Understanding metamaterials through discrete networks

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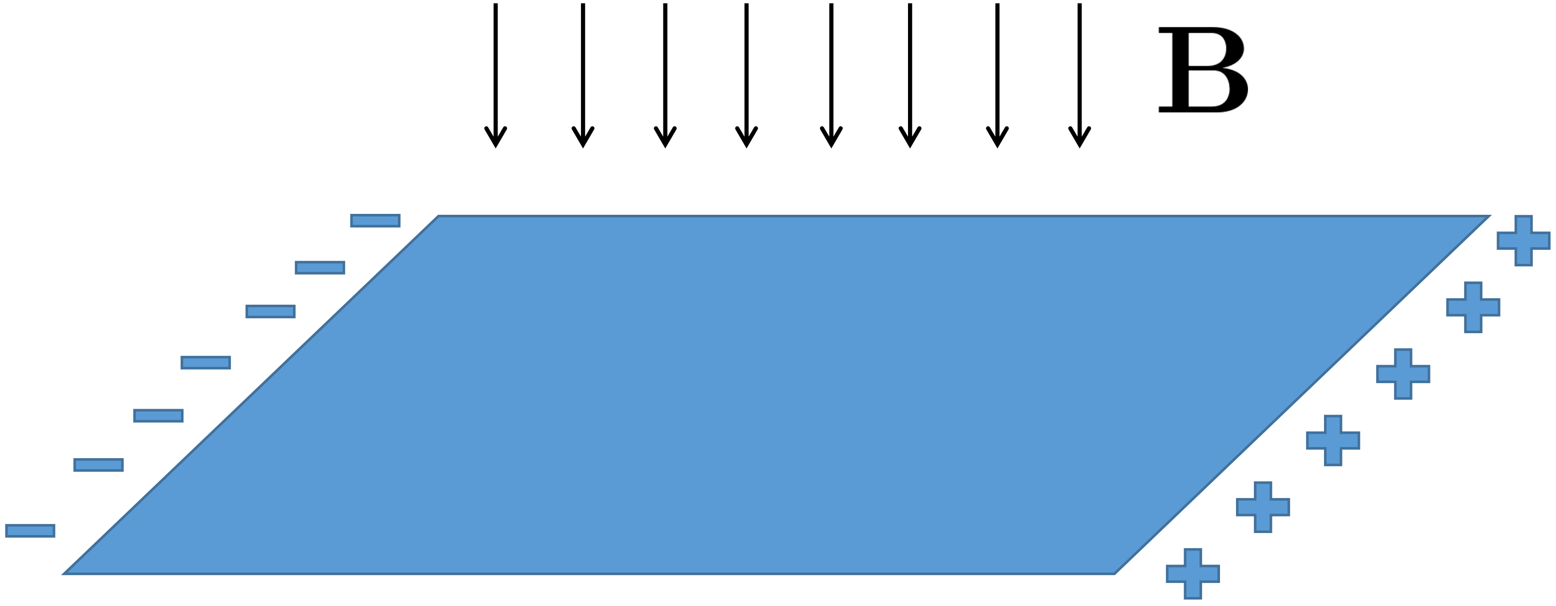
Outline:

- Introduction
- Models for negative, anisotropic, and complex densities
- Models for the Willis equations
- Unimode, and Bimode Affine Materials
- Complete Charaterization of the linear dynamics of mass-spring networks
- Field Patterns: a new type of wave

It's constantly a surprise to find what properties a composite can exhibit.

One interesting example:



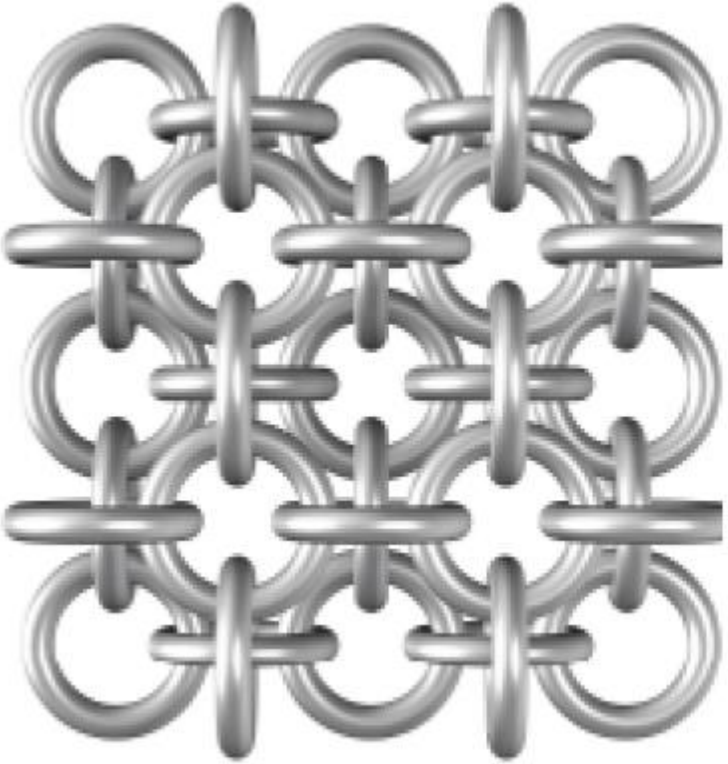
Hall Voltage

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

In elementary physics textbooks one is told that in classical physics the sign of the Hall coefficient tells one the sign of the charge carrier.

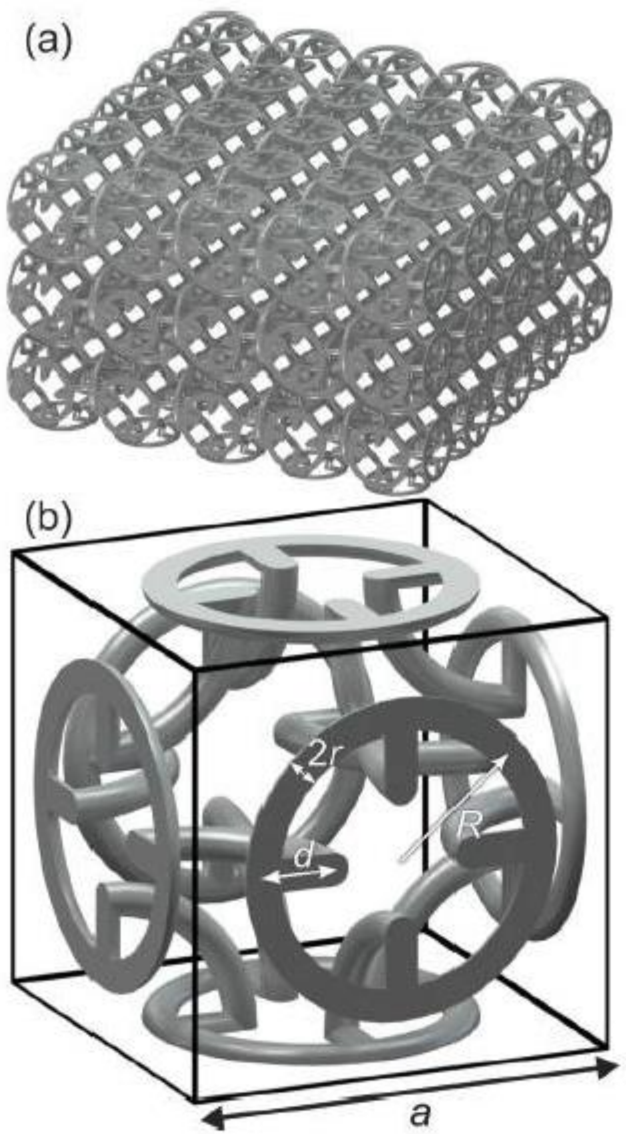
However there is a counterexample!

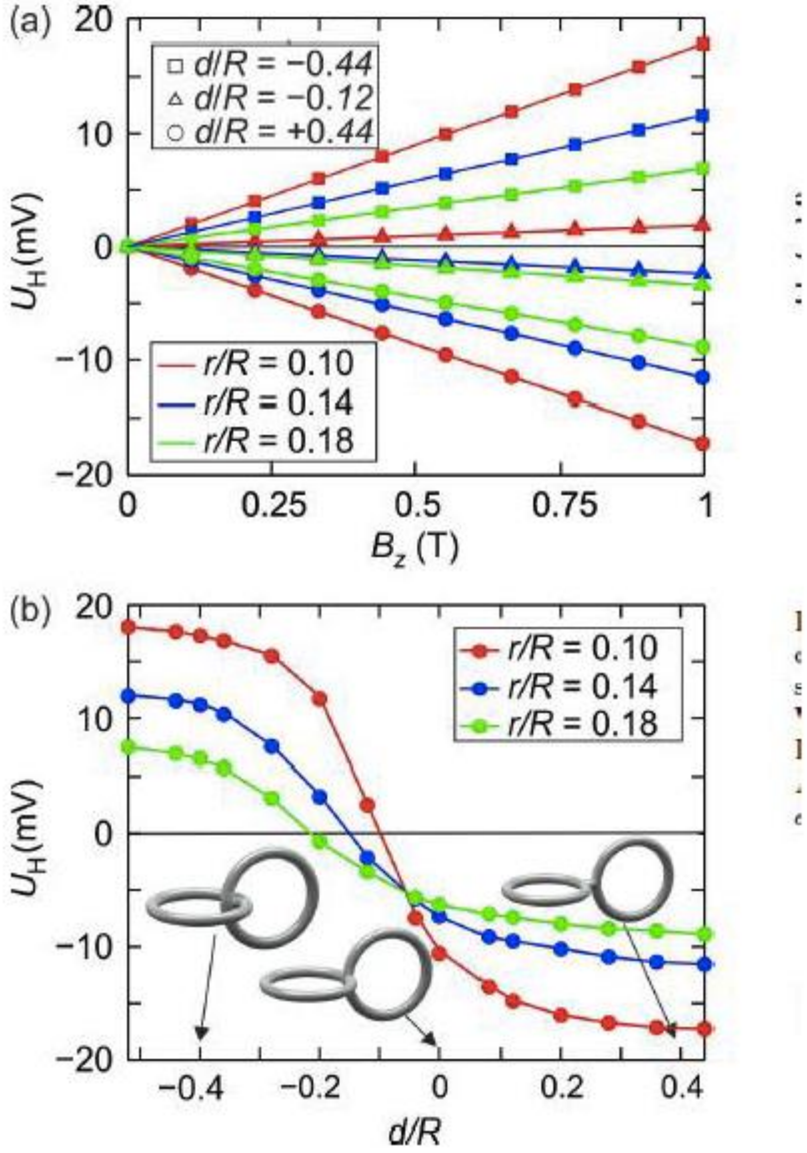
Geometry suggested by artist Dylon Whyte

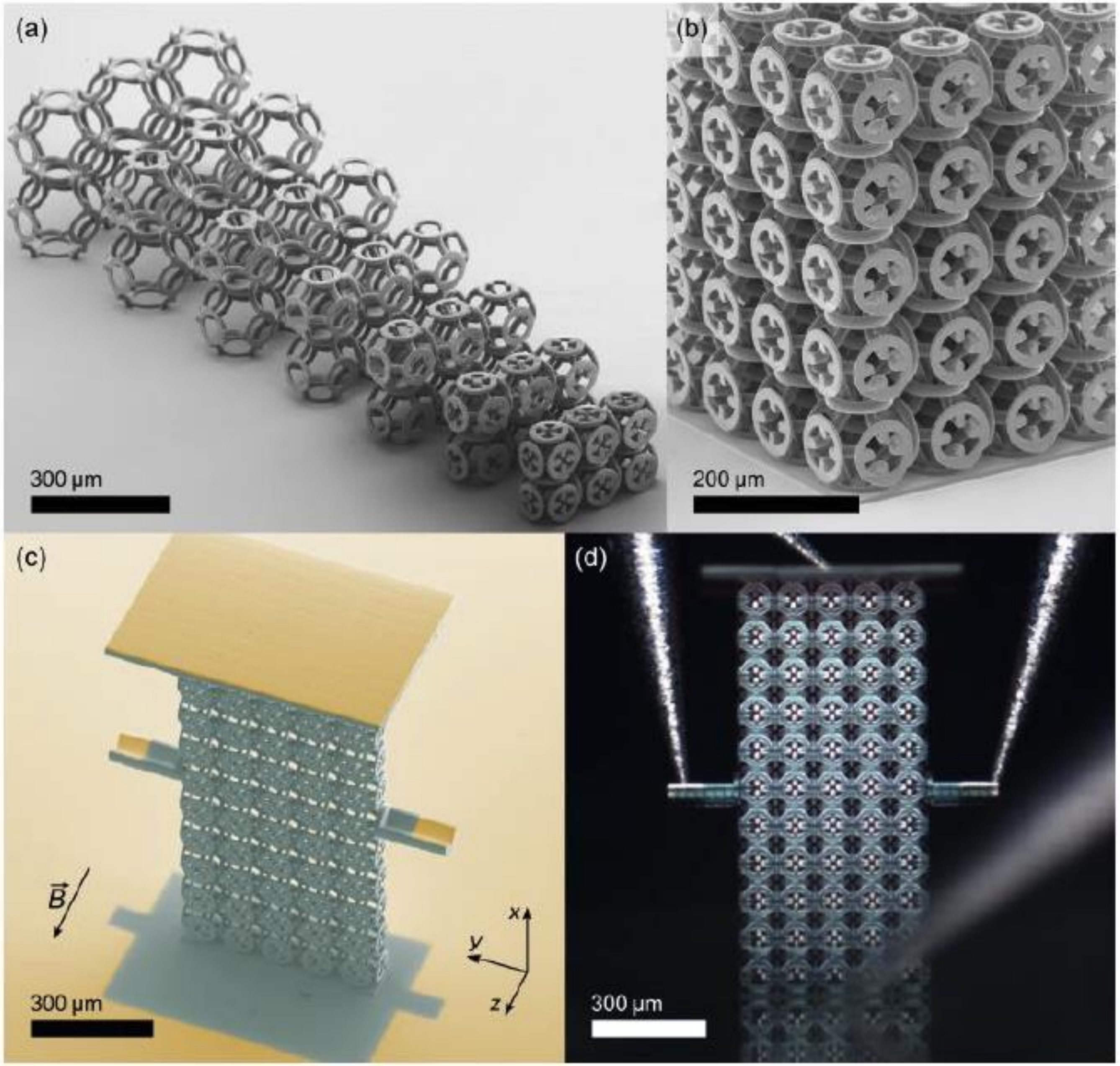


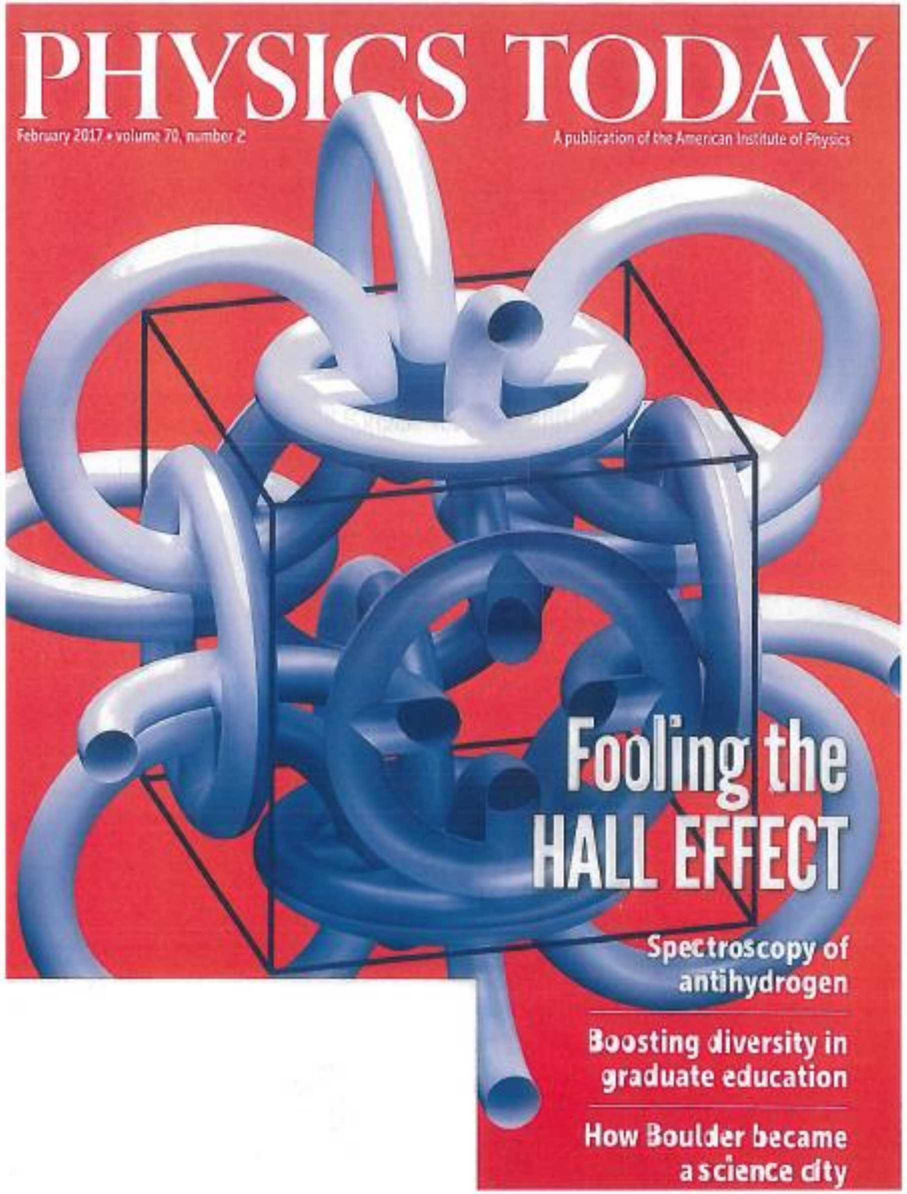
A material with cubic symmetry having a Hall Coefficient opposite to that of the constituents (with Marc Briane)

Simplification of Kadic et.al. (2015)

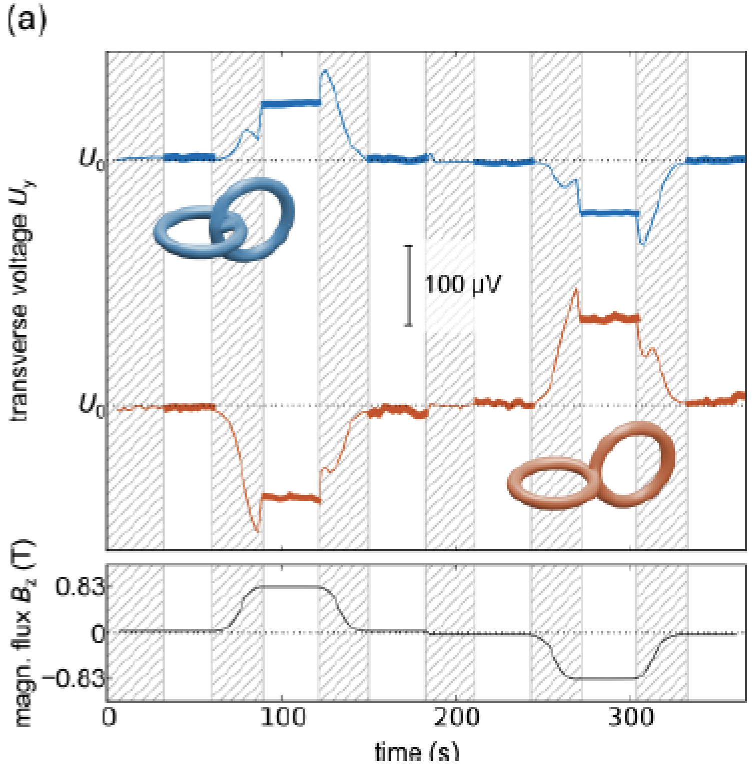


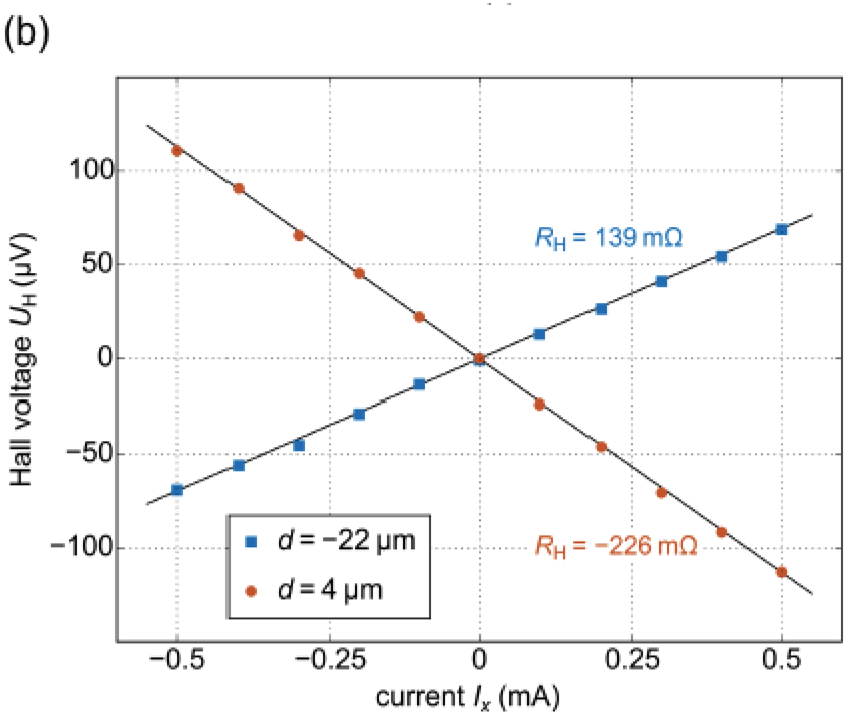


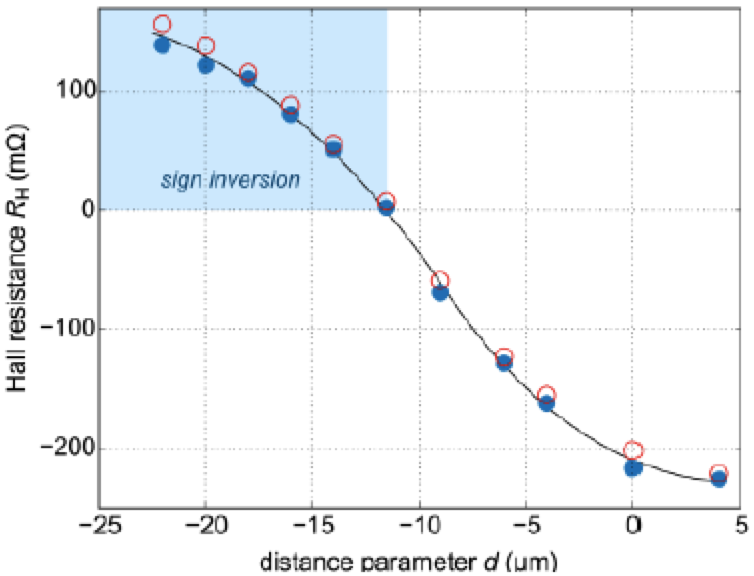




Experimental Realization of Kern, Kadic, Wegener

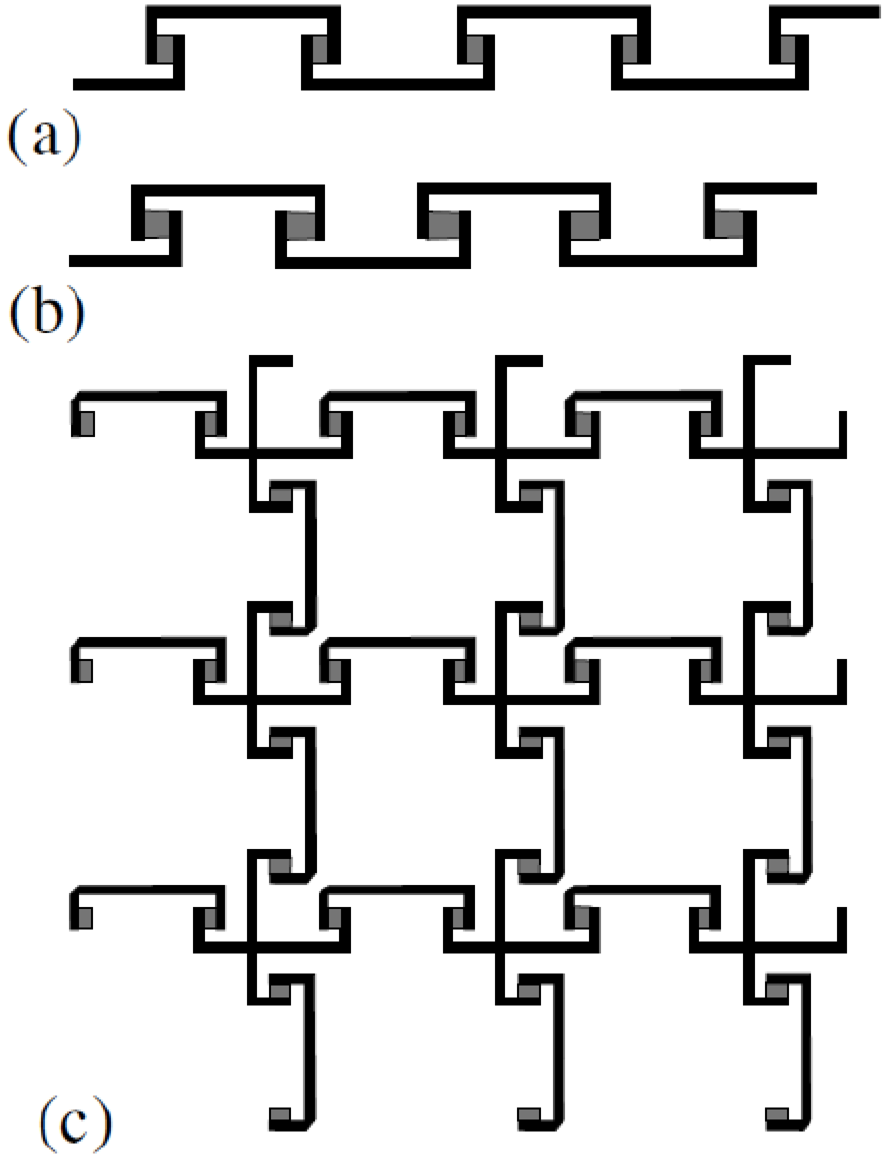






Their experimental results confirming Hall-effect reversal

Another example: negative expansion from positive expansion



Original designs: Lakes (1996); Sigmund & Torquato (1996, 1997)

An important parallel:

Maxwell's Equations:

$$\frac{\partial}{\partial x_i}\left(C_{ijk\ell}\frac{\partial E_\ell}{\partial x_k}\right) = \{\omega^2 \boldsymbol{\varepsilon}\mathbf{E}\}_j$$

$$C_{ijk\ell} = e_{ijm}e_{k\ell n}\{\boldsymbol{\mu}^{-1}\}_{mn}$$

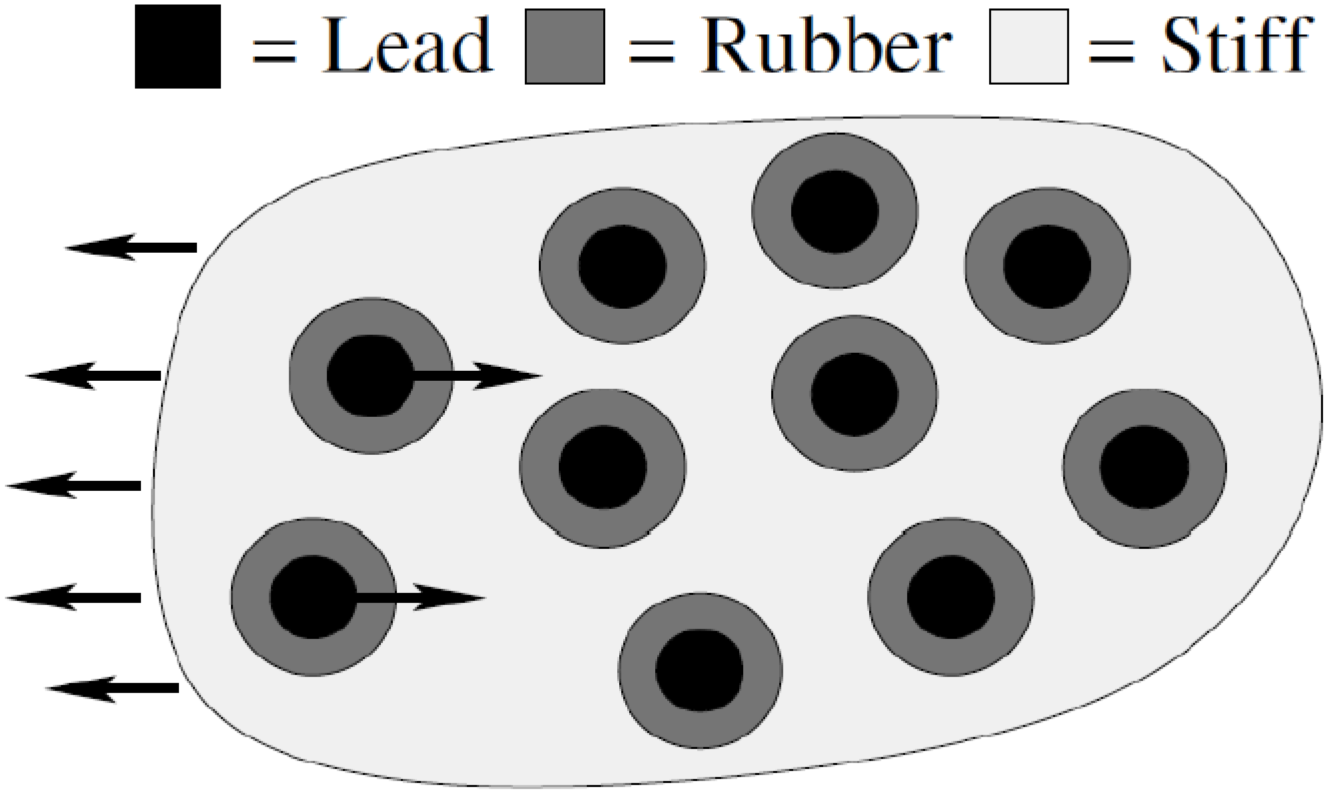
Continuum Elastodynamics:

$$\frac{\partial}{\partial x_i}\left(C_{ijk\ell}\frac{\partial u_\ell}{\partial x_k}\right) = -\{\omega^2 \boldsymbol{\rho}\mathbf{u}\}_j$$

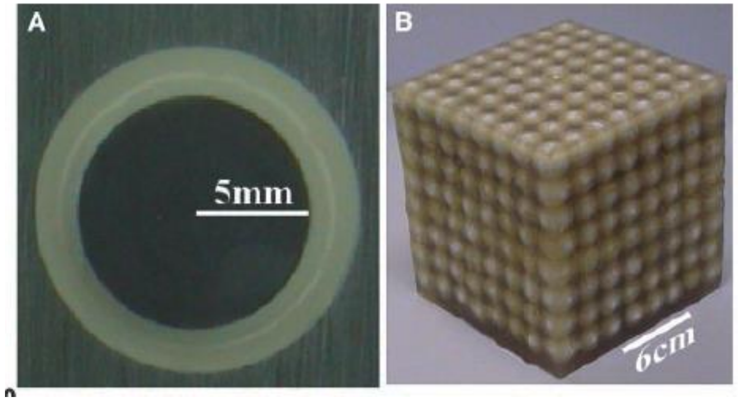
Suggests that $\varepsilon(\omega)$ and $\boldsymbol{\rho}(\omega)$ might have similar properties

Specifically a similar dependence on frequency

Sheng, Zhang, Liu, and Chan (2003) found that materials could exhibit a negative effective density over a range of frequencies

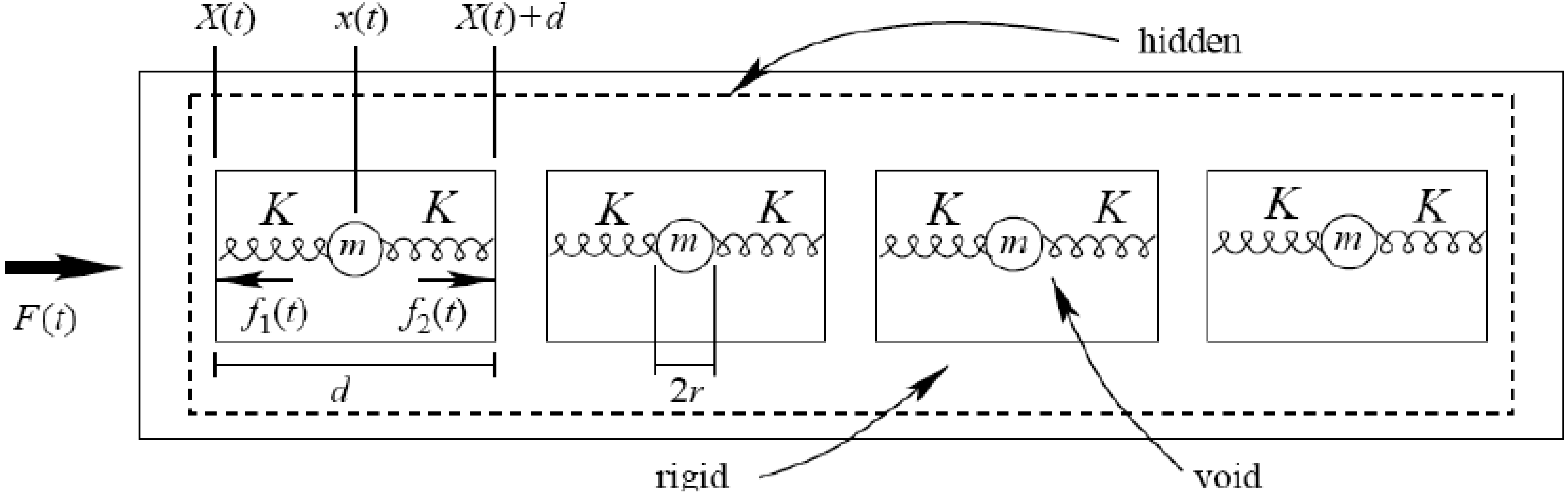


Experiment: Liu et. al (2000)



Mathematically the observation goes back to Zhikov (2000) also Bouchitte & Felbacq (2004)

A simplified one-dimensional model:



$$\hat{P} = M\hat{V}, \quad \text{with} \quad M = M_0 + \frac{2Knm}{2K - m\omega^2},$$

(With John Willis)

Early work recognizing anisotropic and negative densities

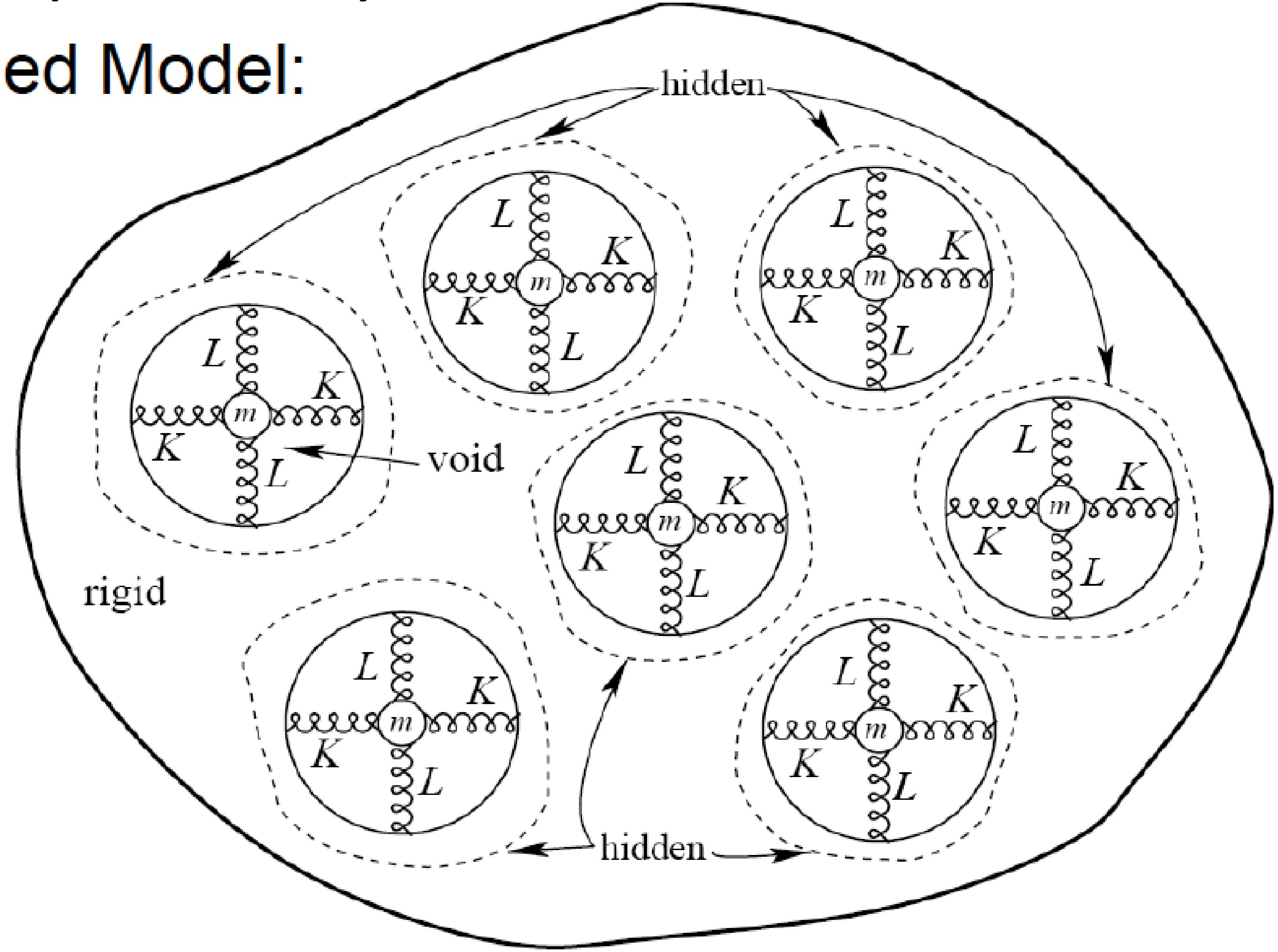
Aurialt and Bonnet (1994, 1995)

“The monochromatic macroscopic behavior is elastic, but with an effective density ϱ^{eff} of tensorial character and depending on the pulsation”

"hatched areas correspond to negative densities ϱ^{eff},
i.e., to stopping bands."

Anisotropic Density

Simplified Model:

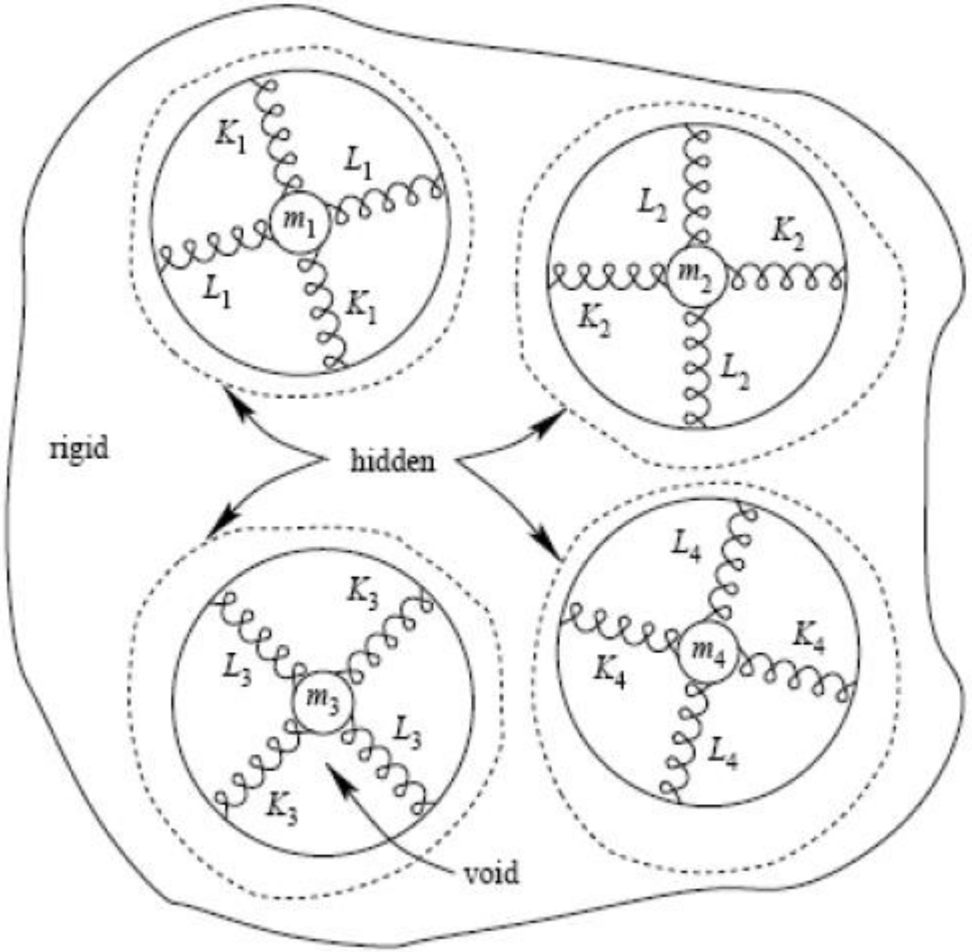


Anisotropic density in layered materials: Schoenberg and Sen (1983)

The springs could have some damping in which case the mass will be complex

(With John Willis)

Seemingly rigid body



Eigenvectors of the effective mass density can rotate with frequency

(With John Willis)

What do we learn?
For materials with microstructure, Newton's law

$$F = ma$$

needs to be replaced by

$$F(t) = \int_{-\infty}^{t} K(t' - t)a(t')\, dt'$$

It takes some time for the internal masses to respond to the macroscopically applied force.

(With John Willis)

Models for the Willis equations

$$\begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathcal{C} & \mathbf{S} \\ \mathbf{D} & \boldsymbol{\rho} \end{pmatrix} \begin{pmatrix} \nabla \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

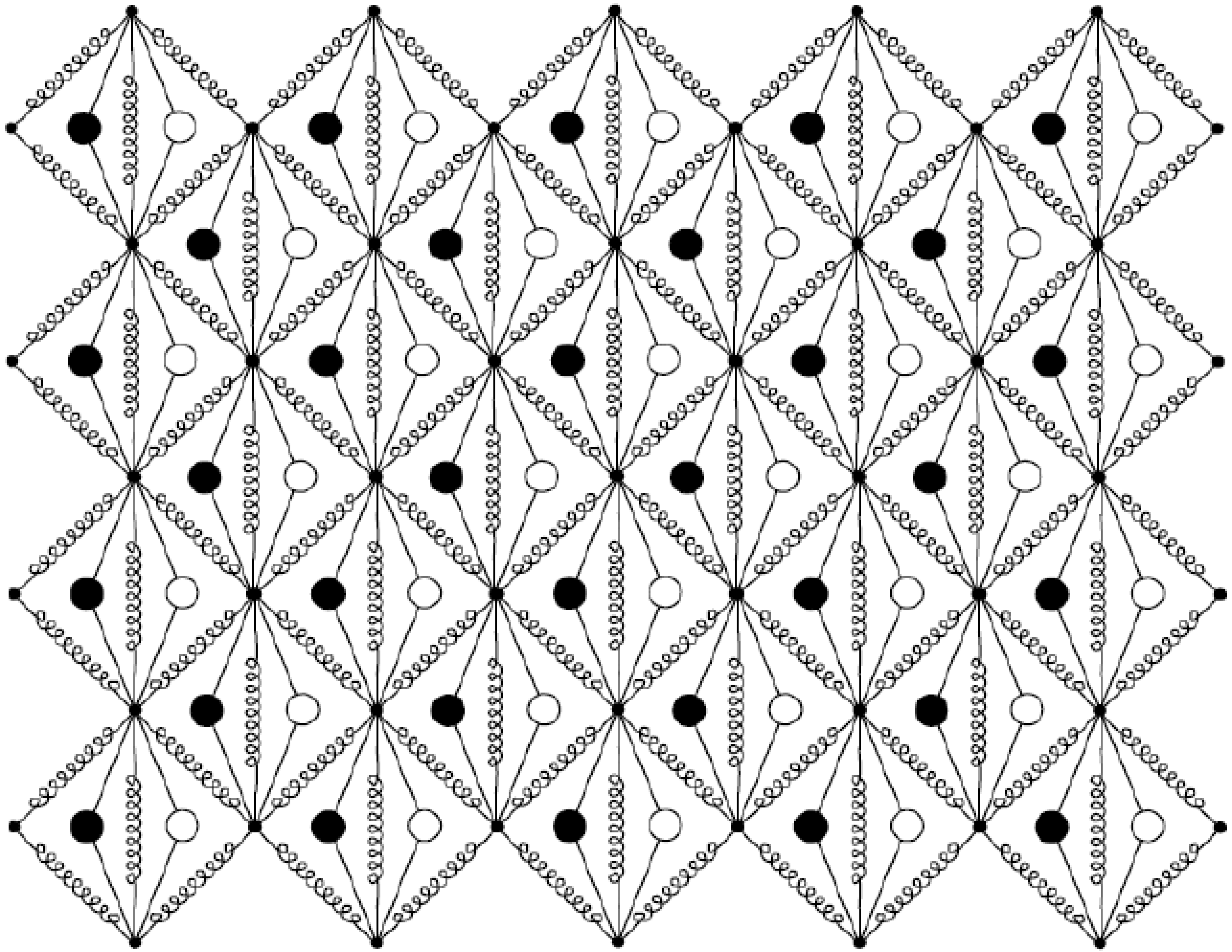
σ –Stress

$\mathbf{p}$–Momentum

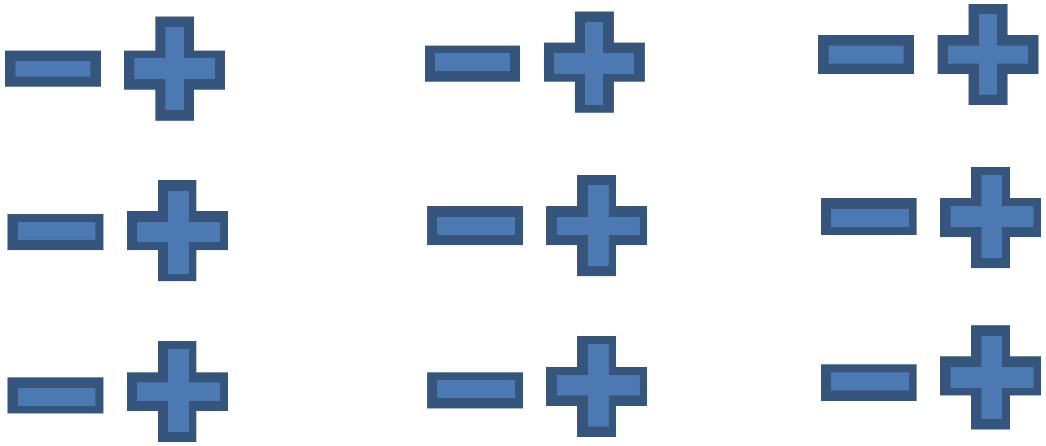
$\mathbf{u}$–Displacement

$\mathbf{v}$–Velocity

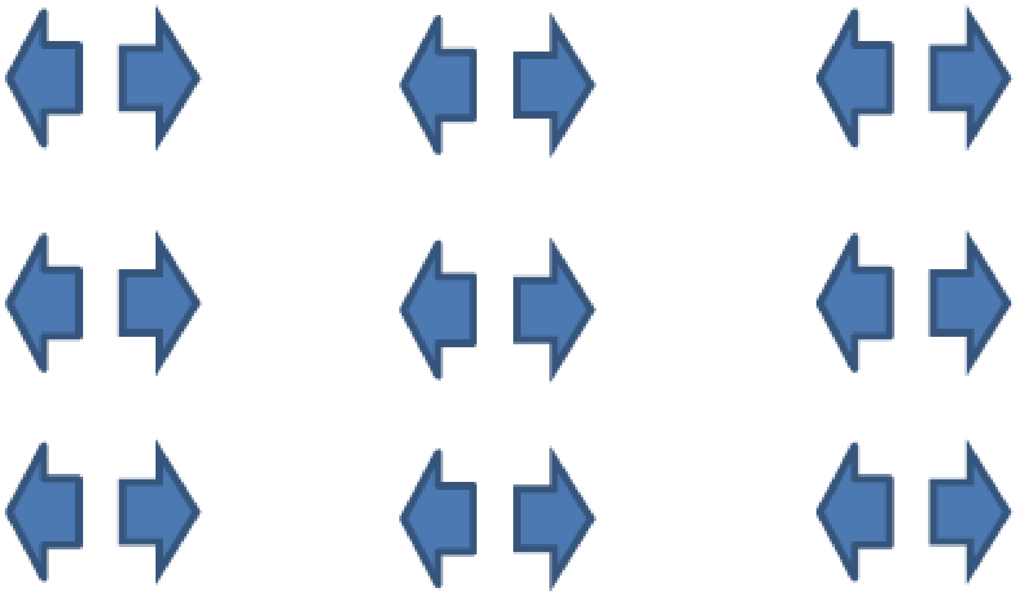
Analog of the bianisotropic equations of electromagnetism



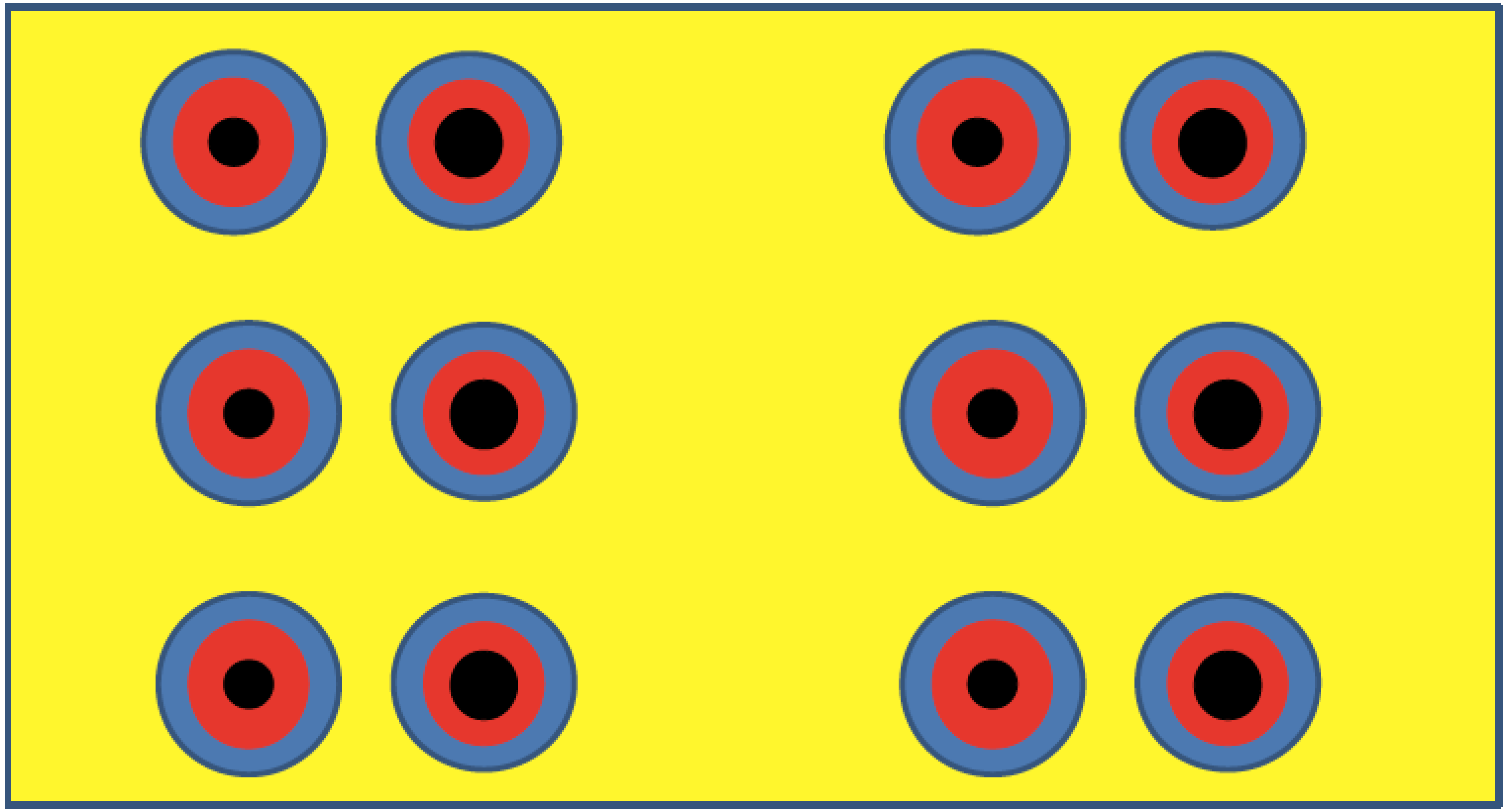
The Black circles have positive effective mass
The White circles have negative effective mass



Electric dipole array
generates
polarization field



Force dipole array
generates
stress field



Yellow=Compliant, Blue=Stiff
Red=Rubber, Black=Lead

Time harmonic acceleration with no strain gives stress: Example of a Willis material

Linear elastic equations under a Galilean transformation

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathcal{C}(\mathbf{x}) & 0 \\ 0 & \rho(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} -\frac{1}{2}\left[\nabla \mathbf{v} + \nabla \mathbf{v}^T\right] \\ \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix} \qquad x_4 = -t,$$

$$\overline{\nabla} = \begin{pmatrix} \nabla \\ \frac{\partial}{\partial x_4} \end{pmatrix} = \begin{pmatrix} \nabla \\ -\frac{\partial}{\partial t} \end{pmatrix}, \quad J_{ik} = -\frac{\partial \sigma_{ik}}{\partial t}, \quad \text{for } i,k = 1,2,3, \quad J_{4k} = -\{\nabla \cdot \boldsymbol{\sigma}\}_k,$$

$$\overline{\nabla} \cdot \mathbf{J} = 0, \quad \mathbf{J} = \mathbf{Z}\overline{\nabla}\mathbf{v}.$$ (looks a bit like conductivity)

Galilean transformation: $\overline{\mathbf{x}}' = \mathbf{A}\overline{\mathbf{x}}, \quad \text{with } \mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{w} \\ 0 & 1 \end{pmatrix},$

$$\begin{pmatrix} \frac{\partial \boldsymbol{\sigma}'}{\partial t'} \\ \nabla' \cdot \boldsymbol{\sigma}' \end{pmatrix} = \begin{pmatrix} \mathcal{I} & \mathbf{w}\mathbf{I} \\ 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{w}(\nabla \cdot \boldsymbol{\sigma})^T \\ \nabla \cdot \boldsymbol{\sigma} \end{pmatrix},$$

$$\begin{pmatrix} -\nabla' \mathbf{v}' \\ \frac{\partial \mathbf{v}'}{\partial t'} \end{pmatrix} = \begin{pmatrix} \mathcal{I} & 0 \\ \mathbf{I}\mathbf{w}^T & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} \end{pmatrix} = \begin{pmatrix} -\nabla \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{w}^T \nabla \mathbf{v} \end{pmatrix},$$

$$\mathbf{Z}'(\overline{\mathbf{x}}') = \begin{pmatrix} \mathcal{I} & \mathbf{w}\mathbf{I} \\ 0 & \mathbf{I} \end{pmatrix} \mathbf{Z}(\mathbf{x}) \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{I}\mathbf{w}^T & \mathbf{I} \end{pmatrix}$$

$$= \begin{pmatrix} -\mathcal{C}(\mathbf{x}) + \mathbf{w}\rho(\mathbf{x})\mathbf{w}^T & \mathbf{w}\rho(\mathbf{x}) \\ \rho(\mathbf{x})\mathbf{w}^T & \rho(\mathbf{x}) \end{pmatrix}.$$

Has Willis type couplings!
Also a non-symmetric stress

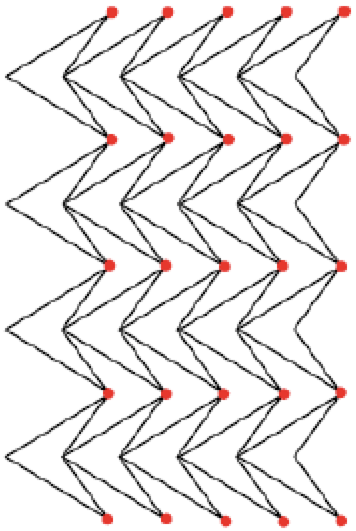
“Transformation Optics” that dates to Dolin (1961)

Also the talk of Guoliang Huang on Monday

Unimode and Bimode Affine Materials

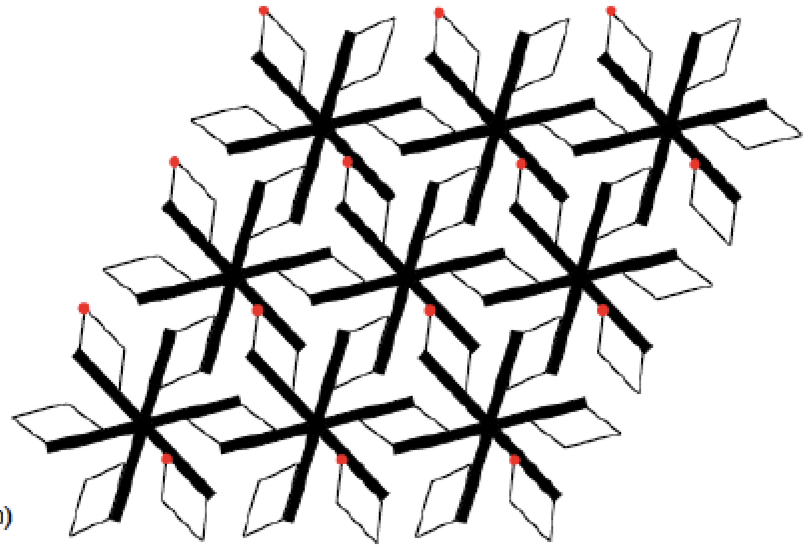
Examples of nonlinear 2d unimode materials

(a)



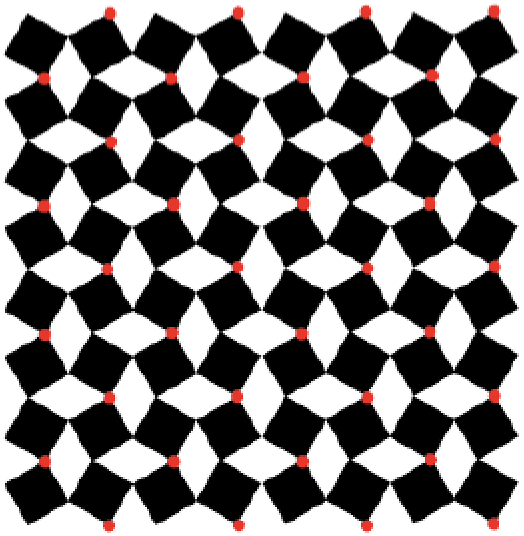
Larsen et. al.

(b)



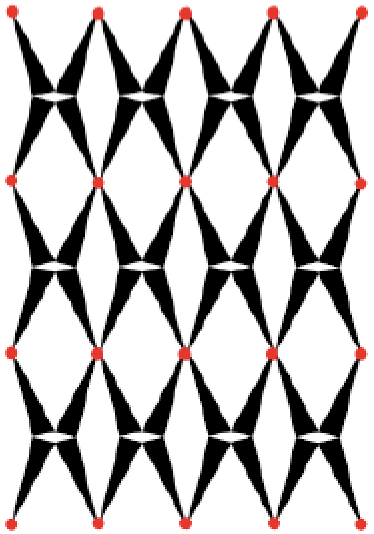
Milton

(c)

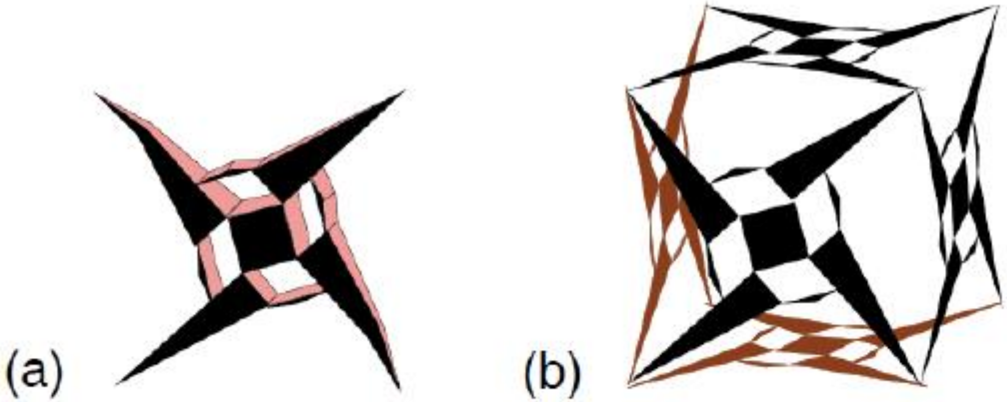


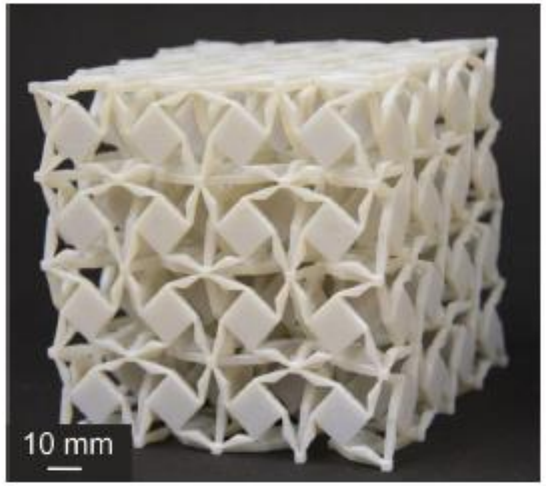
Grima and Evans

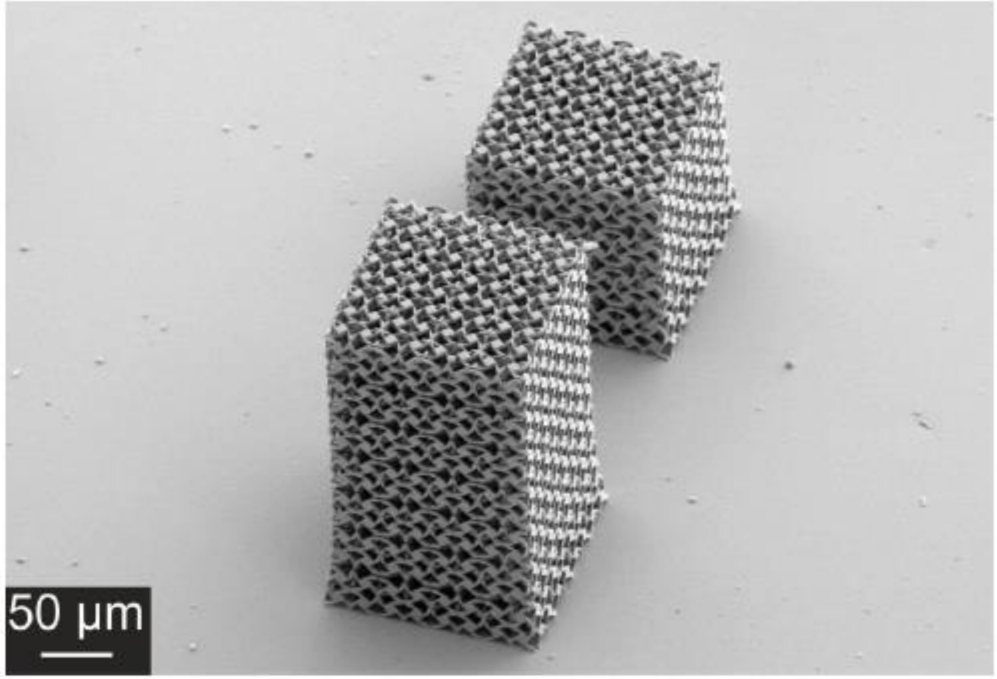
(d)



Three Dimensional Dilational materials

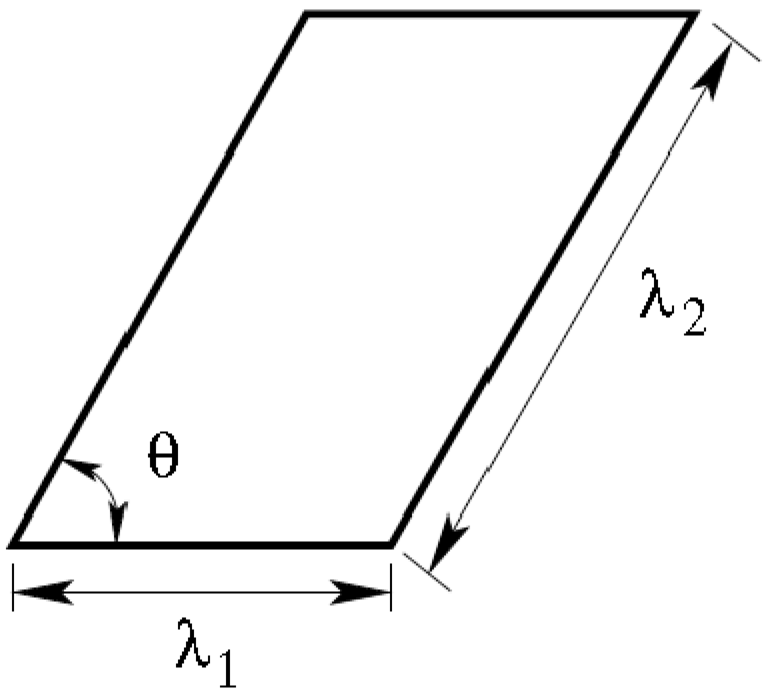


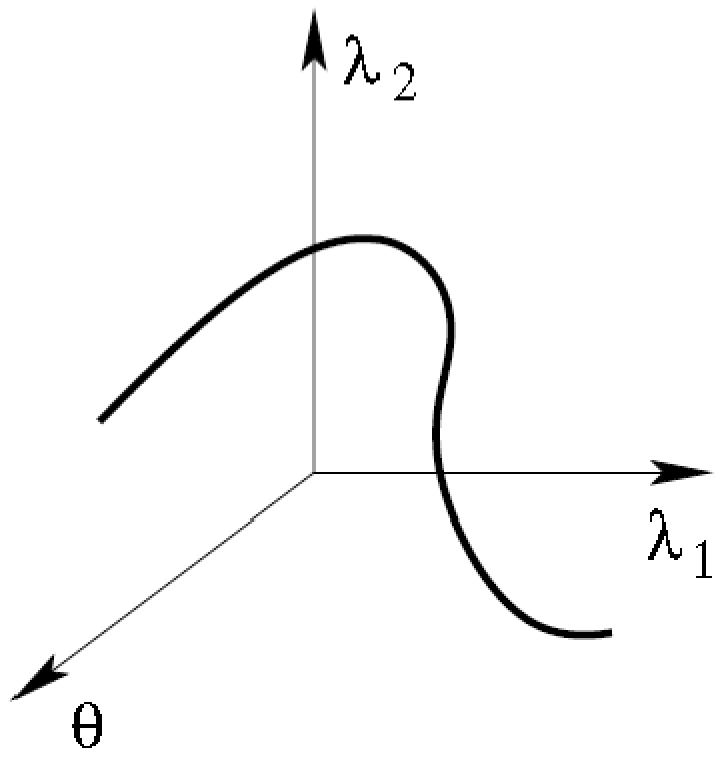




Buckmann,, Schittny, Thiel, Kadic, Milton Wegener (2014)

Unimode:



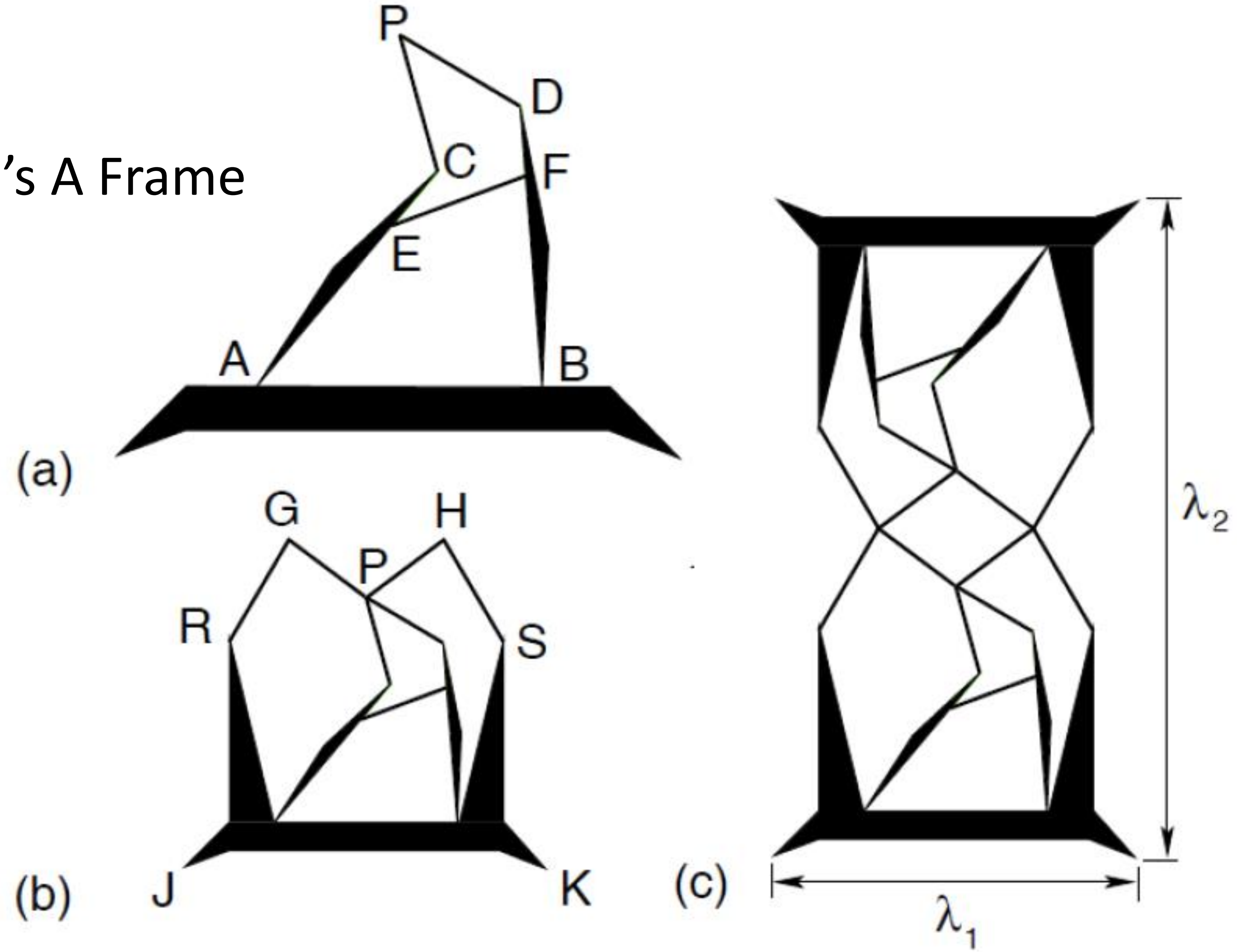


What trajectories $\lambda_1(t) = \lambda_2(t) = \theta(t)$ are realizable? (Answer: any trajectory!)

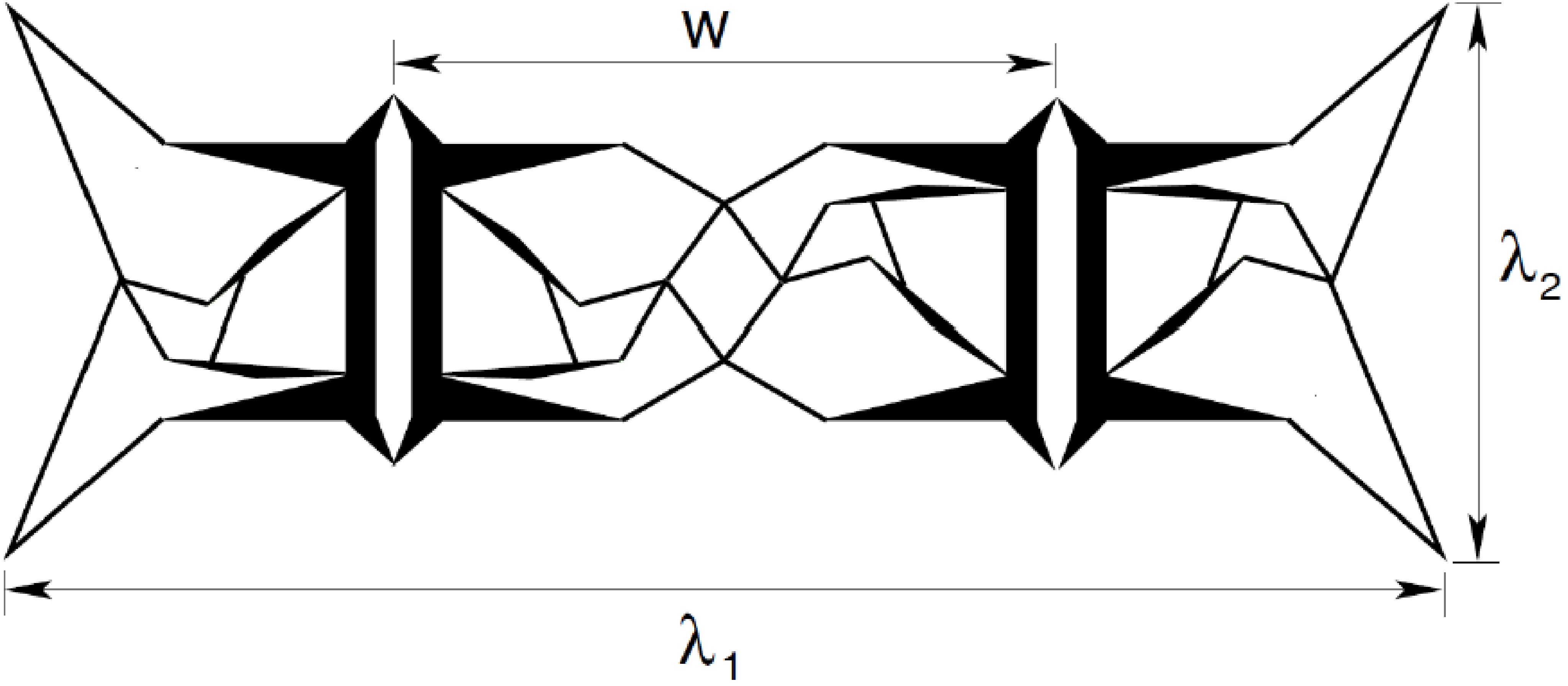
In a bimode material there is a surface of realizable motions.

Cell of the perfect expander: a unimode material

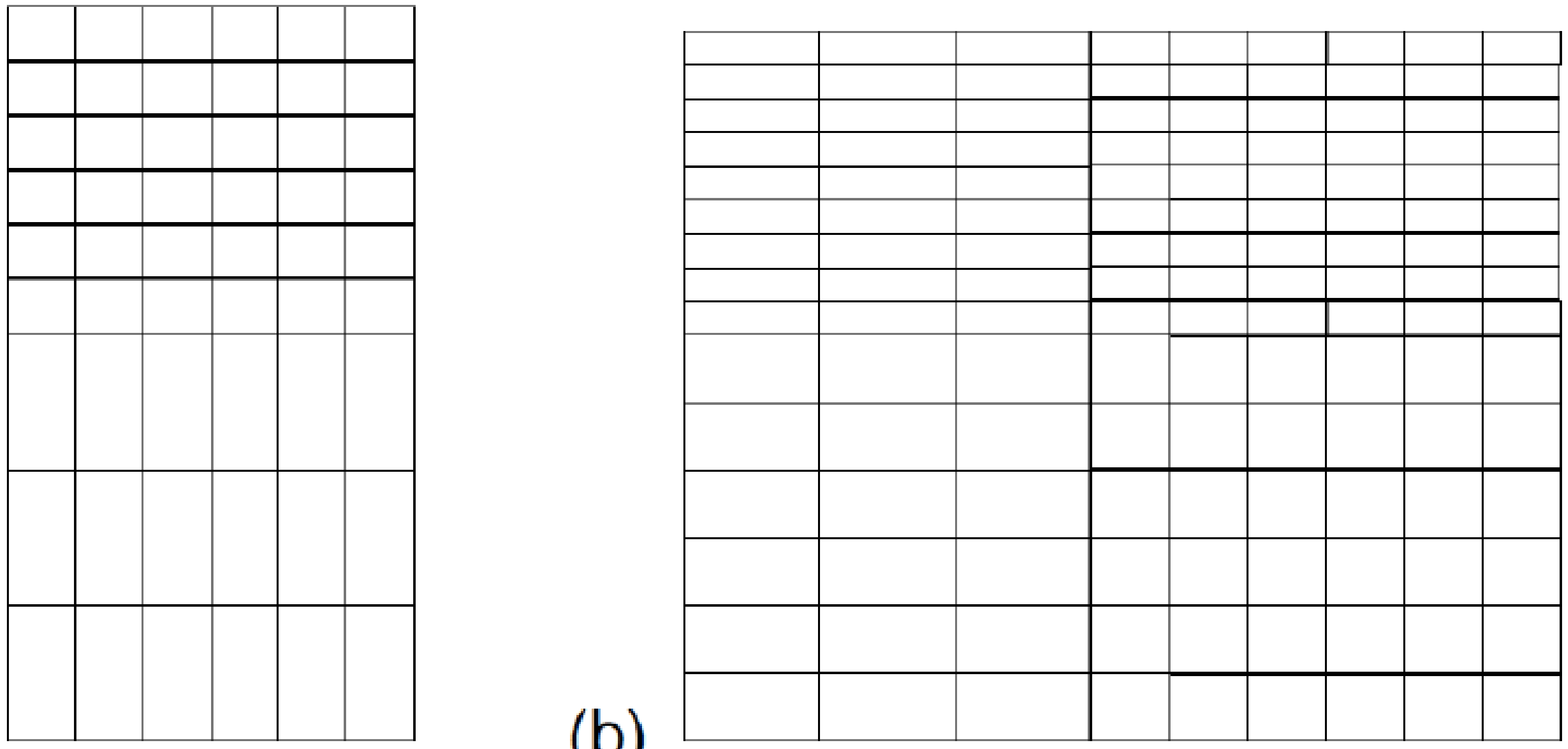
Hart's A Frame



Cell of a bimode material



However neither are affine materials:

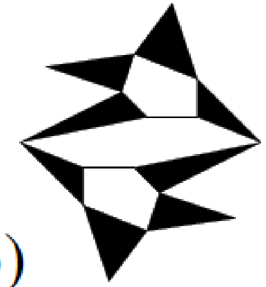


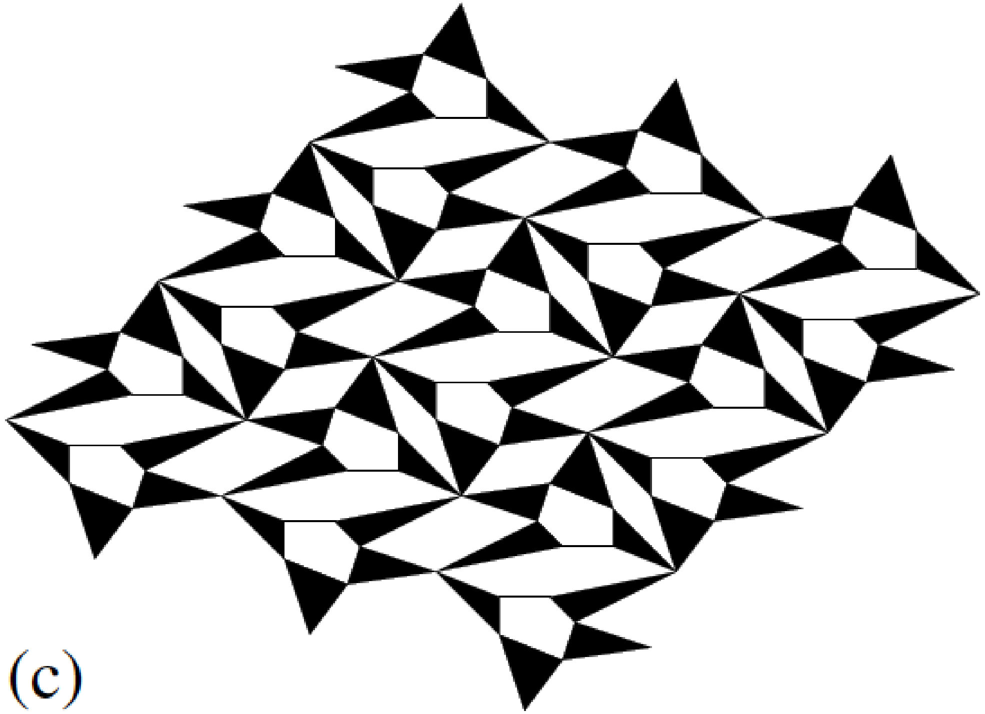
(a) (b)

So can one get affine bimode materials?

Bimode material for which the only easy modes of deformations are affine ones

(a) 

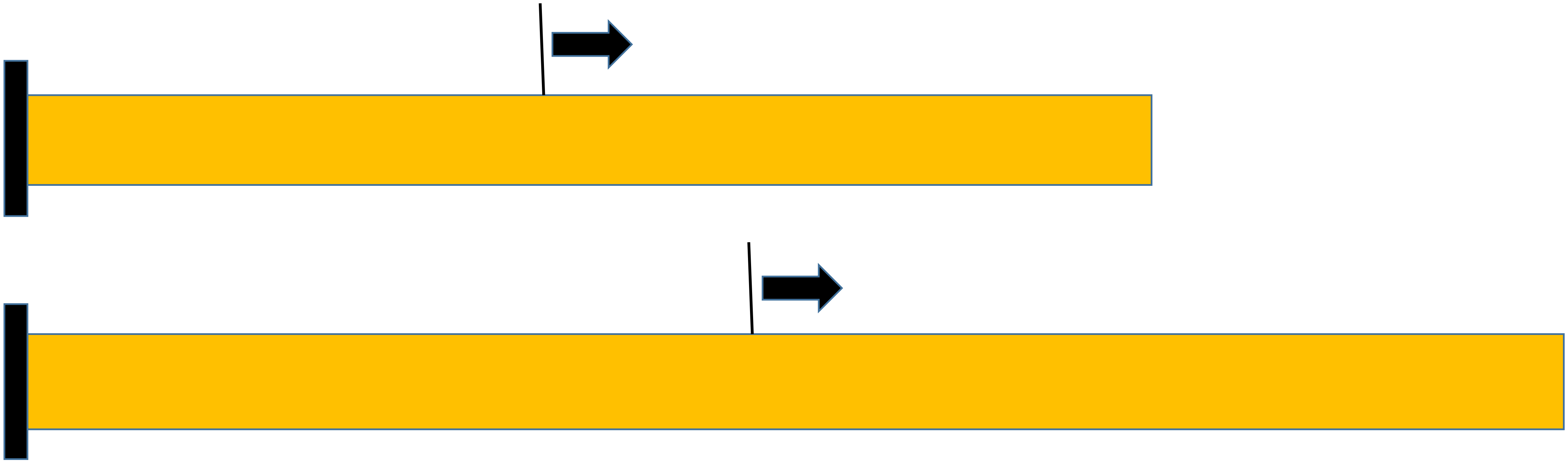
(b) 



(c)

Characteristic Feature of Affine Materials:

They dislike strain gradients

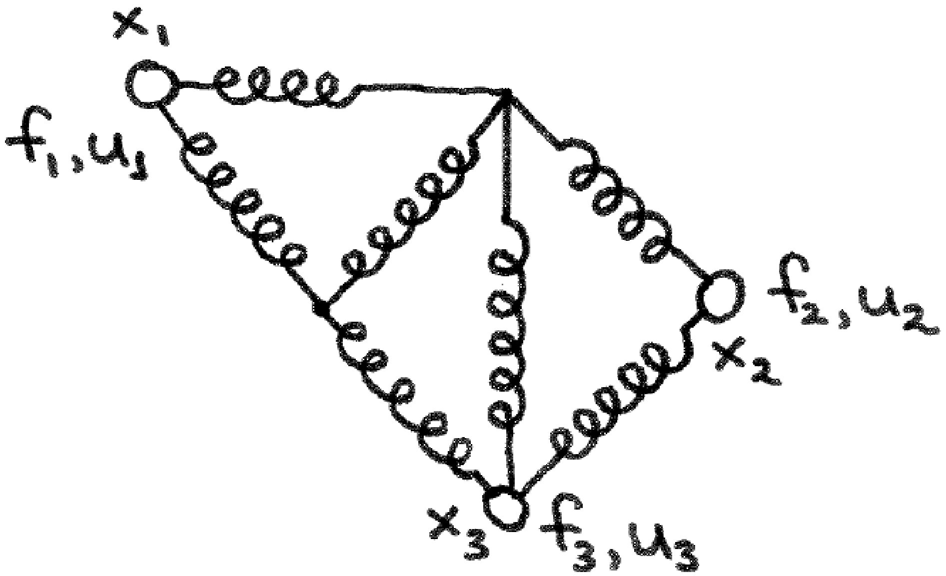


Example of Pierre Seppecher

Dynamic Response of Mass-Spring Networks

In some sense any linear elastic metamaterial can be approximated by a mass-spring network. So what responses can these have? First lets look at statics- no masses.

Multiterminal networks



$$\sum_i \mathbf{f}_i = 0, \quad \sum_i \mathbf{x}_i \times \mathbf{f}_i$$

We say the system of forces is balanced

Relation between the displacements and forces at the terminal nodes

$$\mathbf{f} = \mathbf{W}\mathbf{u}, \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_n \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{pmatrix}$$

1. $\mathbf{W} \geq 0$, $\mathbf{W}$ symmetric
2. Each column of $\mathbf{W}$ is a balanced system of forces

Mass-spring networks (With Guevara-Vasquez and Onofrei)

Any network of has the response matrix

$$\mathbf{W}(\omega) = \mathbf{W}(0) - \omega^2 \mathbf{M} + \sum_{i=1}^{p} \mathbf{C}^{(i)} \frac{\omega^2}{\omega_i^2(\omega^2 - \omega_i^2)} \quad (*)$$

where

- $\mathbf{W}(0)$ satisfies (1) and (2)
- $\mathbf{M}$ is diagonal with the masses at the terminals along the diagonal
- $\mathbf{C}_i \geq 0$

Conversely given any placement of terminals $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$

Given any $\mathbf{W}(0)$ satisfying (1) and (2),

Given any diagonal $\mathbf{M}$ with repeated elements in each block

Given any p

Given any set of matrices $\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(p)} \geq 0$

Given any set of frequencies $\omega_1, \ldots, \omega_p$

One can construct a network which has the response $\mathbf{W}(\omega)$ given by (*).

Proof: Realize each part of (*) separately and superimpose.

Realizing $\mathbf{M}$ is trivial

Realizing $\mathbf{W}(0)$ using a spring network without masses follows the approach of Camar-Eddine and Seppecher (2003) but need to treat the degenerate case and 2d

Remains to realize

$$\mathbf{W}(\omega) = \mathbf{f}\mathbf{f}^T \frac{\omega^2}{\omega^2 - \omega_0^2}, \; \textit{where } \mathbf{f}^T = (\mathbf{f}_1^T, \mathbf{f}_2^T, \dots, \mathbf{f}_n^T).$$

Add two new points $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2}$ and choose $\mathbf{f}_{n+1}$ and $\mathbf{f}_{n+2}$ such that

$$\{(\mathbf{f}_1, \mathbf{x}_1), (\mathbf{f}_2, \mathbf{x}_2), \dots, (\mathbf{f}_{n+1}, \mathbf{x}_{n+1}), (\mathbf{f}_{n+2}, \mathbf{x}_{n+2})\}$$

is a balanced system. Attach a mass m to both of these 2 additional nodes

There is an elastic network with a rank-1 response matrix:

$$\begin{bmatrix} \mathbf{w}_B \\ m\omega^2 \mathbf{u}_I \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{a} \end{bmatrix} \begin{bmatrix} \mathbf{f}^T & \mathbf{a}^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_I \end{bmatrix}$$

$$\mathbf{a} = \begin{pmatrix} \mathbf{f}_{n+1} \\ \mathbf{f}_{n+2} \end{pmatrix} \qquad \mathbf{u}_I = \begin{pmatrix} \mathbf{u}_{n+1} \\ \mathbf{u}_{n+2} \end{pmatrix}$$

Eliminating $\mathbf{u}_I$:

$$\mathbf{w}_B = \mathbf{f}\mathbf{f}^T \mathbf{u}_B \frac{\omega^2}{\omega^2 - \|\mathbf{a}\|^2 / m}.$$

Finally choose

$$m = \|\mathbf{a}\|^2 / \omega_0^2$$

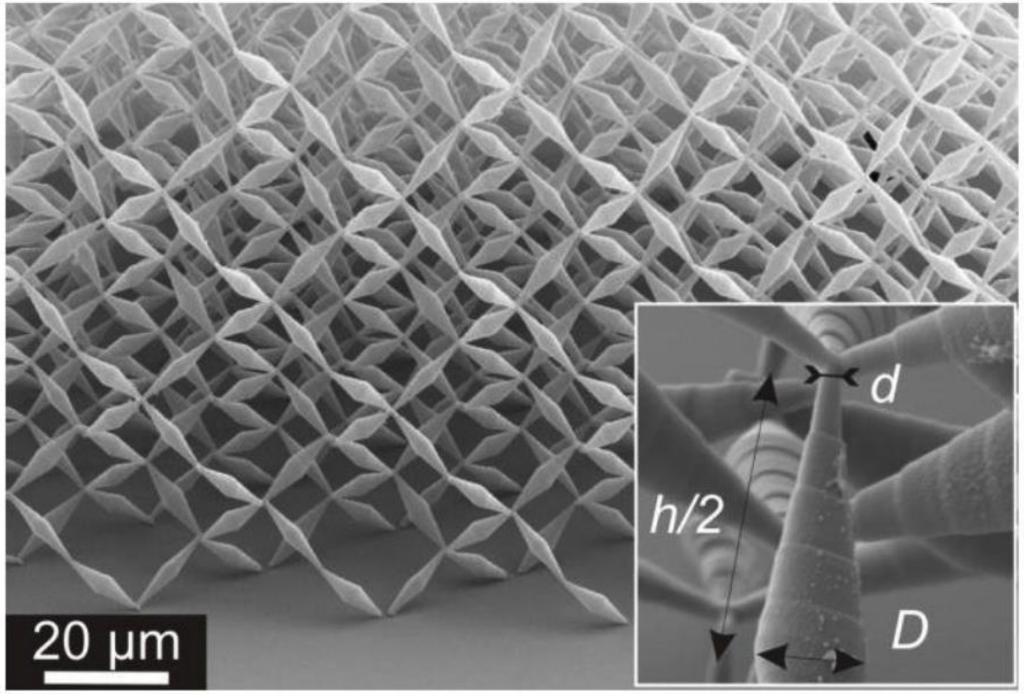
Realizing networks that only support one loading.

Inductive proof of Camar-Eddine and Seppecher (2003)

Here we introduce an alternative approach.

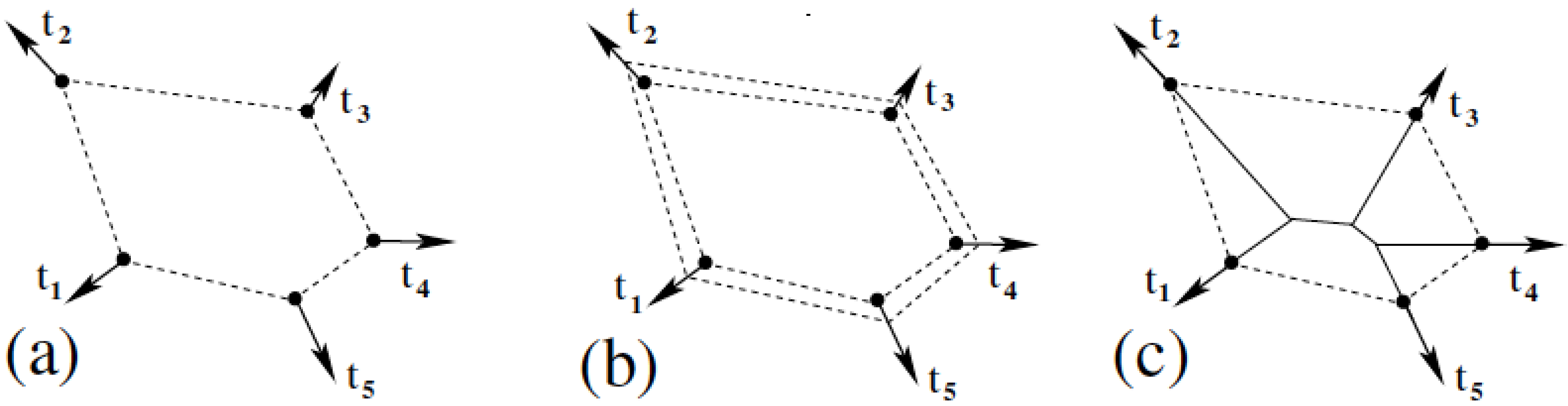
Pentamodes: Introduced with Cherkaev 1995

Realization of Kadic et.al. 2012



Key observation: It only supports a single stress because 4 elements meet at each junction- the tension in one determines the tension in the others, by balance of forces.

The Inverse Problem for inextensible Wire Webs under Tension



$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{R}_\perp^T \nabla\nabla\phi(\mathbf{x})\mathbf{R}_\perp, \quad \mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The Airy Stress Function must have non-negative curvature!

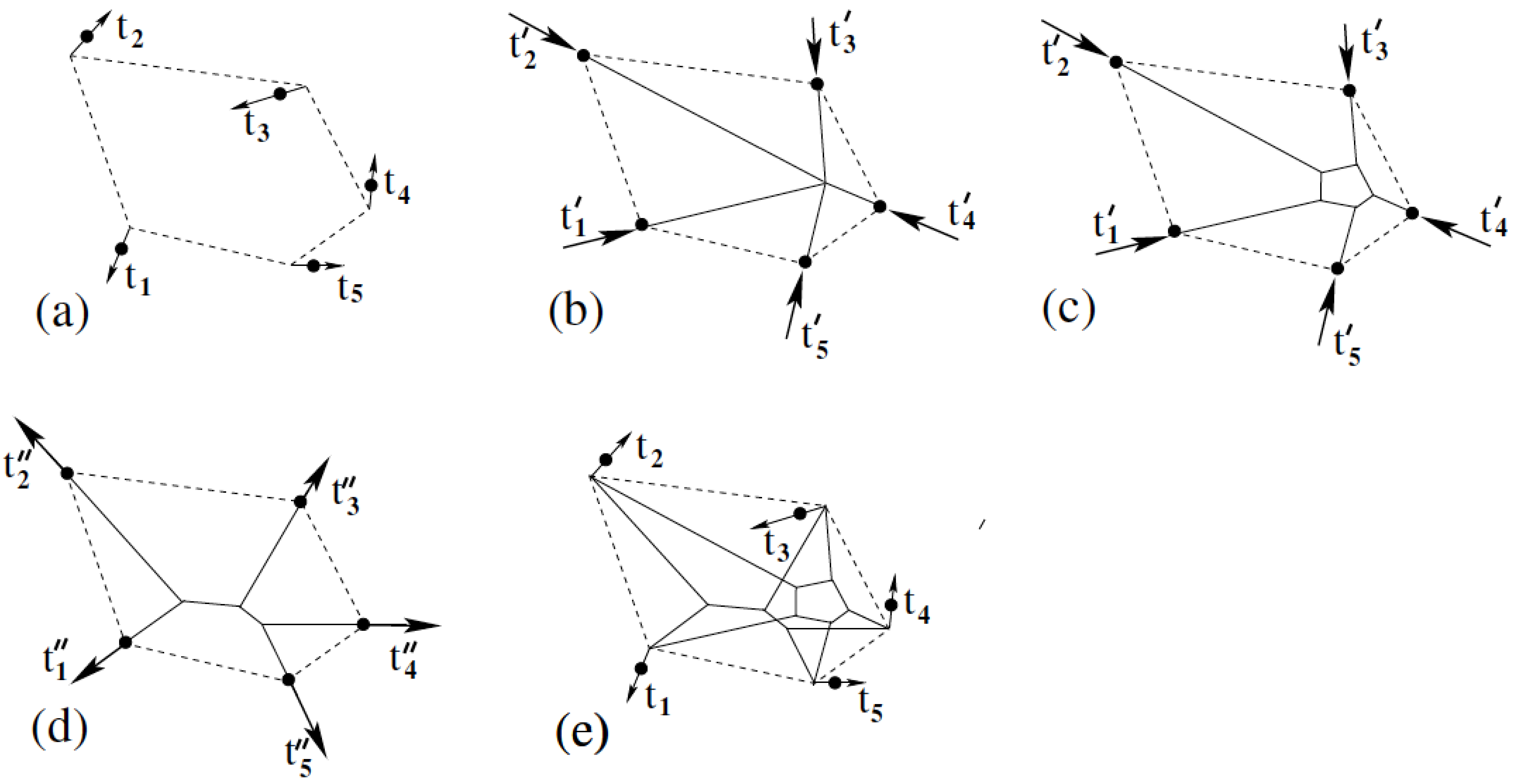
Theorem 1. *A set of points* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *at the vertices of a convex polygon, numbered clockwise, can support balanced forces* $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ *at these vertices, with a truss with all its elements under tension, if and only if for all* i *and* j*,*

$$\sum_{k=j}^{i-1} (\mathbf{x}_k - \mathbf{x}_j) \cdot [\mathbf{R}_\perp \mathbf{t}_k] \geq 0, \tag{1.4}$$

and we have assumed $i > j$*, if necessary by replacing* i *by* $i+n$ *and identifying where necessary* $\mathbf{x}_k$ *and* $\mathbf{t}_k$ *with* $\mathbf{x}_{k-n}$ *and* $\mathbf{t}_{k-n}$*.*

The net anticlockwise torque going clockwise around the boundary must be non-negative

Combining Networks under Compression and Tension to support any balanced set of forces



Field Patterns: a new sort of wave.
Joint work with Ornella Mattei

Space-time microstructures

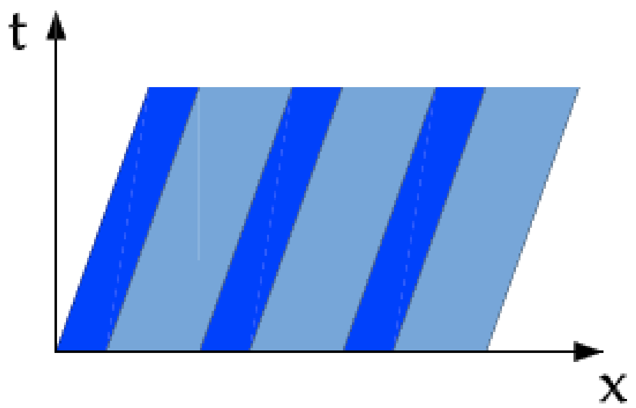
$$(a\,u_t)_t - (b\,u_x)_x = 0$$

Static materials: $a = a(x)$ and $b = b(x)$

Space-time microstructures: $a = a(x, t)$ and $b = b(x, t)$

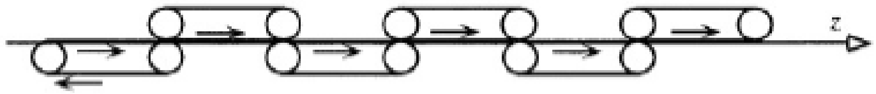
Activated materials:

The property pattern moves



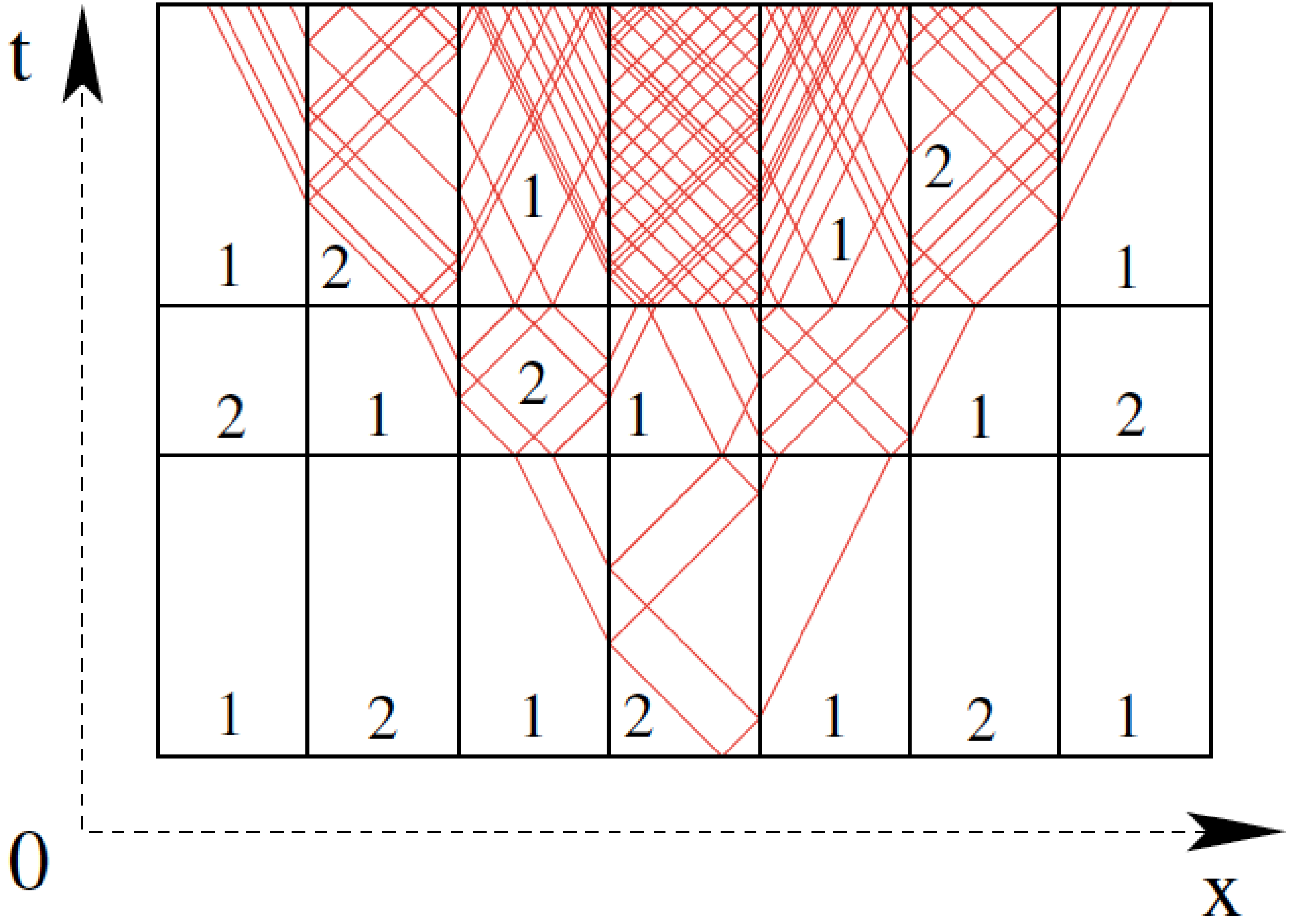
Kinetic materials:

The material itself moves

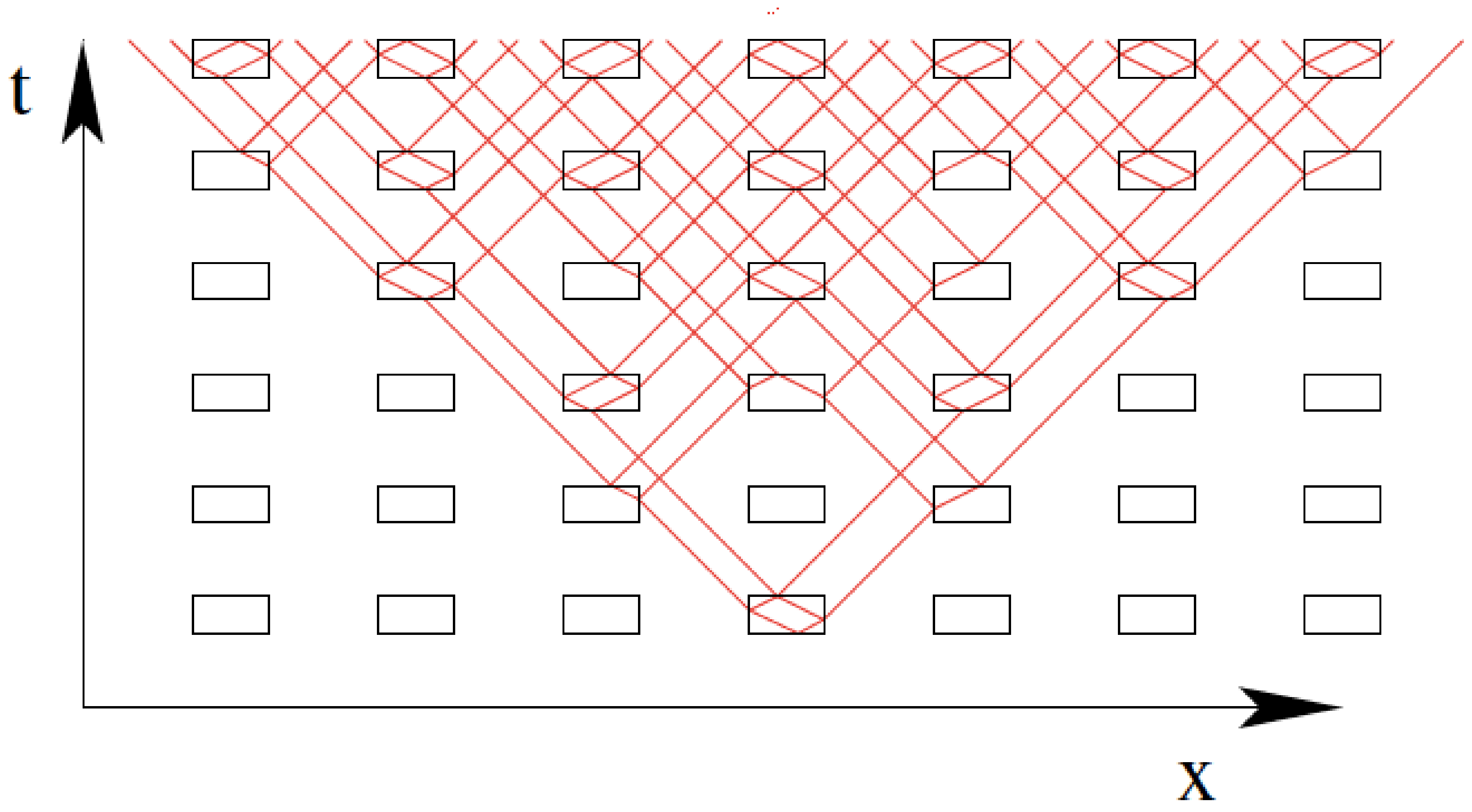


[K.A. Lurie, An Introduction to the Mathematical Theory of Dynamic Materials (2007)]

Green function for a generic space-time microstructure

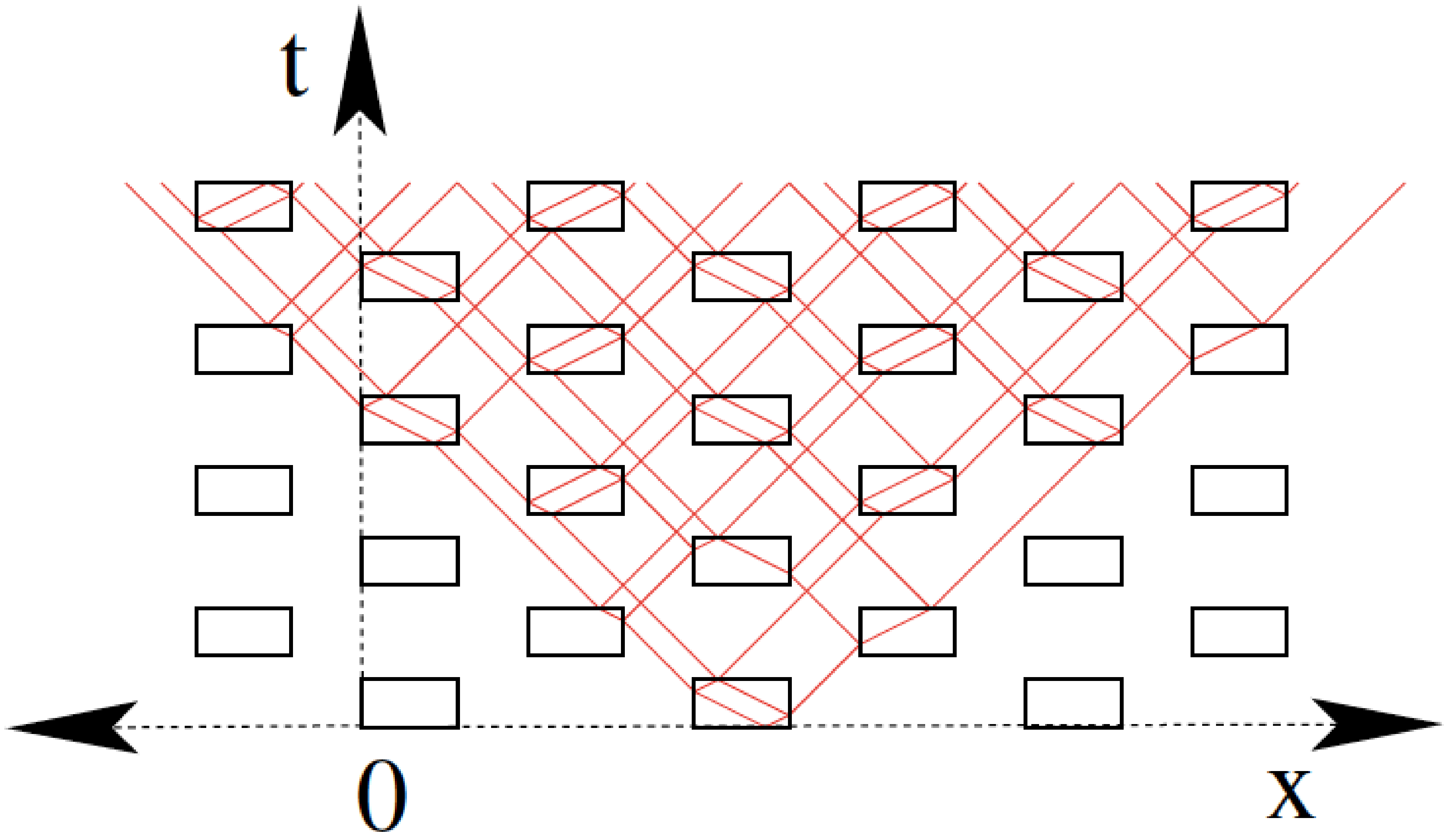


Green function for a special microstructure



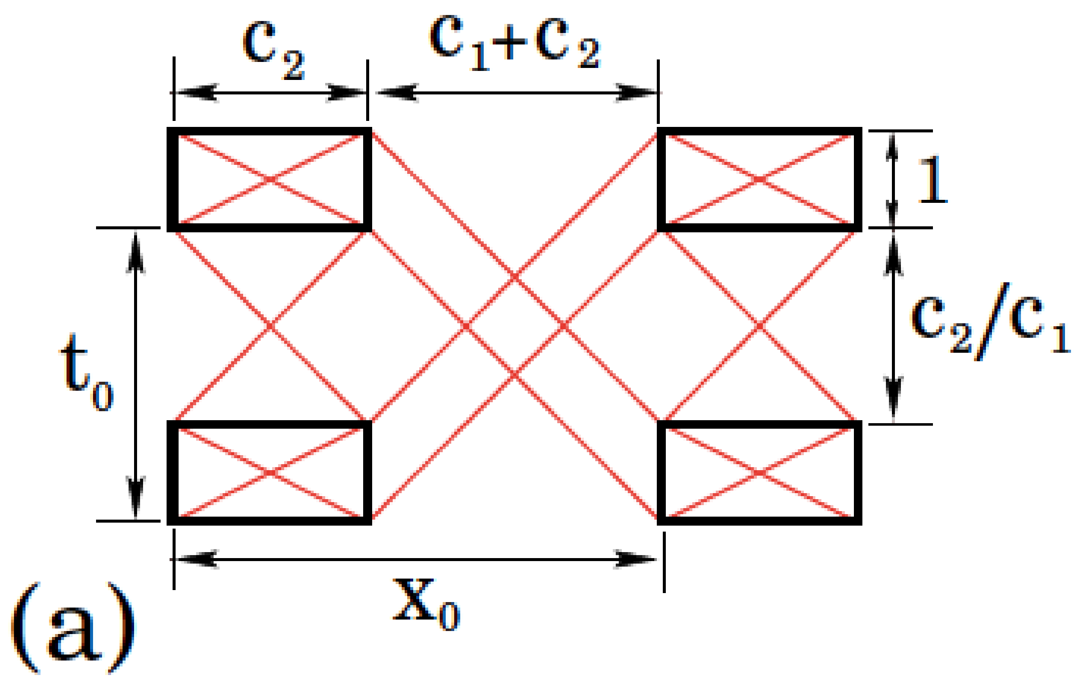
Characteristic lines lie on a pattern–the field pattern.
Note the P-T symmetry of the microstructure.

Green function for another special microstructure

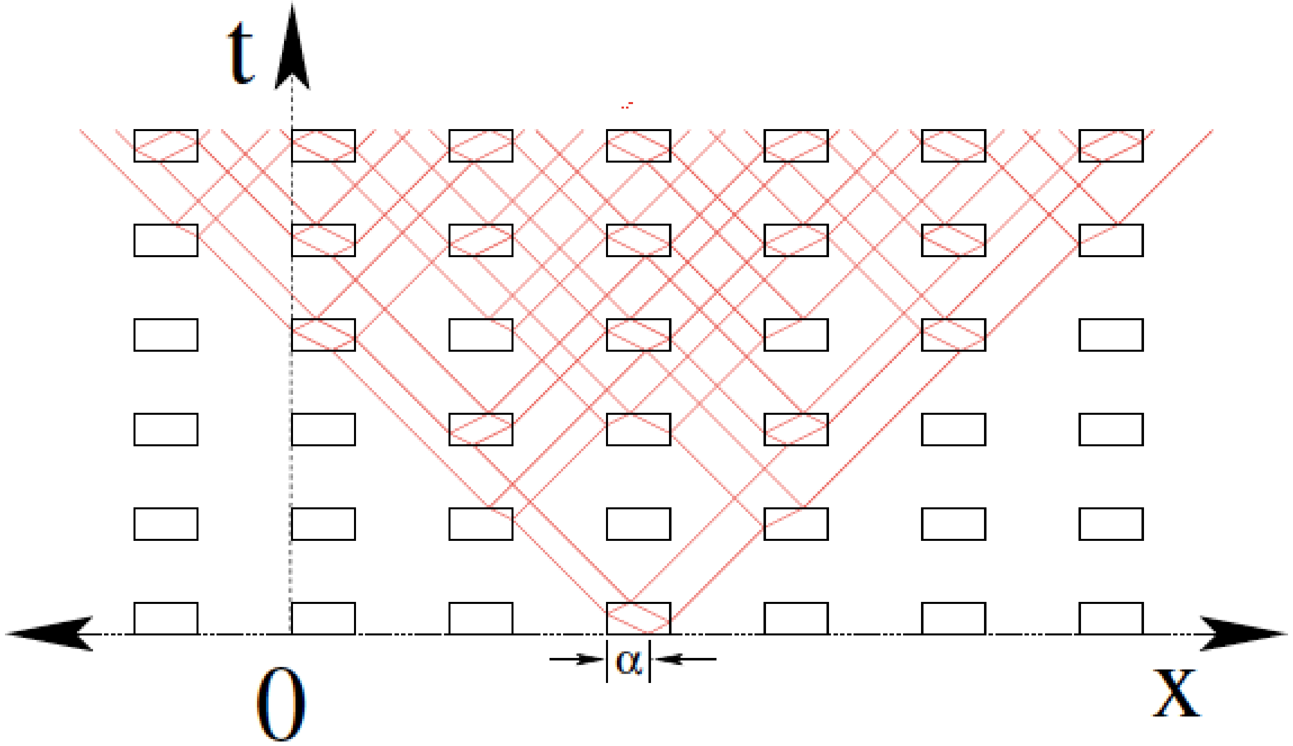


Again, note the P-T symmetry of the microstructure.

Geometry: Relation to Characteristic Lines



Multidimensional nature of field patterns

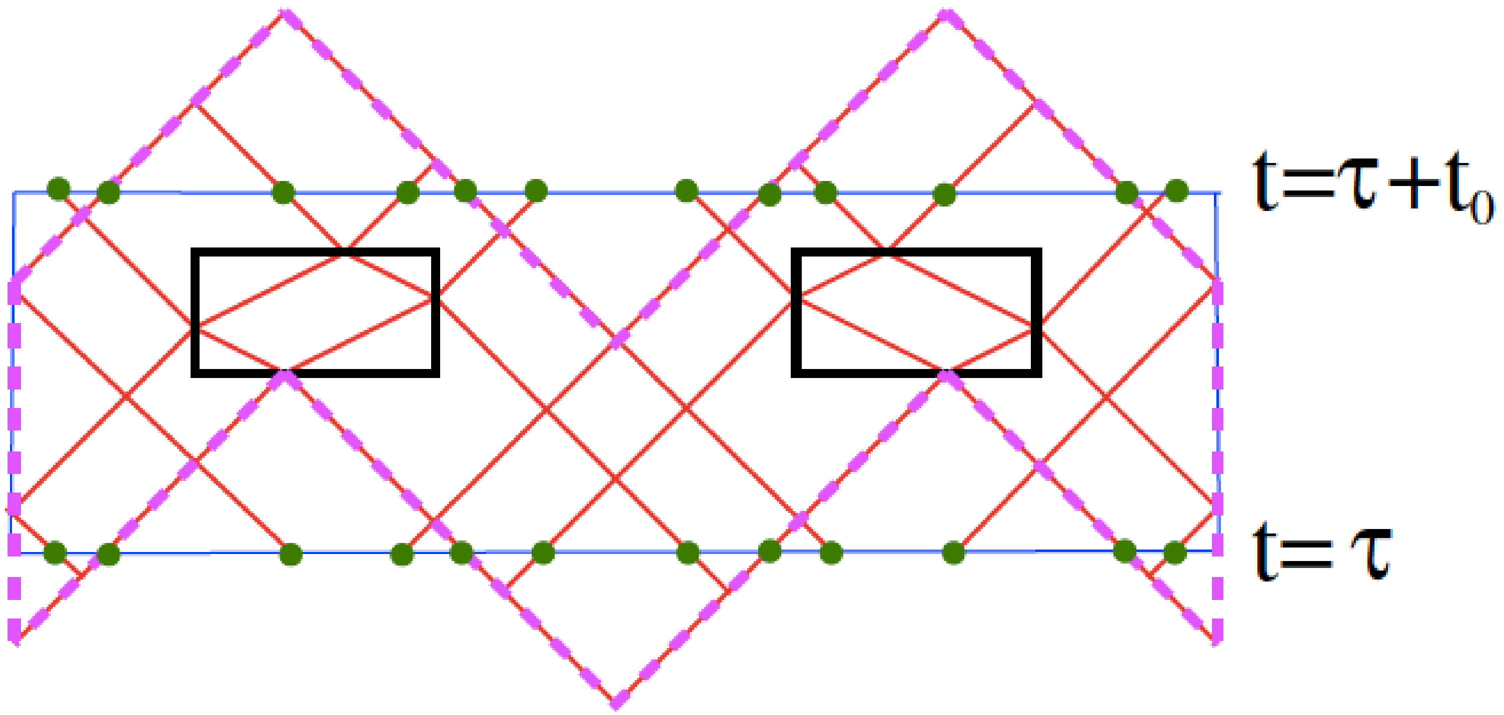


$$V(x,t) = \sum_{i=1}^{m} V_{\alpha_i}(x,t)$$

Multidimensional space: $V(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} V_{\alpha_i}(x_i, t)$

Multidimensional potential: $\mathbf{V}(x,t)$

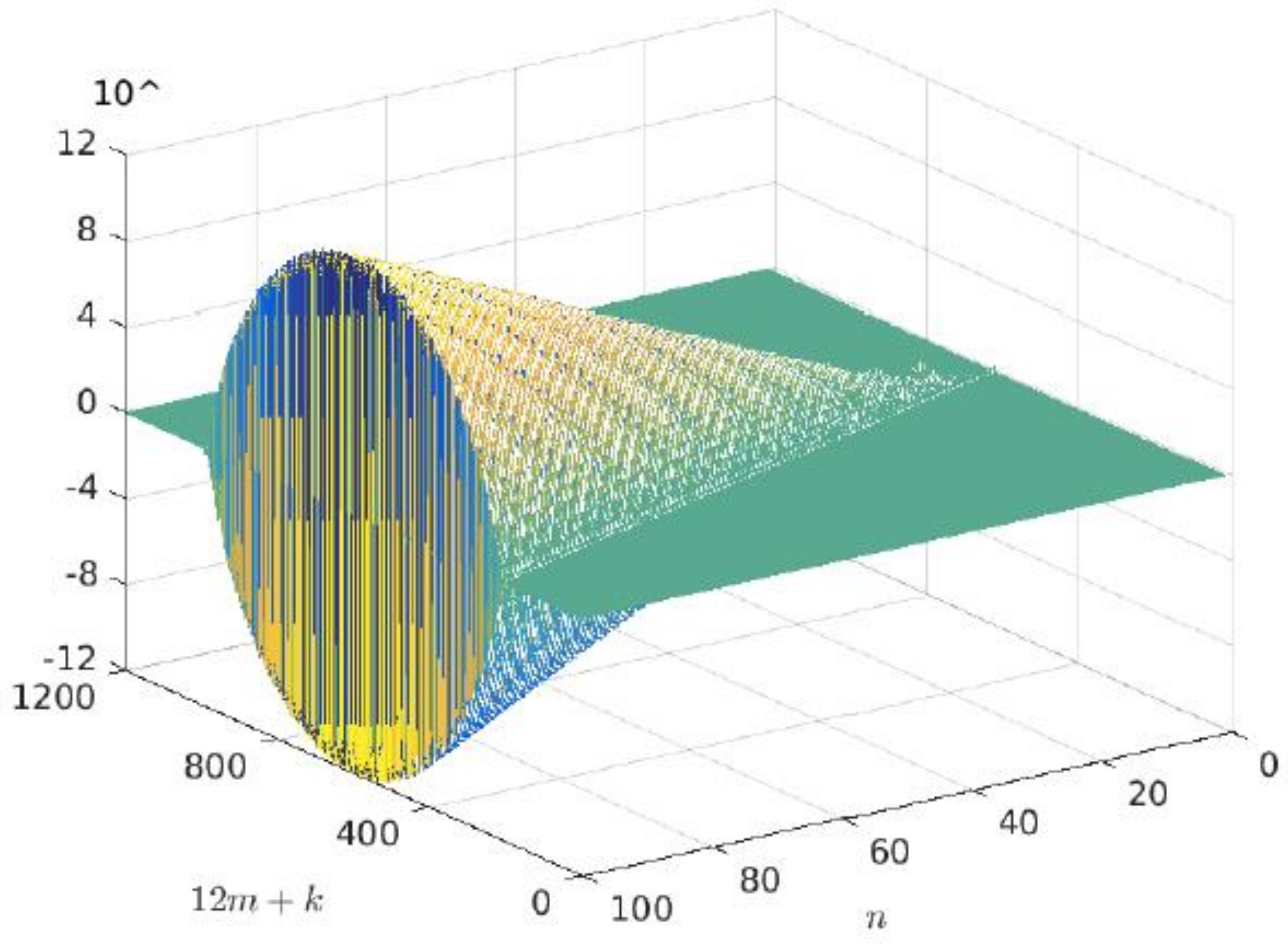
Numerical results: Transfer Matrix



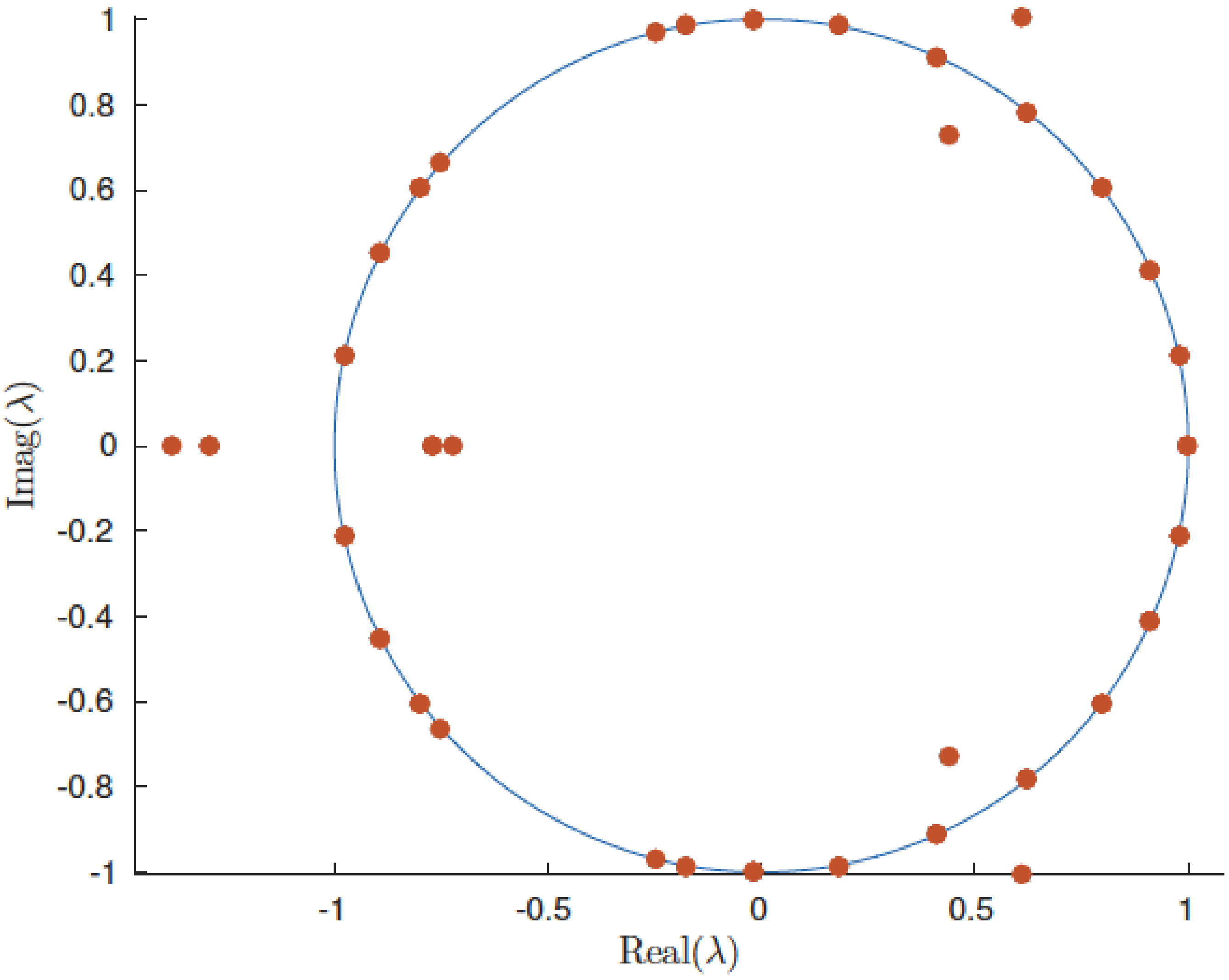
$$j(k,m,n+1) = \sum_{k',m'} T_{(k,m),(k,'m')} j(k',m',n)$$

$$T_{(k,m),(k,'m')} = G_{k,k'}(m-m')$$

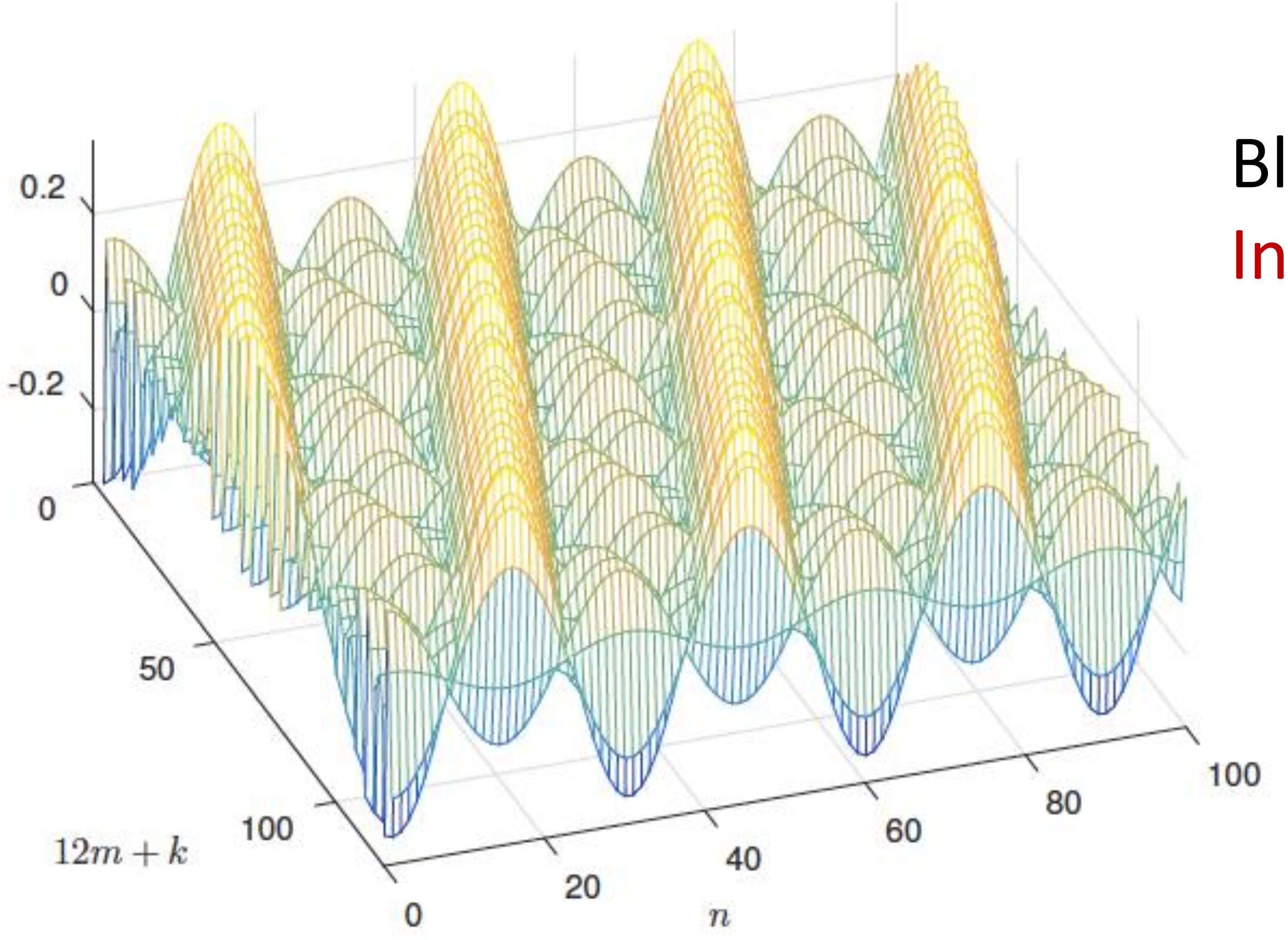
Blow up



Eigenvalues of the transfer matrix



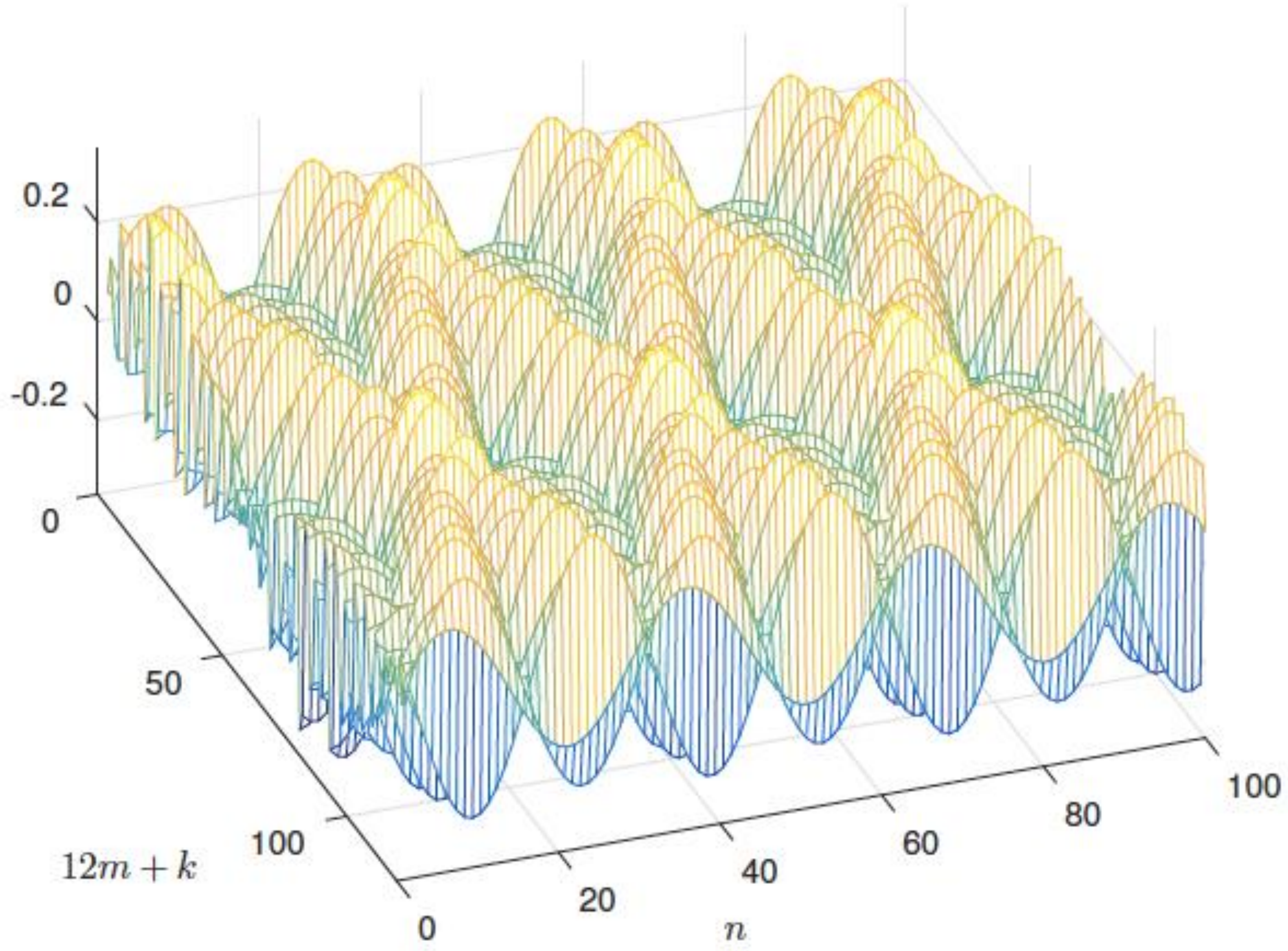
An example of a solution that does not blow up



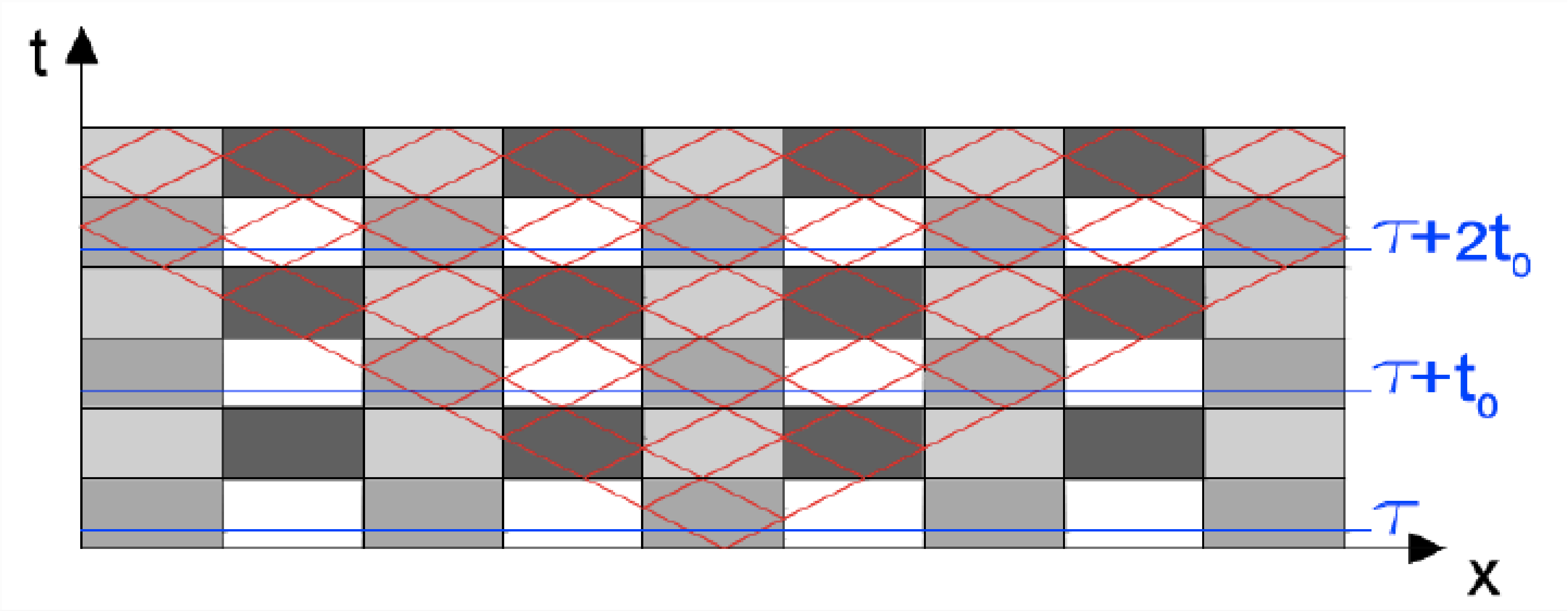
Bloch Wave:

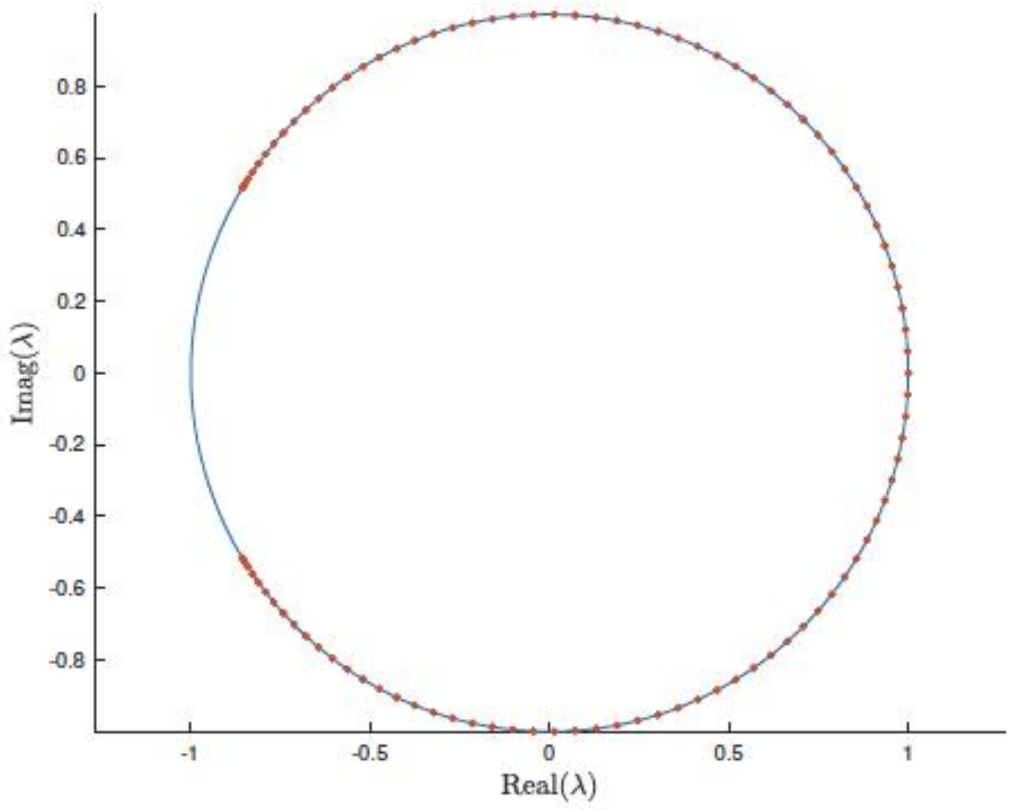
Infinitely Degenerate!

One more solution that does not blow up

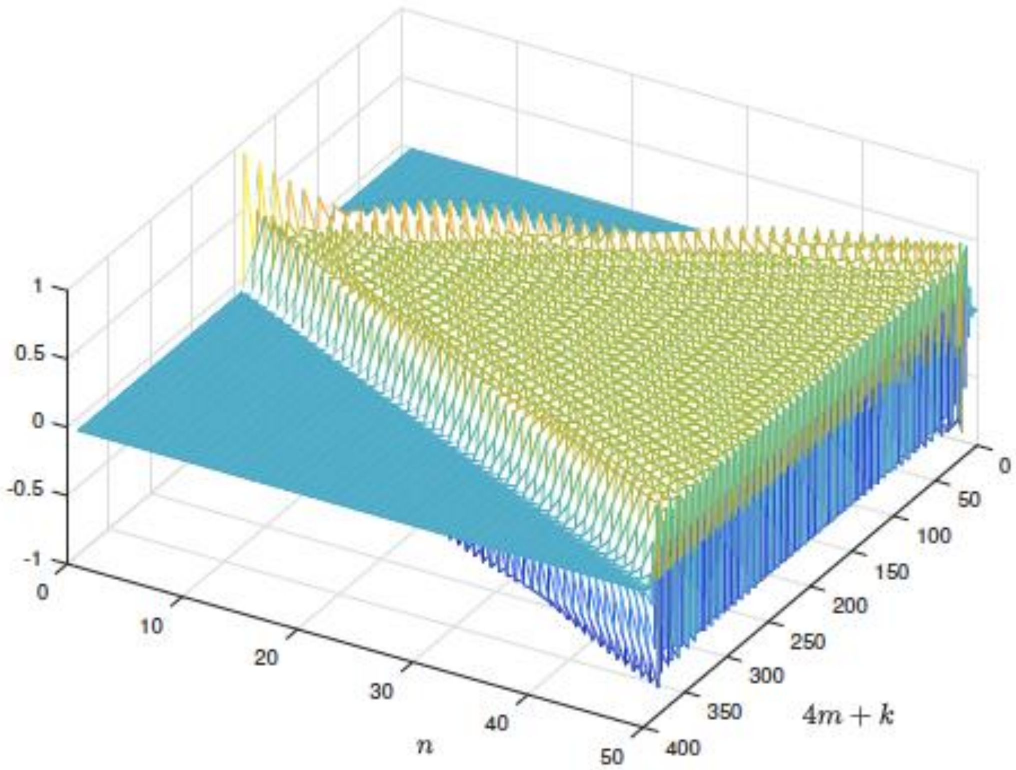


Checkerboard geometries where there is no blow up





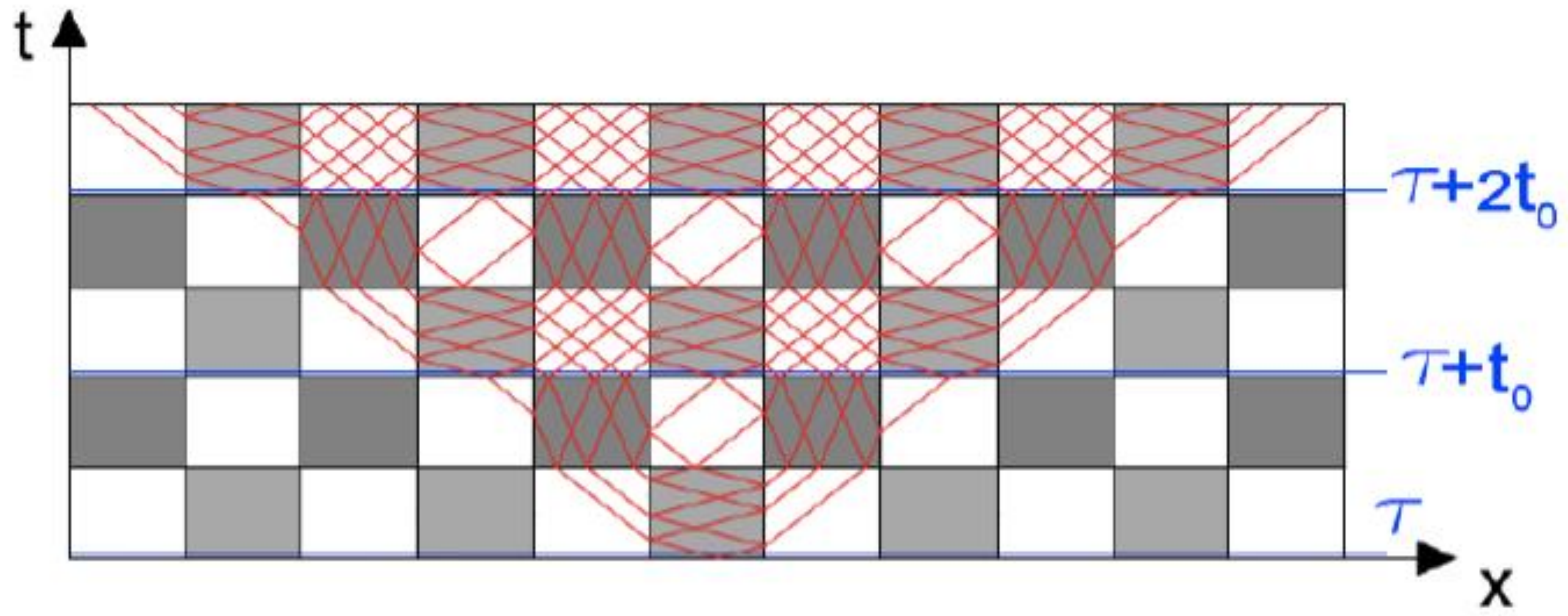
Imag(λ)
Real(λ)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0
0.5
1



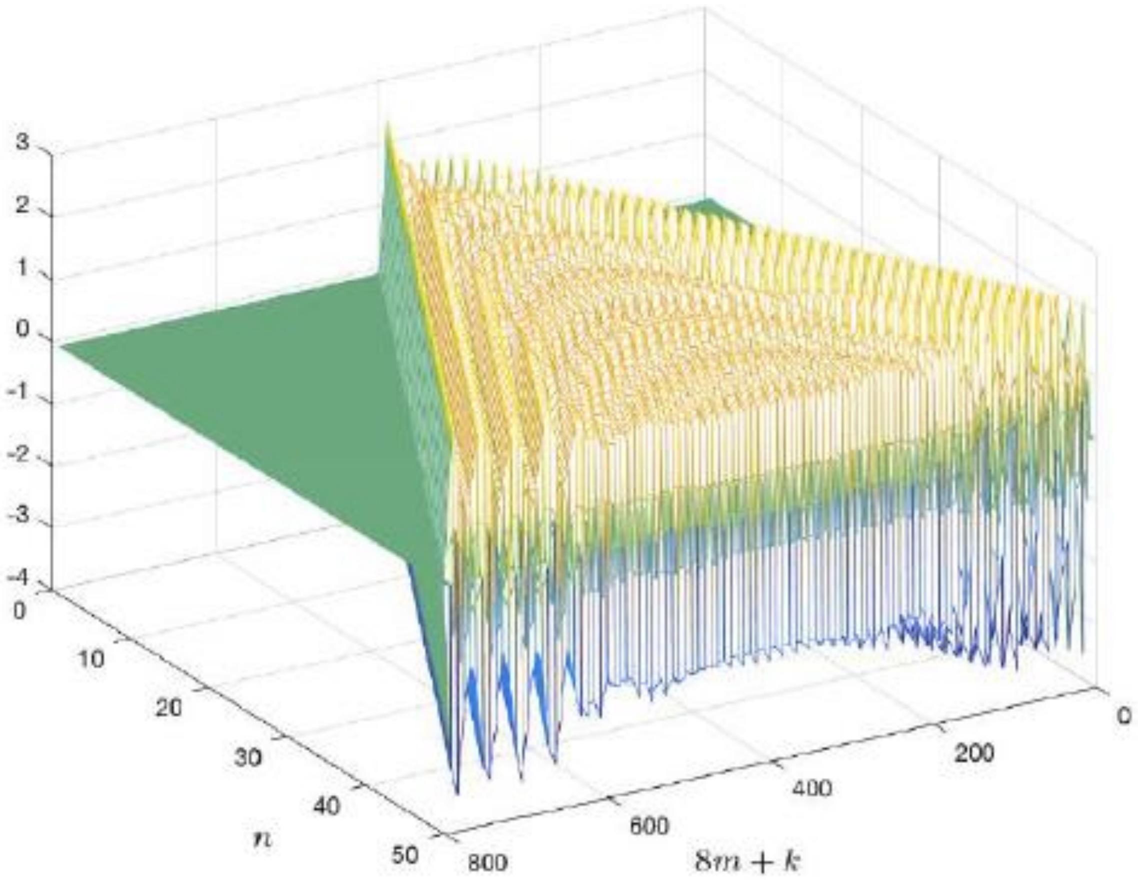
1
0.5
0
-0.5
-1
0
10
20
30
40
50
n
0
50
100
150
200
250
300
350
400
4m + k

A New Wave

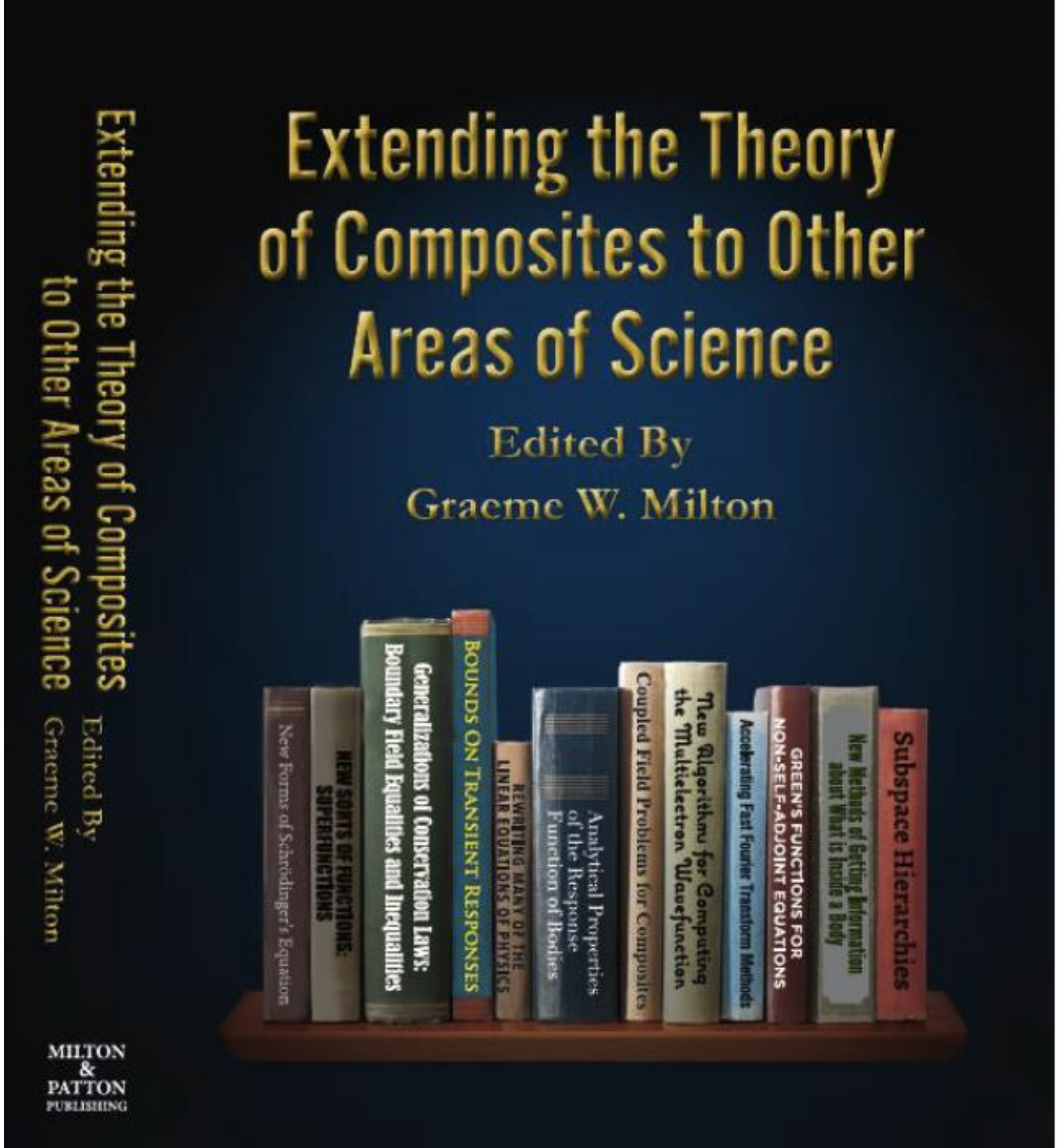
Checkerboard geometries where there is no blow up



Checkerboard geometries where there is no blow up



Breaking time's arrow, ala Boltzmann.



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