, and
the FFT operations only need be done on $(\mathbf{x}_1, \mathbf{x}_2)$, i.e.
only on two electron co-ordinates not all n electrons.

A new perspective on **conservation laws**:
Boundary field equalities and inequalities

$$\text{If } \nabla \cdot \mathbf{Q} = 0 \text{ in } \Omega \text{ then } \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{Q} = 0$$

$$\text{If } \nabla \cdot \mathbf{Q} \geq 0 \text{ in } \Omega \text{ then } \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{Q} \geq 0$$

Requires information about what is happening inside Ω namely that $\nabla \cdot \mathbf{Q} = 0$ or $\nabla \cdot \mathbf{Q} \geq 0$ in Ω.

Are there other boundary field equalities or inequalities that use partial information about what is inside the body?

$$\begin{pmatrix} \mathbf{j}_1(\mathbf{x}) \\ \mathbf{j}_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} a(\mathbf{x})\mathbf{I} & c(\mathbf{x})\mathbf{I} \\ c(\mathbf{x})\mathbf{I} & b(\mathbf{x})\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1(\mathbf{x}) \\ \mathbf{e}_2(\mathbf{x}) \end{pmatrix},$$

$$\nabla \cdot \mathbf{j}_1 = 0, \quad \nabla \cdot \mathbf{j}_2 = 0 \quad \mathbf{e}_1 = -\nabla V_1, \quad \mathbf{e}_2 = -\nabla V_2.$$

$$\mathbf{M}(\mathbf{x}) = \begin{pmatrix} a(\mathbf{x}) & c(\mathbf{x}) \\ c(\mathbf{x}) & b(\mathbf{x}) \end{pmatrix}, \qquad \beta\mathbf{I} \geq \mathbf{M}(\mathbf{x}) \geq \alpha\mathbf{I}, \quad \text{for some } \beta > \alpha > 0,$$

Following the ideas of Straley, Milgrom and Shtrikman suppose there is a matrix $\mathbf{W}$ such that

$$\mathbf{WMW}^T = \begin{pmatrix} a'(\mathbf{x}) & 0 \\ 0 & b'(\mathbf{x}) \end{pmatrix}.$$

$$\begin{pmatrix} V_1(\mathbf{x}) \\ V_2(\mathbf{x}) \end{pmatrix} = \mathbf{W}^T \begin{pmatrix} f(\mathbf{x}) \\ 0 \end{pmatrix} \quad \text{for } \mathbf{x} \in \partial\Omega, \implies W_{21}[\mathbf{n} \cdot \mathbf{j}_1(\mathbf{x})] + W_{22}[\mathbf{n} \cdot \mathbf{j}_2(\mathbf{x})] = 0, \quad \text{for all } \mathbf{x} \in \partial\Omega,$$

In two dimensions suppose

$$c(\mathbf{x}) = 0; \qquad b(\mathbf{x}) = \alpha^2/a(\mathbf{x})$$

Following ideas of Keller, Dykhne and Mendelson, we have the boundary field equality

$$\mathbf{n} \cdot \mathbf{j}_2(\mathbf{x}) = -\alpha \mathbf{t} \cdot \mathbf{e}_1(\mathbf{x}) \quad \text{when} \quad \mathbf{t} \cdot \mathbf{e}_2(\mathbf{x}) = \alpha^{-1} \mathbf{n} \cdot \mathbf{j}_1(\mathbf{x}).$$

Due to the fact that the equations are satisfied with

$$\mathbf{e}_2(\mathbf{x}) = \alpha^{-1} \mathbf{R}_\perp \mathbf{j}_1(\mathbf{x}) \qquad \mathbf{j}_2(\mathbf{x}) = \alpha \mathbf{R}_\perp \mathbf{e}_1(\mathbf{x})$$

$$\text{where} \quad \mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Some boundary field inequalities (with D. Harutyunyan)

Look for functions $f(\mathbf{E}) = \overline{\mathbf{E}} \cdot \mathbf{TE}$ and constants f_0 just dependent on the boundary fields such that

$$\int_\Omega f(\mathbf{E}(\mathbf{x})) \geq f_0$$

for all fields $\mathbf{E}(\mathbf{x})$ satisfying appropriate differential constraints and the boundary conditions. To get f_0 one could solve the Euler Lagrange-equations

$$\mathbf{J}_0(\mathbf{x}) = \mathbf{TE}_0(\mathbf{x}), \quad \mathbf{J}_0 \in \mathcal{J}_\Omega, \quad \mathbf{E}_0 \in \mathcal{E}_\Omega. \qquad f_0 = \int_\Omega \overline{\mathbf{E}_0} \cdot \mathbf{J}_0$$

To establish the inequality one needs to pick a $\mathbb{T}$ such that the volume average of $f(\mathbf{E}) = \overline{\mathbf{E}} \cdot \mathbb{T}\mathbf{E}$ is non-negative for any C-periodic function $\mathbf{E}(\mathbf{x})$ satisfying the appropriate differential constraints which we write as $\mathbf{E} \in \mathcal{E}$

Find $c > 0$ such that $\mathbf{L}(\mathbf{x}) - c\mathbb{T}$ is positive semidefinite for all $\mathbf{x} \in \Omega$

Then we have the boundary field inequality:

$$0 \leq \int_\Omega \overline{\mathbf{E}(\mathbf{x})} \cdot \mathbf{L}(\mathbf{x})\mathbf{E}(\mathbf{x}) - cf(\mathbf{E}(\mathbf{x})) \leq -cf_0 + \int_{\partial\Omega} \mathbf{n} \cdot \tilde{\mathbf{Q}}(\mathbf{x}).$$

New Methods for Imaging

Key idea: making a direct link between Dirichlet-to-Neumann maps for bodies and effective tensors for composites.

Abstract Theory of Composites

Hilbert Space $\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}$

Operator $\mathbf{L} : \mathcal{H} \rightarrow \mathcal{H}$

Given $\mathbf{E}_0 \in \mathcal{U}$

Solve $\mathbf{J}_0 + \mathbf{J} = \mathbf{L}(\mathbf{E}_0 + \mathbf{E})$

With $\mathbf{J}_0 \in \mathcal{U}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{E} \in \mathcal{E},$

Then $\mathbf{J}_0 = \mathbf{L}_* \mathbf{E}_0$ defines $\mathbf{L}_* : \mathcal{U} \rightarrow \mathcal{U}$

Example: Conducting Composites

$\mathcal{H}$ - Periodic fields that are square integrable over the unit cell

$\mathcal{U}$ - Constant vector fields

$\mathcal{E}$ - Gradients of periodic potentials

$\mathcal{J}$ - Fields with zero divergence and zero average value

$\mathbf{E}_0 + \mathbf{E}(\mathbf{x})$ - Total electric field

$\mathbf{J}_0 + \mathbf{J}(\mathbf{x})$ - Total current field

$\mathbf{L} = \boldsymbol{\sigma}(\mathbf{x})$ - Local conductivity

$\mathbf{L}_* = \boldsymbol{\sigma}_*$ - Effective conductivity

Variational principles if $\mathbf{L}$ is self-adjoint and positive definite:

$$(\mathbf{J}_0, \mathbf{L}_*^{-1}\mathbf{J}_0) = \inf_{\underline{\mathbf{J}} \in \mathcal{J}} (\mathbf{J}_0 + \underline{\mathbf{J}}, \mathbf{L}^{-1}(\mathbf{J}_0 + \underline{\mathbf{J}}))$$

$$(\mathbf{E}_0, \mathbf{L}_*\mathbf{E}_0) = \inf_{\underline{\mathbf{E}} \in \mathcal{E}} (\mathbf{E}_0 + \underline{\mathbf{E}}, \mathbf{L}(\mathbf{E}_0 + \underline{\mathbf{E}}))$$

Leading to the elementary bounds:

$$\mathbf{L}_* \geq 0, \quad \mathbf{L}_* \leq \boldsymbol{\Gamma}_0 \mathbf{L} \boldsymbol{\Gamma}_0, \quad \mathbf{L}_*^{-1} \leq \boldsymbol{\Gamma}_0 \mathbf{L}^{-1} \boldsymbol{\Gamma}_0,$$

$\boldsymbol{\Gamma}_0$ is the projection onto $\mathcal{U}$

Formula for the effective operator

$$\mathbf{L}_* = \boldsymbol{\Gamma}_0 \mathbf{L}[\mathbf{I} + \boldsymbol{\Gamma}_1(\mathbf{L}/\sigma_0 - \mathbf{I})]^{-1}\boldsymbol{\Gamma}_0.$$

where $\boldsymbol{\Gamma}_1$ is the projection onto $\mathcal{E}$.
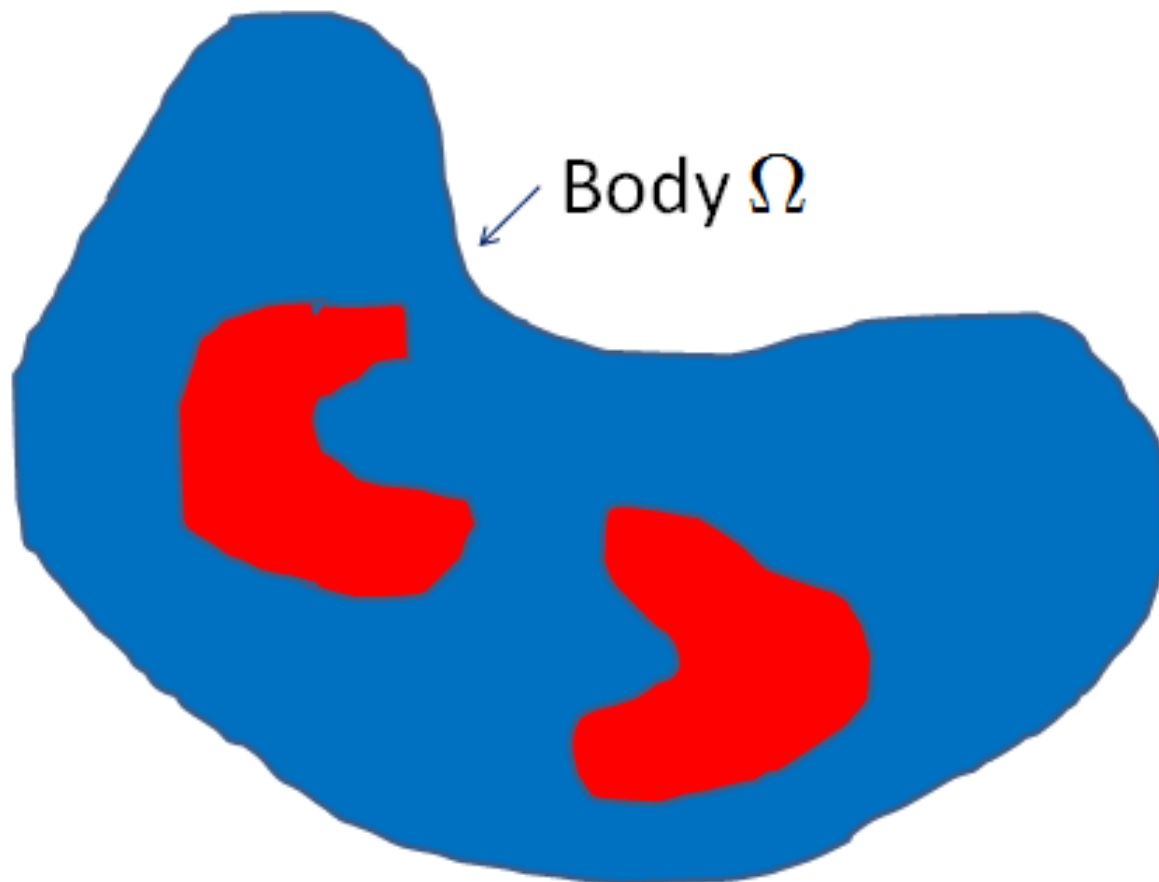
Leads to series expansions:

$$\mathbf{L}_* = \sum_{j=0}^{\infty} \boldsymbol{\Gamma}_0 \mathbf{L}[\boldsymbol{\Gamma}_1(\mathbf{I} - \mathbf{L}/\sigma_0)]^j \boldsymbol{\Gamma}_0,$$

$$\mathbf{J}_0 = \sum_{j=0}^{\infty} \boldsymbol{\Gamma}_0 \mathbf{L}[\mathbf{I} - \boldsymbol{\Gamma}_1(\mathbf{L}/\sigma_0)]^j \mathbf{E}_0,$$

$$\mathbf{E} = \sum_{j=0}^{\infty} [\boldsymbol{\Gamma}_1(\mathbf{I} - \mathbf{L}/\sigma_0)]^j \mathbf{E}_0,$$

$$\mathbf{J} = \sum_{j=0}^{\infty} \boldsymbol{\Gamma}_2 \mathbf{L}[\boldsymbol{\Gamma}_1(\mathbf{I} - \mathbf{L}/\sigma_0)]^j \mathbf{E}_0.$$
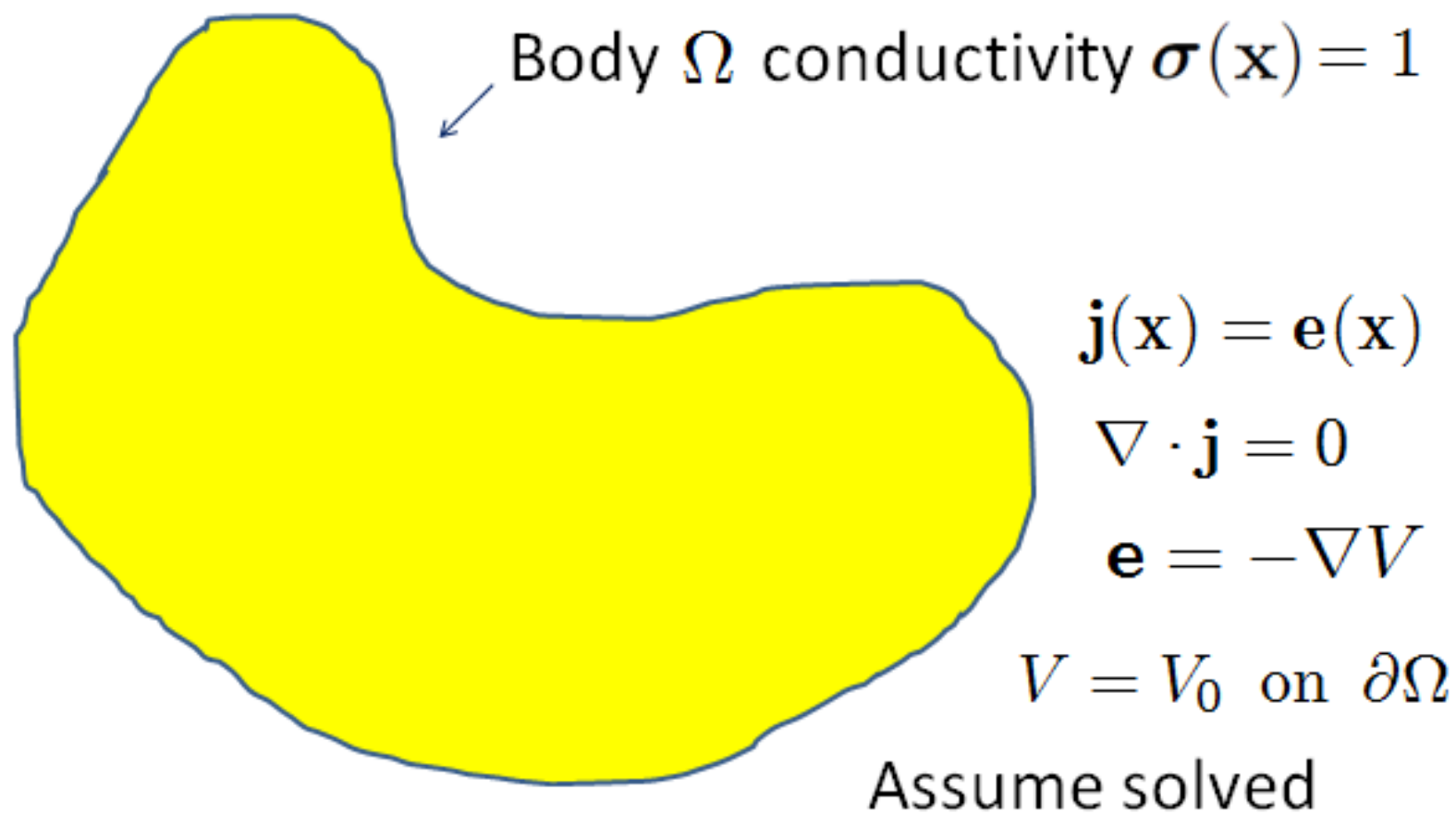
Dirichlet-to-Neumann Map

Body Ω

Specify boundary potential $V_0(\mathbf{x})$

Measure current flux $\mathbf{n} \cdot \mathbf{j}(\mathbf{x})$

We want to reformulate it as a problem in the abstract theory of composites, so we can apply the machinery of the theory of composites.

Remove boundary conditions, by expressing the problem in terms of the fields that solve the problem when Ω is filled with a homogeneous material

Body Ω conductivity $\boldsymbol{\sigma}(\mathbf{x}) = 1$

$$\mathbf{j}(\mathbf{x}) = \mathbf{e}(\mathbf{x})$$

$$\nabla \cdot \mathbf{j} = 0$$

$$\mathbf{e} = -\nabla V$$

$$V = V_0 \ \text{on} \ \partial\Omega$$

Assume solved

Now let

$\mathcal{U}$ consist of those fields $\mathbf{j}(\mathbf{x}) = \mathbf{e}(\mathbf{x})$ that solve the equations as the boundary potential $V_0(\mathbf{x})$ varies.

$\mathcal{E}$ consist of fields $\mathbf{E} = -\nabla V$ with $V(\mathbf{x}) = 0$ on $\partial\Omega$

$\mathcal{J}$ consist of fields $\mathbf{J}$ with $\nabla \cdot \mathbf{J} = 0$ and $\mathbf{n} \cdot \mathbf{J} = 0$ on $\partial\Omega$

Three spaces are orthogonal

Note that fields $\mathbf{j}(\mathbf{x}) = \mathbf{e}(\mathbf{x})$ in $\mathcal{U}$ can be parameterized either by the boundary values of $V = V_0$ on $\partial\Omega$ or by the boundary values of $\mathbf{n} \cdot \mathbf{j}(\mathbf{x})$.

The abstract problem in composites consists in finding for a given field $\mathbf{e}(\mathbf{x})$ in $\mathcal{U}$ (with associated boundary potential $V_0(\mathbf{x})$) the fields which solve:

$$\mathbf{j}'(\mathbf{x}) + \mathbf{J}(\mathbf{x}) = \sigma(\mathbf{x})[\mathbf{e}(\mathbf{x}) + \mathbf{E}(\mathbf{x})]$$

with

$$\mathbf{j}'(\mathbf{x}) \in \mathcal{U}, \quad \mathbf{J}(\mathbf{x}) \in \mathcal{J}, \quad \mathbf{E}(\mathbf{x}) \in \mathcal{E}$$

which is exactly the conductivity problem we would solve for the Dirichlet problem.

Furthermore if we knew the effective operator

$$\mathbf{L}_*: \mathcal{U} \to \mathcal{U}$$

Then we have

$$\mathbf{j}' = \mathbf{L}_* \mathbf{e}$$

and the boundary values of $\mathbf{n} \cdot \mathbf{j}'(\mathbf{x})$ allow us to determine the Dirichlet-to-Neumann map assuming the fields in $\mathcal{U}$ have been numerically calculated

Analyticity properties of effective tensors as functions of the moduli of the component materials (Bergman, Milton, Golden and Papanicolaou) extend to the Dirichlet-to-Neumann map

$$\sigma(\mathbf{x}) = \mathbf{R}(\mathbf{x})^T [\textstyle\sum_{i=1}^{n} \chi_i(\mathbf{x})\sigma_i] \mathbf{R}(\mathbf{x})$$

The Dirichlet-to-Neumann map is a Herglotz function of the matrices $\sigma_1, \sigma_2, \ldots \sigma_n$ in the domain where these have positive definite imaginary parts, modulo a rotation in the complex plane.

Easiest to prove using an approach of Bruno:
The truncated series expansion,

$$\mathbf{L}_* \approx \sum_{j=0}^{m} \mathbf{\Gamma}_0 \mathbf{L}[\mathbf{\Gamma}_1(\mathbf{I} - \mathbf{L}/\sigma_0)]^j \mathbf{\Gamma}_0$$

with $\mathbf{L} = \boldsymbol{\sigma}(\mathbf{x})$ is a polynomial in the matrix elements of $\sigma_1, \sigma_2, \ldots \sigma_n$ and hence $\mathbf{L}_*$ will be an analytic function of them in the domain of convergence of the series

One obtains integral representation formulas for $\mathbf{L}_*$ and hence for the Dirichlet to Neumann map.

Time Harmonic Equations:

Acoustics:

$$\underbrace{\begin{pmatrix} -i\mathbf{v} \\ -i\nabla \cdot \mathbf{v} \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -(\omega\rho)^{-1} & 0 \\ 0 & \omega/\kappa \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla P \\ P \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Elastodynamics:

$$\underbrace{\begin{pmatrix} -\boldsymbol{\sigma}(\mathbf{x})/\omega \\ i\mathbf{p}(\mathbf{x}) \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -\mathbf{C}/\omega & 0 \\ 0 & \omega\rho \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla\mathbf{u} \\ \mathbf{u} \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Maxwell:

$$\underbrace{\begin{pmatrix} -i\mathbf{h} \\ i\nabla \times \mathbf{h} \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} -[\omega\mu(\mathbf{x})]^{-1} & 0 \\ 0 & \omega\varepsilon(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}} \underbrace{\begin{pmatrix} \nabla \times \mathbf{e} \\ \mathbf{e} \end{pmatrix}}_{\mathcal{F}},$$

Schrödinger equation, $\begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla \cdot \mathbf{q}(\mathbf{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{A} & 0 \\ 0 & E - V(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} \nabla\psi(\mathbf{x}) \\ \psi(\mathbf{x}) \end{pmatrix},$

In all these examples Z has positive semidefinite imaginary part (that often can be made positive definite by a slight rotation in the complex plane)

The analog of the variational principles of Cherkaev and Gibiansky are then the variational principles of

Milton, Seppecher, and Bouchitte (2009)
Milton and Willis (2010)

For acoustics, electromagnetism, elastodynamics

Minimization variational principles for electromagnetism at fixed, possibly complex, frequency in lossy materials.

Maxwell's equations: (here we assume real ω)

$$\nabla \times \mathbf{E} = i\omega\mathbf{B}, \quad \nabla \times \mathbf{H} = -i\omega\mathbf{D}$$

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H},$$

Let

$$\mathcal{E} = \begin{pmatrix} \varepsilon'' + \varepsilon'(\varepsilon'')^{-1}\varepsilon' & \varepsilon'(\varepsilon'')^{-1} \\ (\varepsilon'')^{-1}\varepsilon' & (\varepsilon'')^{-1} \end{pmatrix},$$

When $\boldsymbol{\mu}$ is real : $\qquad Y(\mathbf{E}') = \inf_{\underline{\mathbf{E}}'} Y(\underline{\mathbf{E}}'),$

$$Y(\underline{\mathbf{E}}') = \int_\Omega \left(\underset{-\nabla \times \mu^{-1}(\nabla \times \underline{\mathbf{E}}')/\omega}{\overset{\omega \underline{\mathbf{E}}'}{}} \right) \cdot \varepsilon \left(\underset{-\nabla \times \mu^{-1}(\nabla \times \underline{\mathbf{E}}')/\omega \cdot}{\overset{\omega \underline{\mathbf{E}}'}{}} \right)$$

The infimum is over fields with prescribed tangential components of

$$\underline{\mathbf{E}}' \text{ and } \mu^{-1}\nabla \times \underline{\mathbf{E}}' \text{ at } \partial\Omega$$

Unusual boundary conditions (BC), but can be replaced by more normal BC: see paper with John Willis.

For electromagnetism, acoustics and elastodynamics, the Dirichlet-to-Neumann map is a Herglotz function of the matrices $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n$ of the component materials

For electromagnetism an alternative and rigorous proof was obtained with Maxence Cassier and Aaron Welters (in the new book)

Some inverse problems for two-component bodies

Electromagnetism:

suppose μ_1, μ_2 are real and frequency independent

Look for special complex frequencies where

$$\varepsilon_1(\omega) = \varepsilon_2(\omega)$$

Extrapolate (using representation formulas or bounds) measurements at different frequencies, or transient responses, to the neighborhood of these special frequencies

Quasistatic Elastodynamics:

Extrapolate to frequencies where $\mu_1(\omega_0^k) = \mu_2(\omega_0^k)$

Quasistatic Electromagnetism:

Extrapolate to ratios $\varepsilon_1/\varepsilon_2$ close to 1

Rigorous Upper and Lower Bounds on the Stress Relaxation in cylindrical composites in antiplane elasticity

Note collapse of bounds: they can be used in an inverse way to almost exactly recover the volume fraction from measurements at this particular time. Analogously for Dirichlet to Neumann map, one expects to recover geometry

With Ornella Mattei in new book,

Legend:
- No information
- Volume fraction
- Volume fraction and isotropy

Axis labels: $\overline{\sigma}_{12}(t)$ (vertical), t (horizontal)
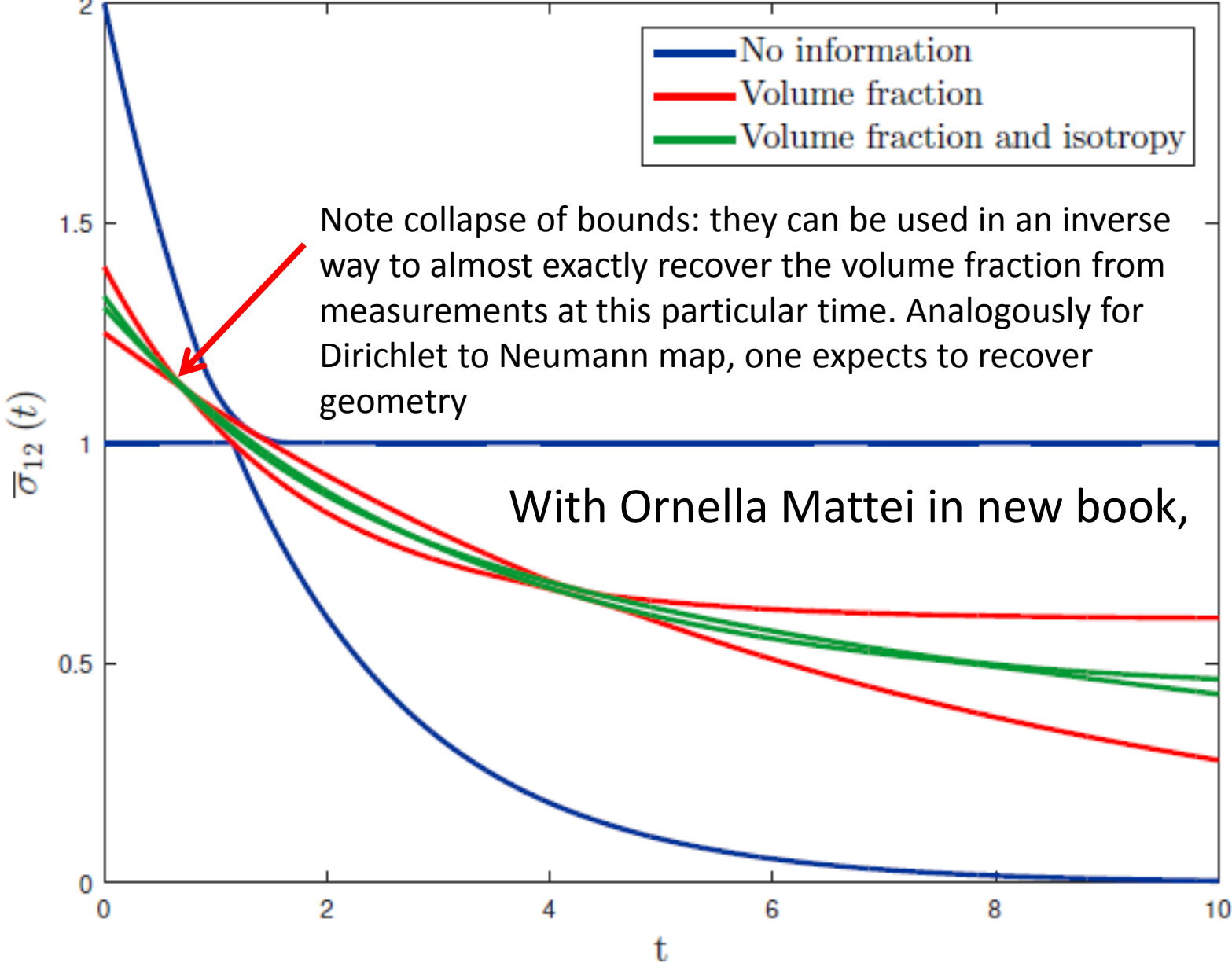
Generalizing the concept of function to

Superfunctions!

Adding resistor networks

(a)

+

(b)

=

(c)
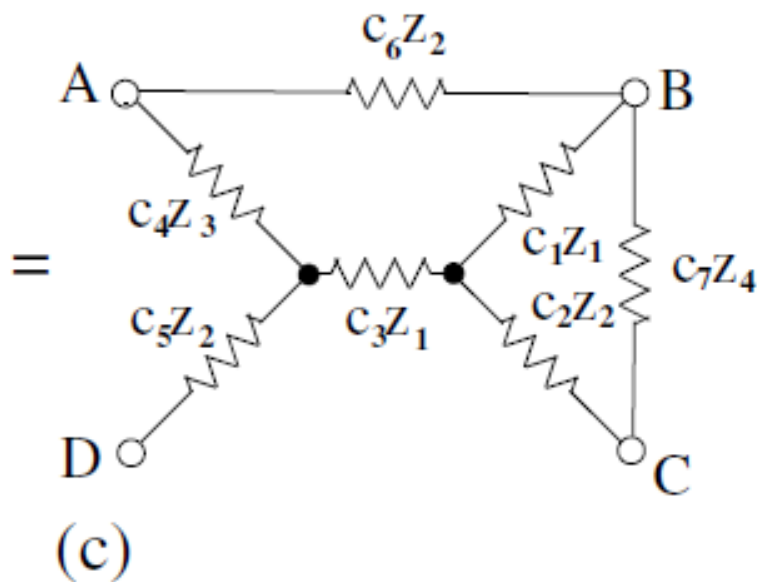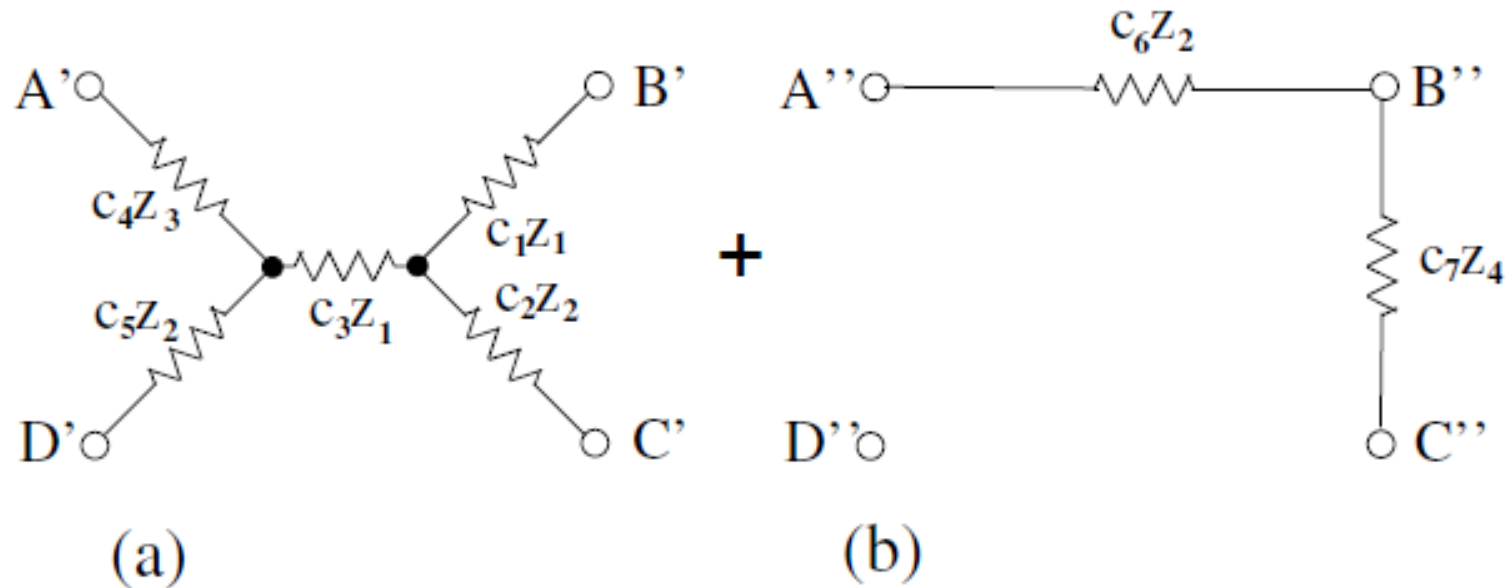
Multiplying resistor networks
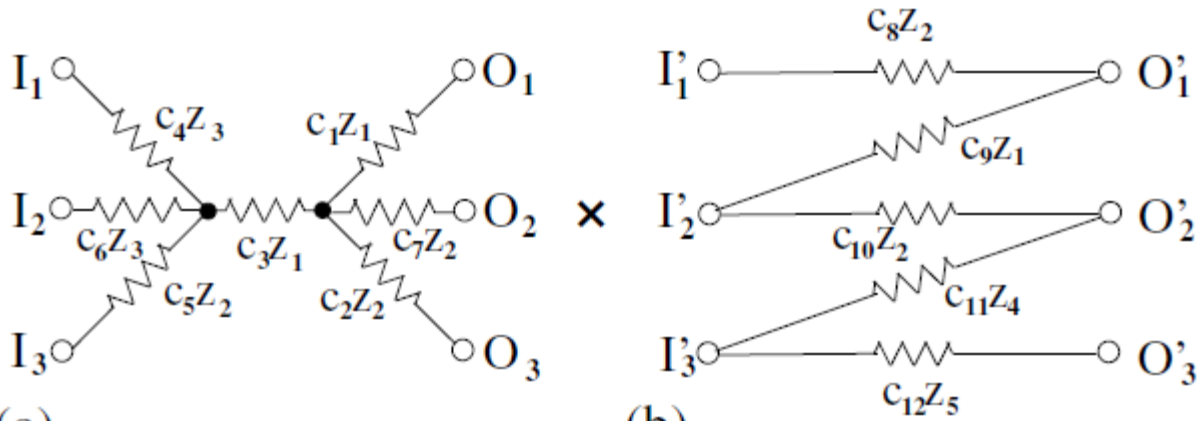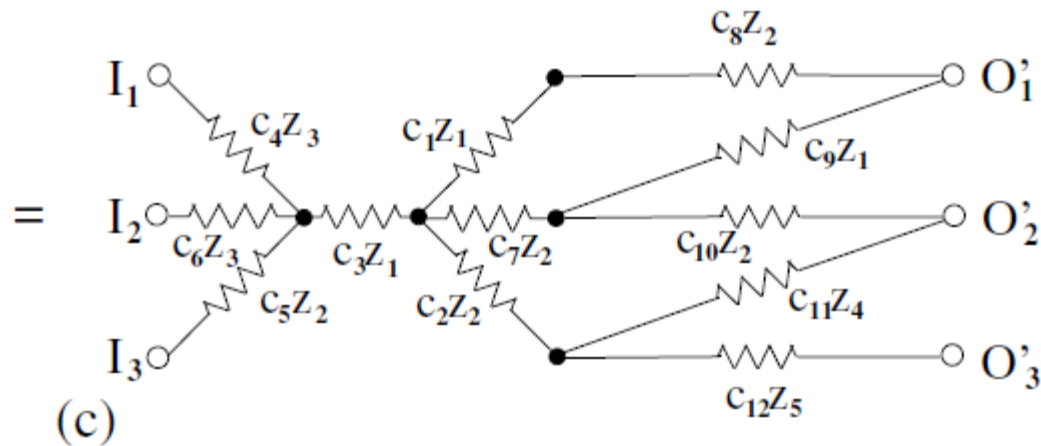
Substitution of networks
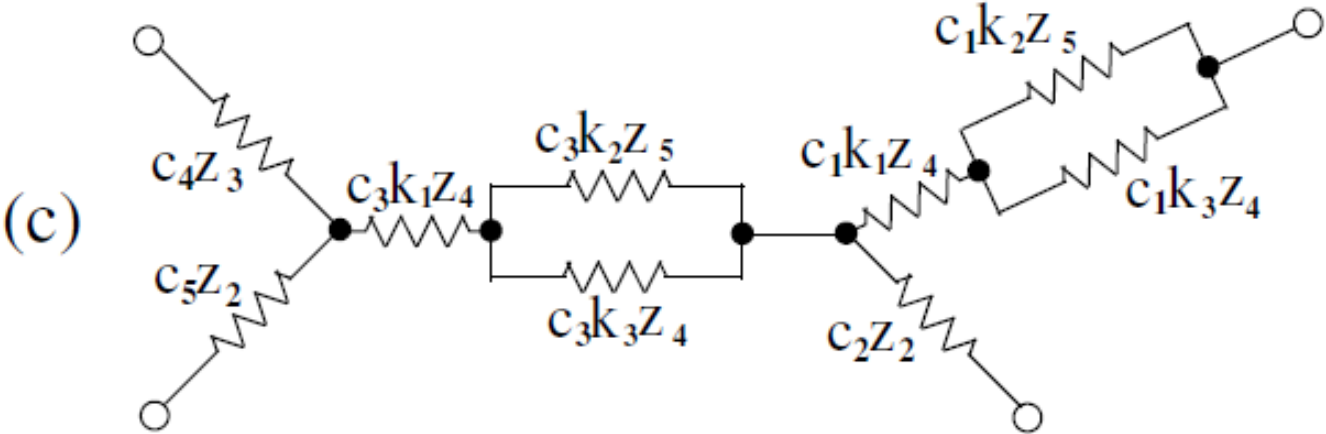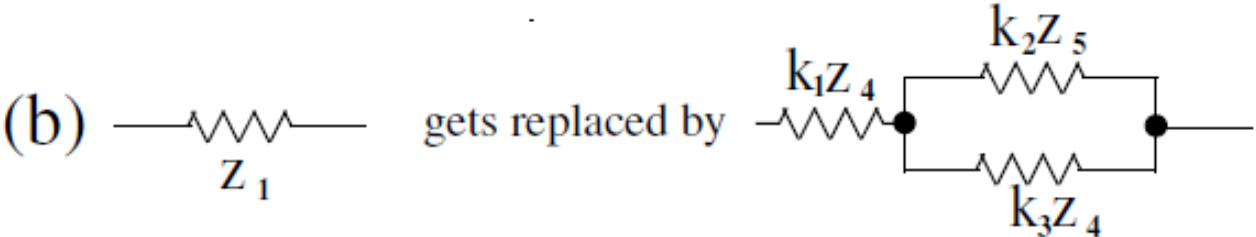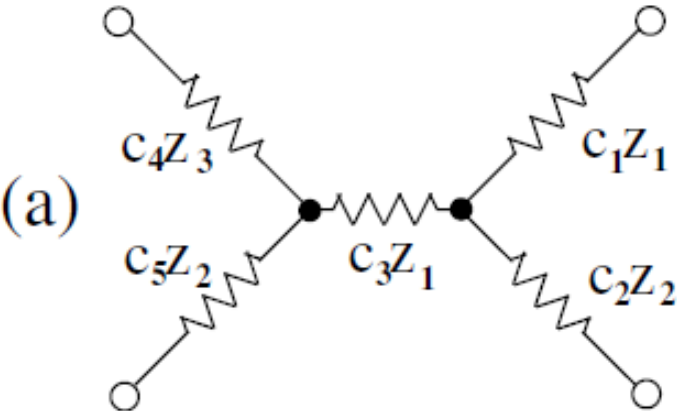
We should consider a resistor network in conjunction with its batteries
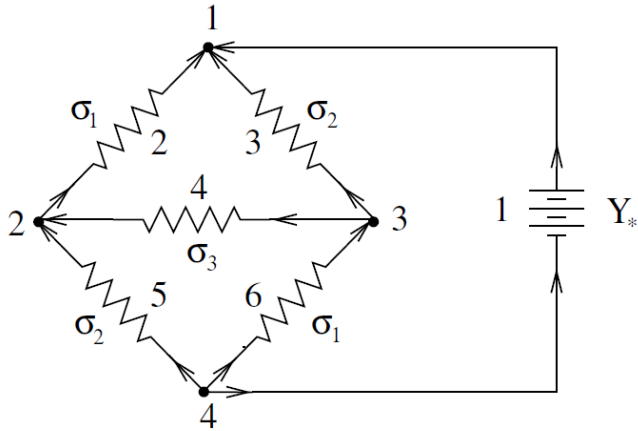

(a)

(b)
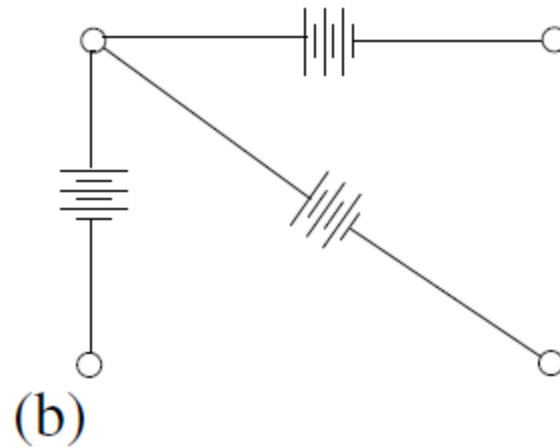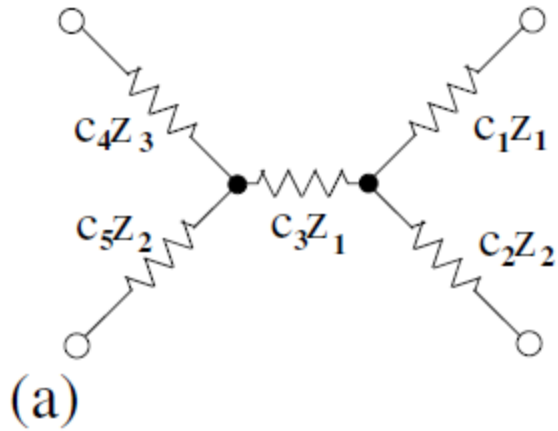
Space $\mathcal{H}$ Space $\mathcal{V}$

Combined Space $\mathcal{K} = \mathcal{H} \oplus \mathcal{V}$

Incidence Matrices:

$$M = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$M_{ij} = +1$ if the arrow of bond i points towards node j,

$\quad\;\; = -1$ if the arrow of bond i points away from node j,

$\quad\;\; = 0$ if bond i and node j are not connected.

Two natural subspaces:
 $\mathcal{J}$ the null space of M^T (current vectors)
 $\mathcal{E}$ the range of M (potential drops)
These are orthogonal spaces and $\mathcal{K} = \mathcal{E} \oplus \mathcal{J}$

Other spaces:

Divide the bonds in $\mathcal{H}$ into n groups (representing the different impedances).

Define $\mathcal{P}_i$ as the space of vectors P with elements P_j that are zero if bond j is not in group i.

The projection Λ_i onto the space $\mathcal{P}_i$ is diagonal and has elements

$$\{\Lambda_i\}_{jk} = 1 \text{ if } j = k \text{ and bond } j \text{ is in group } i,$$
$$= 0 \text{ otherwise.}$$

Thus $\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$

This is an orthogonal subspace collection Y(n)

Y(n) subspace collection:

$$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$$
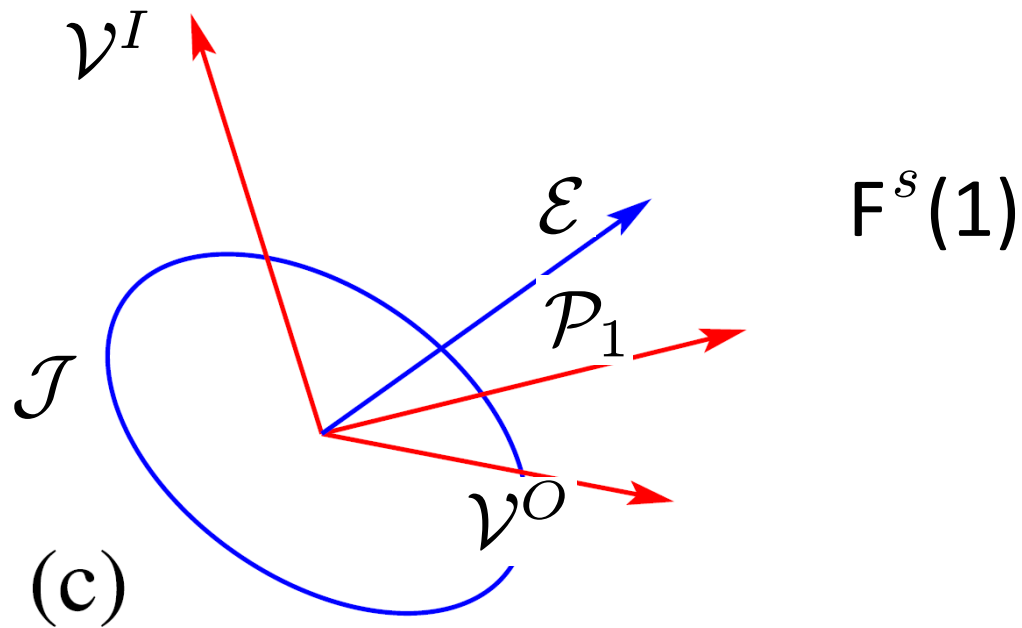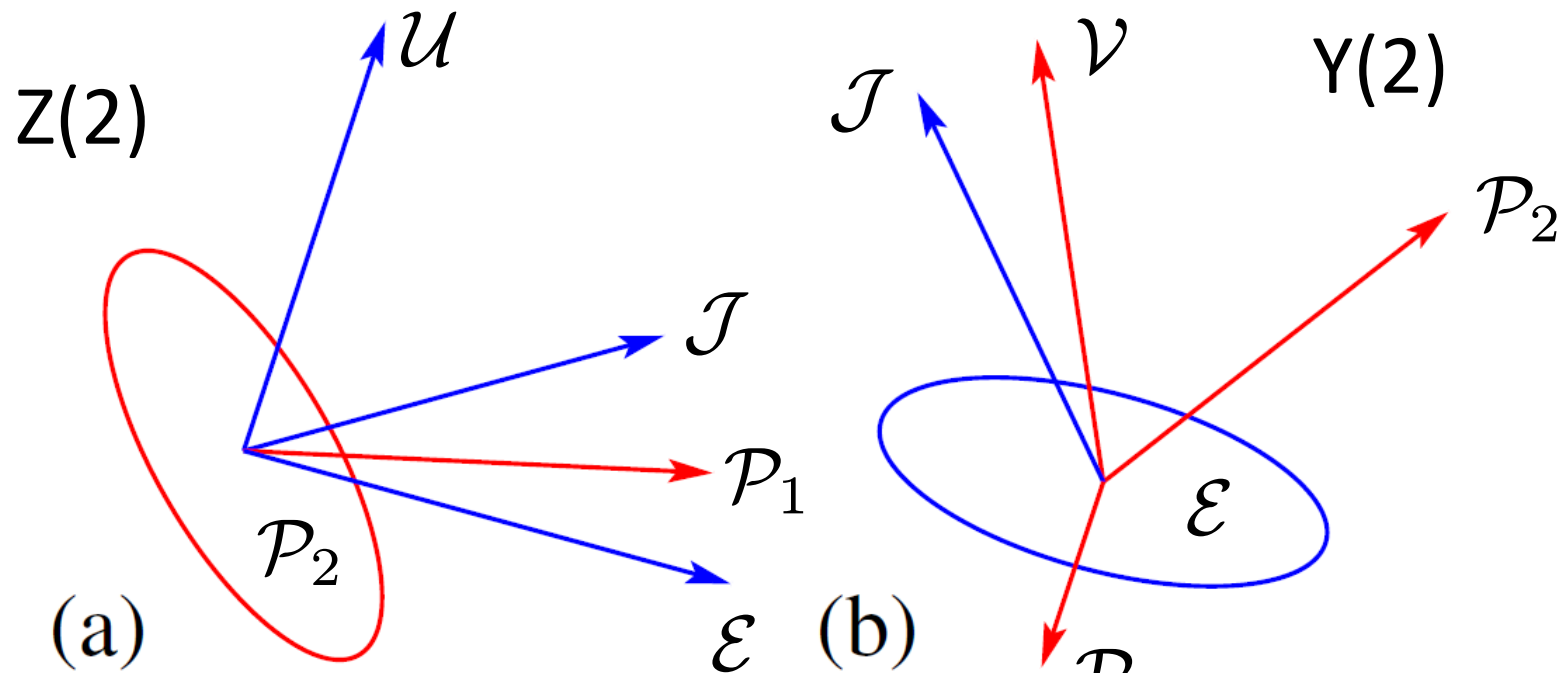
Z(n) subspace collection:

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$$

Superfunction $F^{\mathcal{S}}$(n): Y(n) subspace collection with

$$\mathcal{V} = \mathcal{V}^I \oplus \mathcal{V}^O.$$

Subspace collections need not have orthogonal subspaces

Z(2)

$\mathcal{U}$

$\mathcal{J}$

$\mathcal{P}_1$

$\mathcal{P}_2$

(a)

$\mathcal{E}$

Y(2)

$\mathcal{V}$

$\mathcal{J}$

$\mathcal{P}_2$

$\mathcal{E}$

(b)

$\mathcal{P}_1$

Key: Allow nonorthogonal Subspace collections. Then we have a whole algebra.

$\mathcal{V}^I$

$\mathcal{E}$

$\mathcal{P}_1$

$F^s(1)$

$\mathcal{J}$

$\mathcal{V}^O$

(c)

Application: Accelerating some Fast Fourier Transform Methods in two-component composites

The effective conductivity σ_* is an analytic function of the component conductivities σ_1 and σ_2
With $\sigma_2 = 1$, $\sigma_*(\sigma_1)$ has the properties of a Stieltjes function:
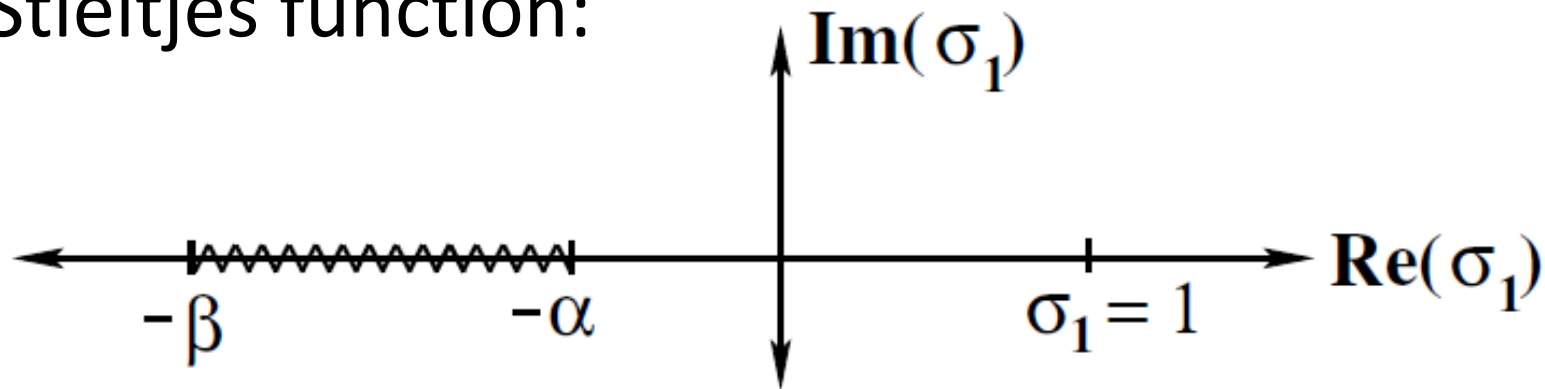
Bergman 1978 (pioneer, but faulty arguments)
Milton 1981 (limit of resistor networks)
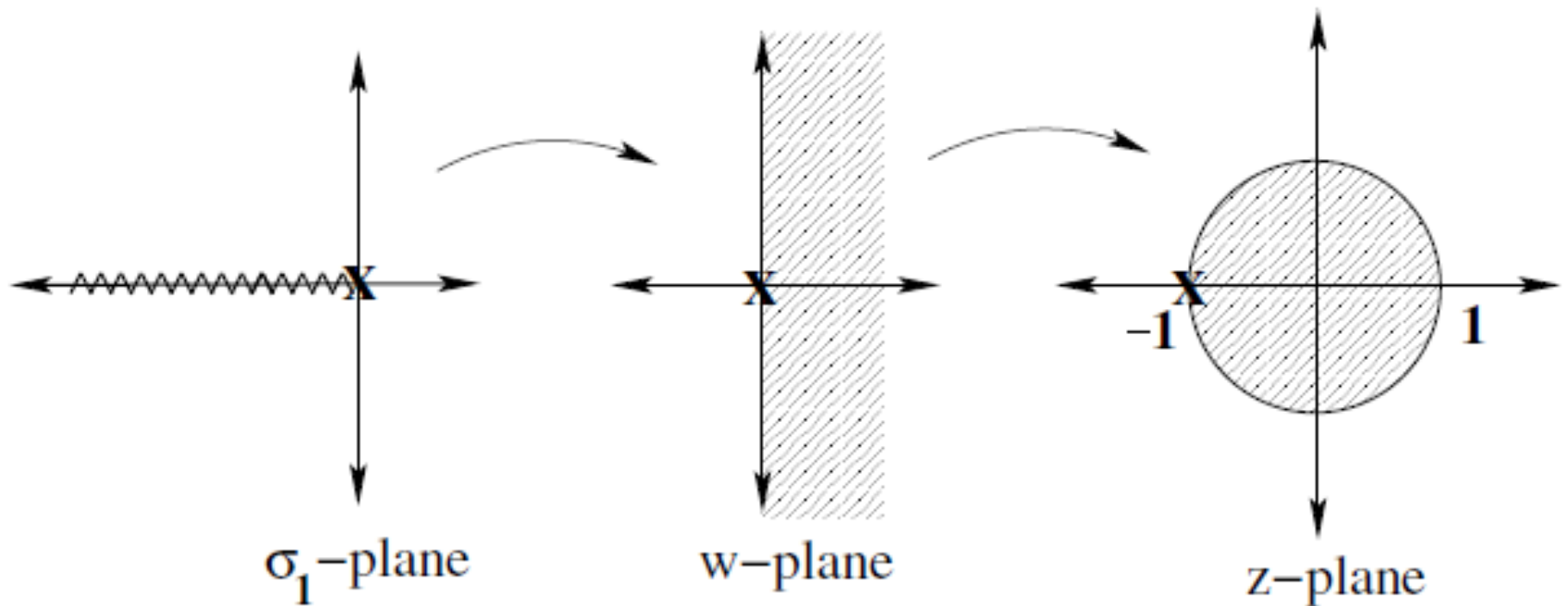Golden and Papanicolaou 1983 (rigorous proof)

Numerical scheme of Moulinec and Suquet (1994)

$$\sigma_* / \sigma_0 = 1 + \sum_{n=1}^{\infty} a_n \left(\frac{\sigma_1 - 1}{\sigma_1 + 1} \right)^n.$$
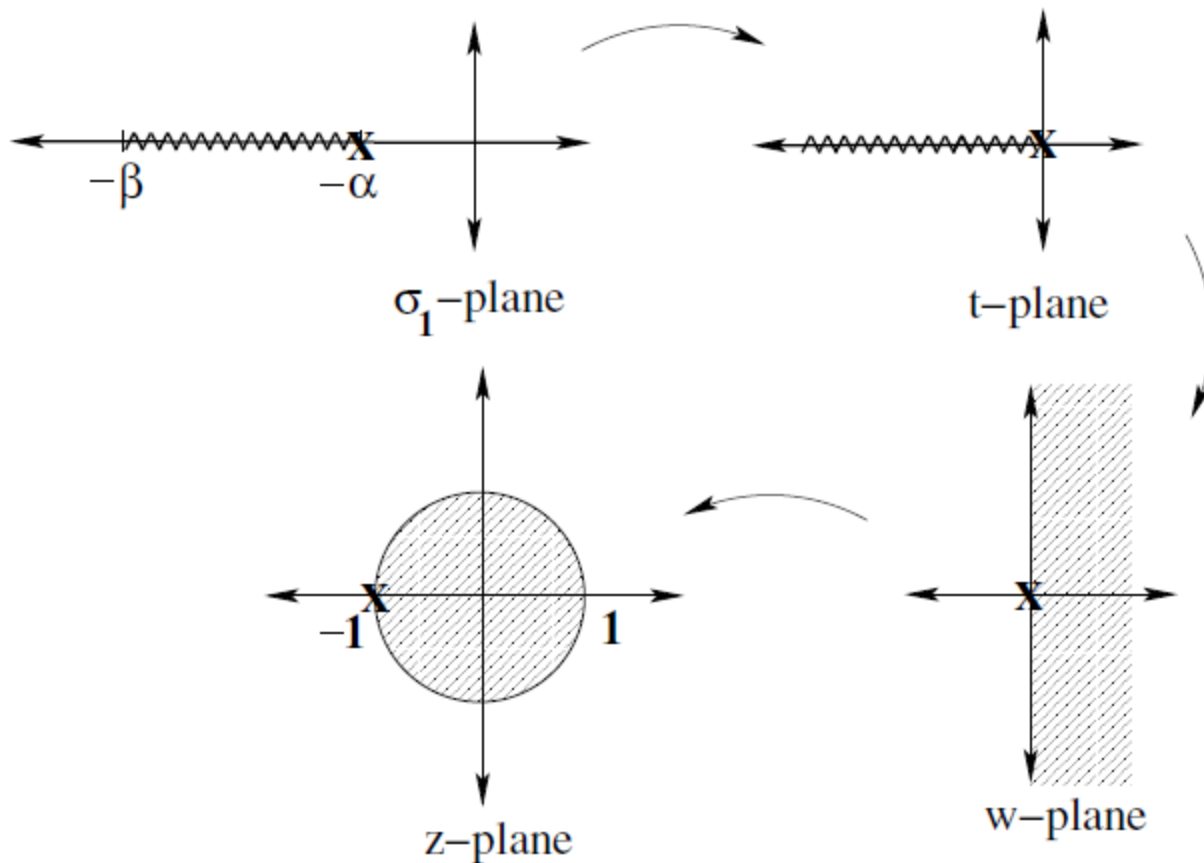

σ_1–plane   z–plane
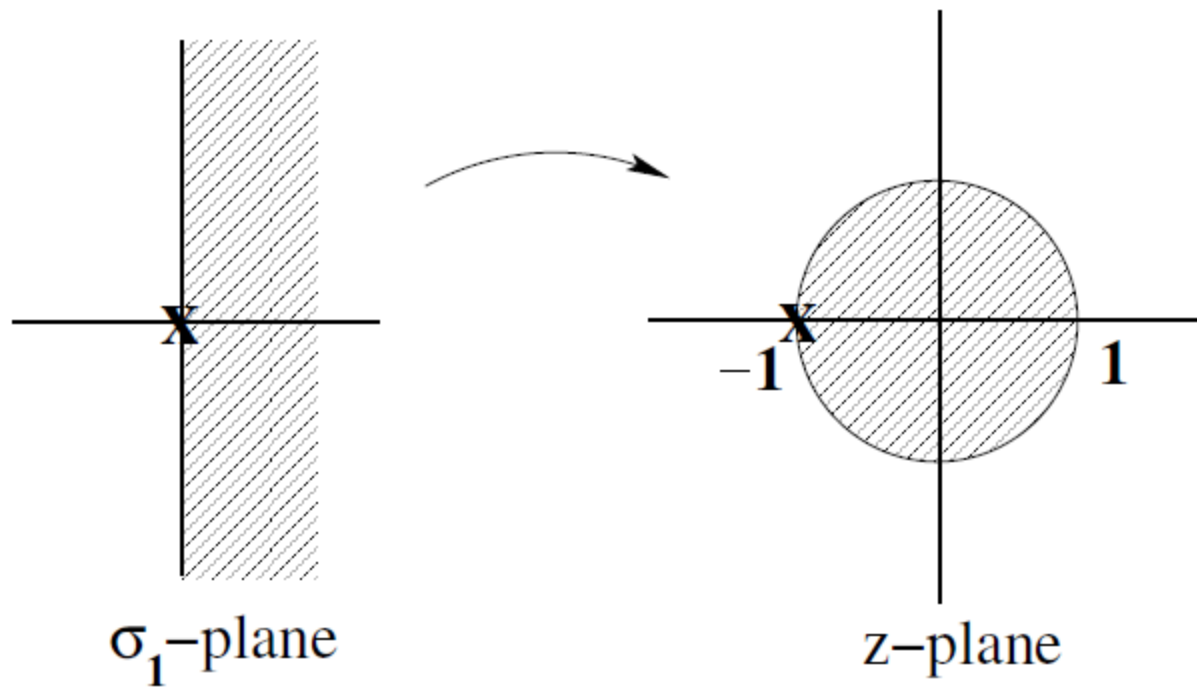
Numerical scheme of Eyre and Milton (1999)

$$\sigma_*/\sqrt{\sigma_1} = 1 + \sum_{n=1}^{\infty} b_n \left(\frac{\sqrt{\sigma_1} - 1}{\sqrt{\sigma_1} + 1} \right)^n.$$
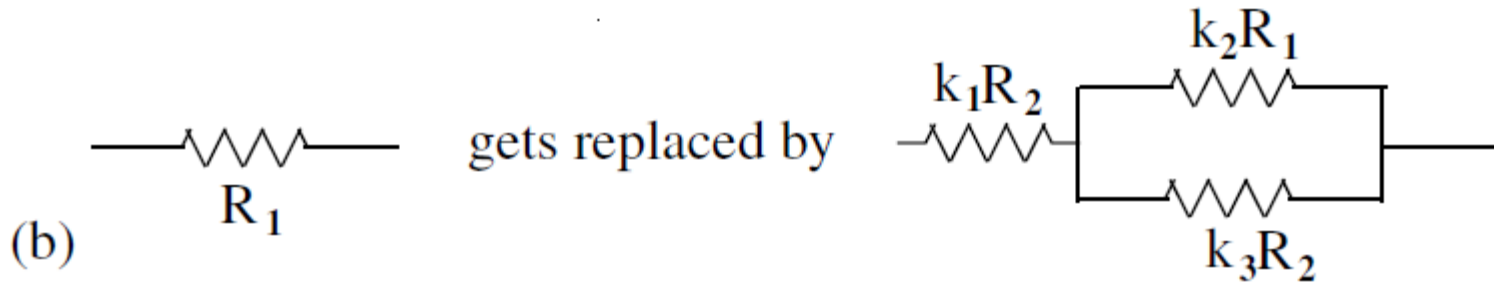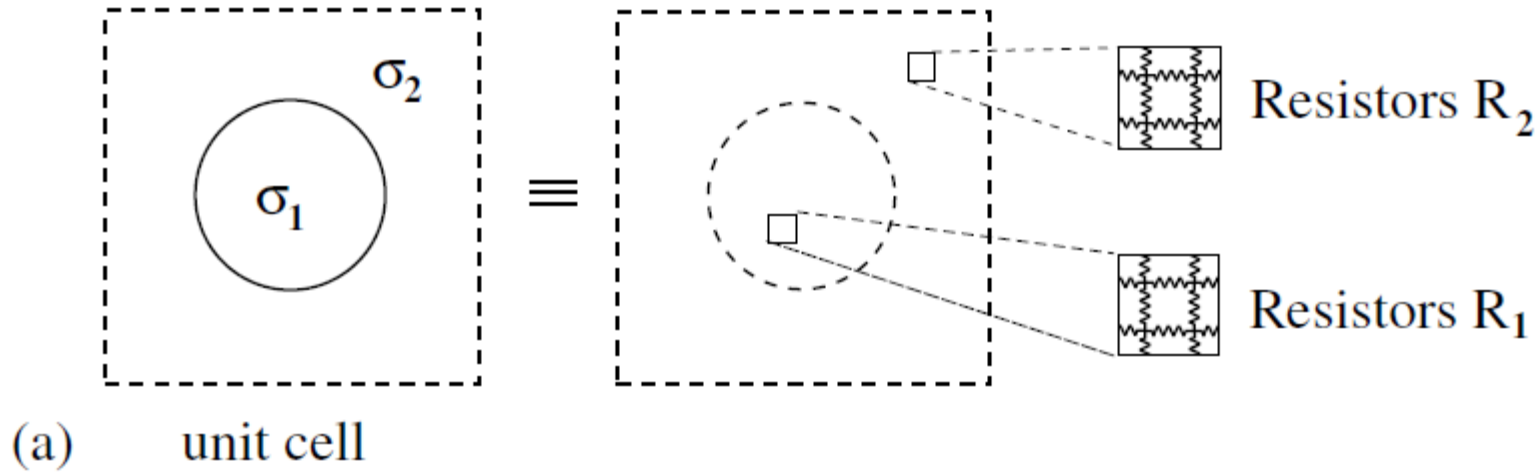

σ_1–plane

w–plane

z–plane

Ideal scheme:

But we want to do this transformation at the level of the subspace collection, to recover the fields

At a discrete level

(a)   unit cell

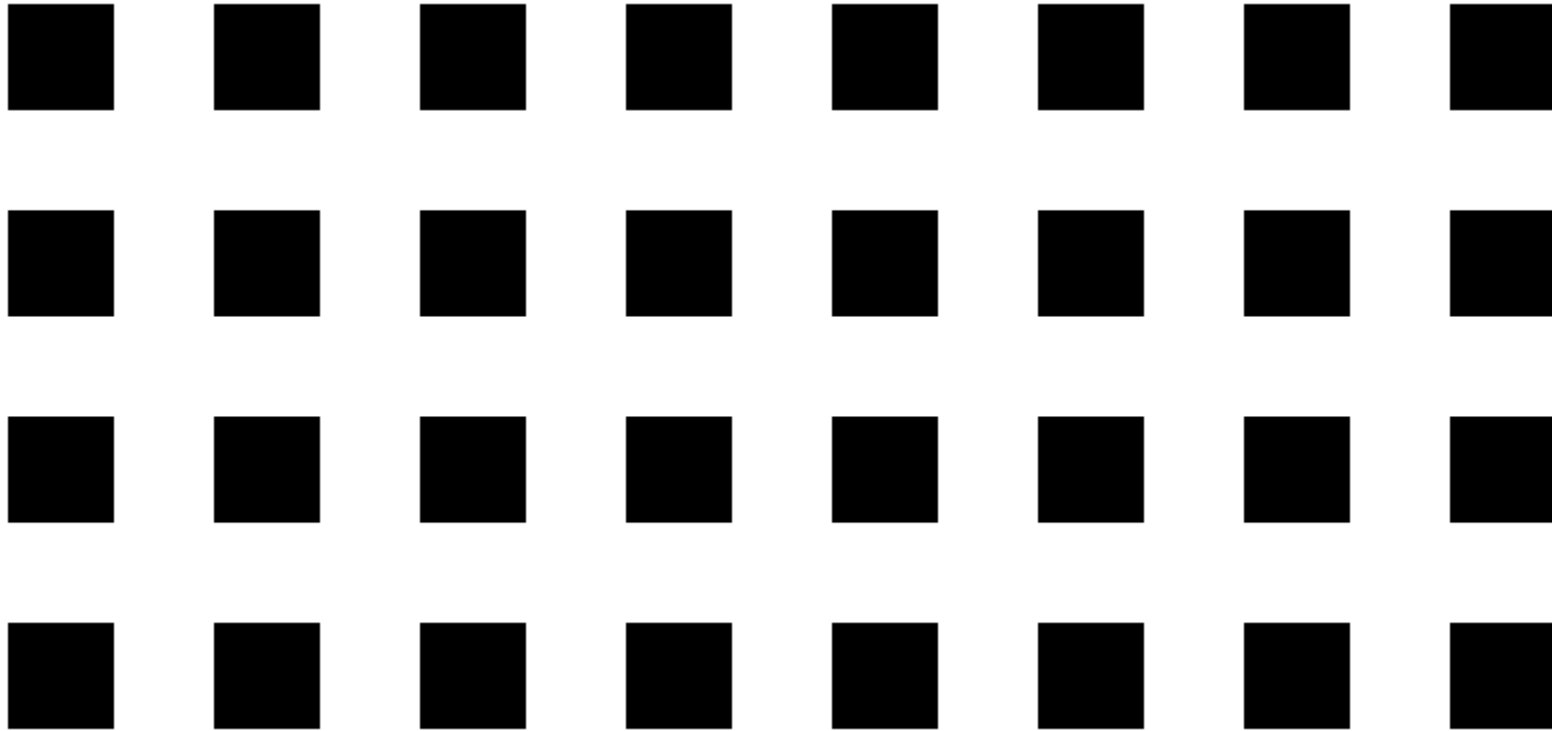(b) R_1 gets replaced by

Problem: this substitution shortens the branch cut instead of lengthening it.
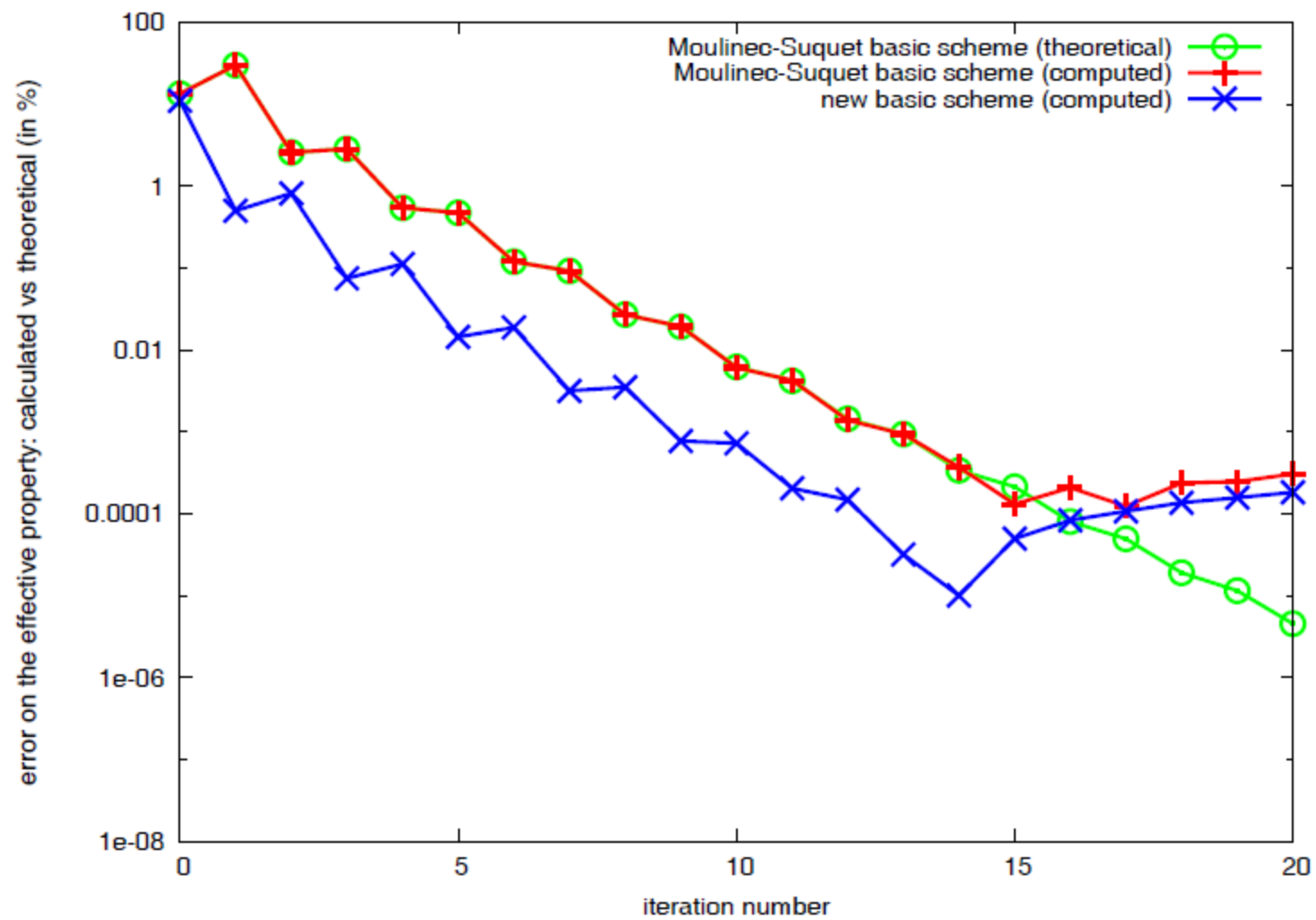
Solution:

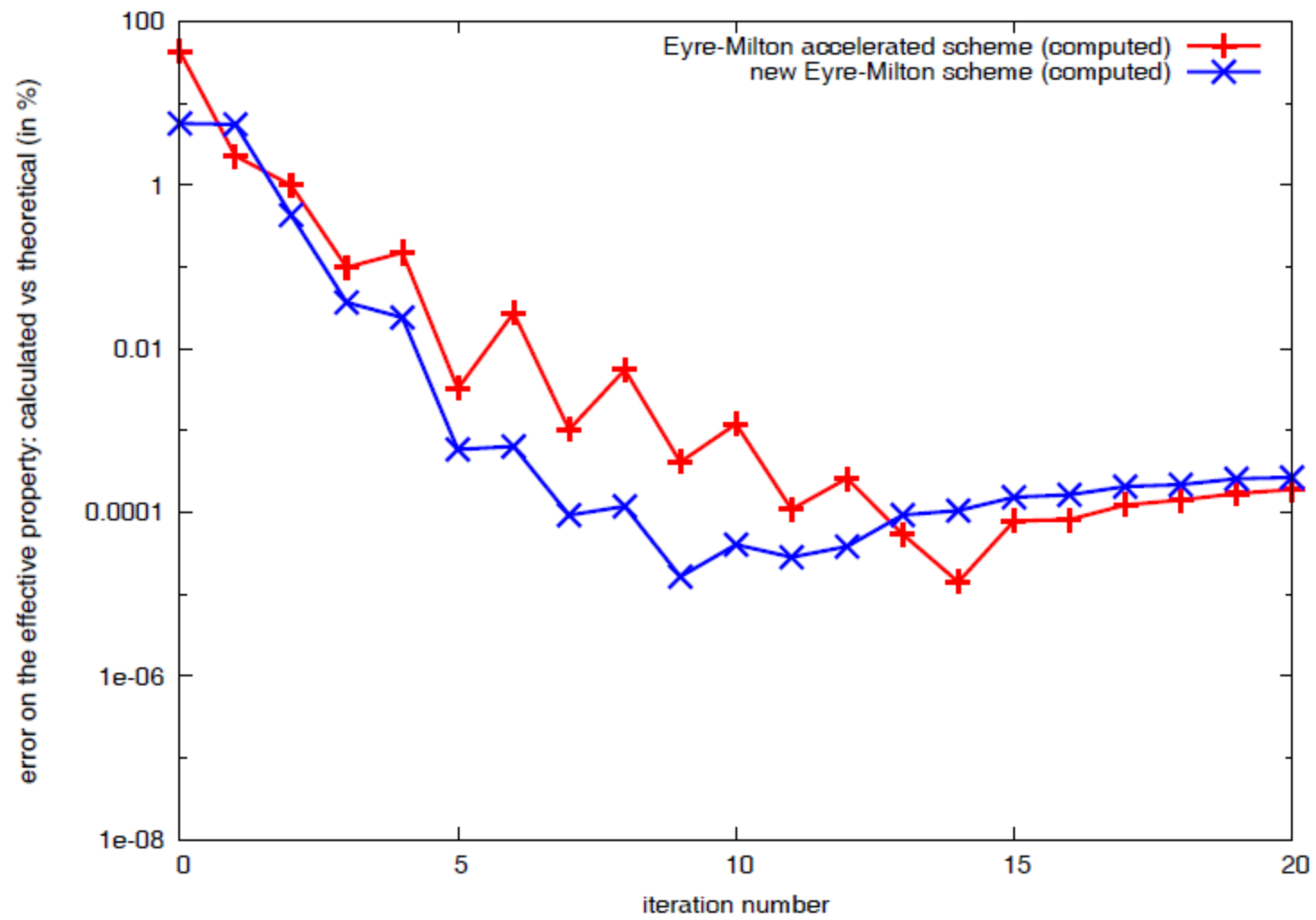Substitute non-orthogonal subspace collections

Model example: a square array of squares at 25% volume fraction

Obnosov's exact formula

$$\sigma_* = \sqrt{(1 + 3\sigma_1)/(3 + \sigma_1)},$$

This last summer was what I call a **super-creative period** of my life.

This summer is again a super-creative period of my life. Perhaps more than last summer.

Another super-creative period was when I was an undergraduate doing an Honors Thesis with Ross McPhedran at Sydney University.
Then, and in the months that followed,
I published 10 papers, some alone, some with coauthors.

One especially significant paper was my very first paper (web-search: Bergman-Milton bounds).

Bounds on the complex dielectric constant of a composite material

I was the sole author. I submitted it to Nature, I think in January 1980. The work was related to work that David Bergman, Tel-Aviv had done, so my advisor Ross McPhedran sent David Bergman a copy of my paper and my honors thesis, now available on research gate:

(https://www.researchgate.net/publication/275657229_Report_TP-79-1)

I had found bounds on the complex dielectric constant. My constructions were different in two and three dimensions, because in two-dimensions I used the Keller-Dykhne duality formula.

David Bergman was also working on this problem independently at Schlumberger (I am confident that is true), but he did not use The Keller-Dykhne duality formula.

David Bergman submitted his research to Physical Review Letters in early March 1980.
My paper was rejected from Nature, resubmitted to Appl.Phys.Lett.

Curiously the first version of his paper said the 3d bounds for isotropic media were obtained by doubly coated sphere assemblages, but did not mention my result that the 2d bounds were bounds were obtained by doubly coated cylinder assemblages.

Then he found he was wrong, revised the paper, and claimed in the published version that my assertion that the 2d bounds were obtained by doubly coated cylinders was wrong, and that is not true.

Later he even published a proof of my 2d-bounds, I guess hoping to get some credit for what I had already done.

I have often wondered: how many other ideas in my first paper had found their way into Bergman's paper without proper attribution?

This is why I cannot tolerate bad scientific practices.

There is more to the saga. In January 1981 I headed to Cornell to do a Ph.D in Physics. What followed en-route in San-Francisco is just such an unbelievable story, I myself have trouble believing it, but it is true.

Some of the players in this story:

My father, John (who dropped everything to "rescue me").
Albert J. Sievers (Nichols Professor of Physics at Cornell).
Evelyn Einstein (granddaughter of Albert Einstein).
Laura Greene (now President Elect of the APS).
Zack Schlesinger, and his parents Kurt and Barbara.
Hans Bethe (Nobel Prize Winner, Cornell, helped develop the atomic and hydrogen bombs, later rallied for peace).

It took just six weeks out of my life, but it took 4 months to recover.
I have told some people but find it difficult as memories resurface.
BUT AGAIN IT IS A STORY THAT MUST BE TOLD, soon.

Epilogue: the story with David Bergman continued:
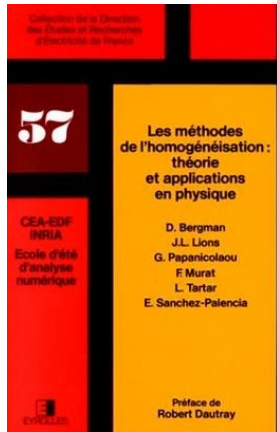
His 128 page article in this 1985 book did not contain even one reference to my work.

His paper **Hierarchies of Stieltjes Functions and Their Application to the Calculation of Bounds for the Dielectric Constant of a Two-Component Composite Medium** (1993)

duplicates, but does not reference, many of the results in

Bounds on the transport and optical properties of a two-component composite material

G. W. Milton
Department of Theoretical Physics, The University of Sydney, Sydney NSW 2006, Australia

even though it did reference the companion paper. In a wider context many of these bounds are known from Nevanlinna-Pick interpolation.

* This slide was inserted after the lecture

Thank you!

Thank you!

Thank you!

Thank you!

Thank you!