On the field recursion method for two-component composites

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Motivation

- For two-component composites, the bounds calculated from the field equation recursion method, variational principles, or the analytic method.
- The real power of the recursion method: multiphase composites – Obtain bounds on complex effective moduli where other methods fail & many of the bounds are sharp.
- Advantage: applicable to any problem formulizable in the abstract theory of composites and it produces bounds in a systematic way.
- Two-component composite with isotropic phase: (i) wanted a rigorous derivation of the recursion method using the abstract theory of composites.
- (ii) Wanted to significantly reduce, at each level of induction, the necessary assumptions required to proceed to the next level.
- (iii) Wanted a simpification of many of the mathematical aspects of the method.

Hilbert space orthogonal-triple, Z-problem, & effective operator

- Hilbert space: $\mathcal{H}$; inner product: $(\cdot,\cdot)$.
- Orthogonal-triple decomposition: $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$.
- Constitutive relations: For a given bounded linear operator $L \in \mathcal{L}(\mathcal{H})$,

$$J = LE, \text{ where } E \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{E},\ J \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{J}.$$

- Z-problem: Given $E_0 \in \mathcal{U}$, find $(J_0, E, J) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ such that

$$J_0 + J = L(E_0 + E).$$

- Effective operator: If there exists an $Z_* \in \mathcal{L}(\mathcal{U})$ such that

$$J_0 = L_* E_0$$

for every $E_0 \in \mathcal{U}$, where (J_0, E, J) is a solution of the Z-problem associated with E_0, then we call L_* an effective operator.

Z-problem solution & effective operator as a Schur complement

- Express L as a 3×3 block operator matrix with respect to $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$:

$$L = \begin{bmatrix} L_{00} & L_{01} & L_{02} \\ L_{10} & L_{11} & L_{12} \\ L_{20} & L_{21} & L_{22} \end{bmatrix}, \; L_{\mathcal{U} \overset{\perp}{\oplus} \mathcal{E}} := \begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix}, \; L_{\mathcal{E}} := L_{11}.$$

- The Z-problem for a given $E_0 \in \mathcal{U}$, equivalent to solving:

$$\begin{cases} J_0 = L_{00} E_0 + L_{01} E, \\ 0 = L_{10} E_0 + L_{11} E, \\ J = L_{20} E_0 + L_{21} E. \end{cases}$$

  for $(J_0, E, J) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$.

- If L_{11}^{-1} exists then a unique solution exists to the Z-problem, the effective operator L_* exists and is unique, and

$$J = L_{20} E_0 + L_{21} E, \quad E = -L_{11}^{-1} L_{10} E_0, \quad J_0 = L_* E_0, \tag{1}$$

$$L_* = L_{\mathcal{U} \overset{\perp}{\oplus} \mathcal{E}} / L_{\mathcal{E}} = L_{00} - L_{01} L_{11}^{-1} L_{10} \text{ (Schur complement)} \tag{2}$$

Orthogonal Z(2) subspace collection

- Orthogonal $Z(2)$ subspace collection: $\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \mathcal{P}_j$, $j = 1, 2$, where

$$\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J} = \mathcal{P}_1 \overset{\perp}{\oplus} \mathcal{P}_2.$$

- Orthogonal projections: $\Gamma_0, \Gamma_1, \Gamma_2, \Lambda_j$, $j = 1, 2$, where

$$\Gamma_0 \mathcal{H} = \mathcal{U},\ \Gamma_1 \mathcal{H} = \mathcal{E},\ \Gamma_2 \mathcal{H} = \mathcal{J},\ \Lambda_j \mathcal{H} = \mathcal{P}_j.$$

- Two-component composite:

$$L = L(L_1, L_2) = L_1 \Lambda_1 + L_2 \Lambda_2,$$

  where L_1, $L_2 \in \mathcal{L}(\mathcal{H})$.
- Isotropic, two-phase:

$$L = L(L_1, L_2) = L_1 \Lambda_1 + L_2 \Lambda_2,\ L_1, L_2 \in \mathbb{C}.$$

Properties: Homogeneity, Normalization, & Herglotz

- Schur complement formula (2) implies homogeneity, normalization, & Herglotz properties.
- Homogeneity property:

$$L_*(cL_1, cL_2) = cL_*(L_1, L_2), \ \forall c \in \mathbb{C} \setminus \{0\}.$$

- Normalization property:

$$L_*(1, 1) = I_{\mathcal{U}}.$$

- Herglotz property: The function $L_* : (\mathbb{C}^+)^2 \to \mathcal{L}(\mathcal{U})$ is analytic and

$$\operatorname{Im} L_*(L_1, L_2) > 0, \ \forall (L_1, L_2) \in \left(\mathbb{C}^+\right)^2,$$

where

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \ \operatorname{Im} M = \frac{M - M^*}{2i}, \ M \in \mathcal{L}(\mathcal{H}).$$

Quintessential example: conductivity equation

Framework for conductivity equation ($\nabla \cdot \sigma \nabla u = 0$) of a two-component periodic composite with unit cell $\mathcal{D} = [0, 2\pi]^d$ ($d = 2, 3$).

- Hilbert space: $\mathcal{H} = \left[L^2_{\text{per}}(\mathcal{D})\right]^d$; inner product and average
$$(E, F) = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} E(x)^T \overline{F(x)} dx, \quad \langle F \rangle = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} F(x)\, dx, \quad \forall E, F \in \mathcal{H}.$$
- Orthogonal-triple (Hodge) decomposition: $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$, where
$$\begin{aligned} \mathcal{U} &= \left\{ U \in \mathcal{H} | U \equiv C, \text{ for some } C \in \mathbb{C}^d \right\}; \\ \mathcal{E} &= \left\{ E \in \mathcal{H} | \nabla \times E = 0 \text{ and } \langle E \rangle = 0 \right\}; \\ \mathcal{J} &= \left\{ J \in \mathcal{H} | \nabla \cdot J = 0 \text{ and } \langle J \rangle = 0 \right\}. \end{aligned}$$
- Constitutive relation: Multiplication operator $\sigma = \sigma(x) \in M_d\left(L^\infty_{\text{per}}(\mathcal{D})\right)$,
$$J = \sigma E, \text{ where } E \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{E}, \; J \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{J}.$$
- Z-problem: Given $E_0 \in \mathcal{U}$, find $(J_0, E, J) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ such that
$$J_0 + J = \sigma(E_0 + E).$$
- Effective operator $\sigma_* \in \mathcal{L}(\mathcal{U})$:
$$J_0 = \sigma_* E_0 \text{ or, equivalently, } \langle J_0 + J \rangle = \sigma_* \langle E_0 + E \rangle.$$

Quintessential example: conductivity equation

- Two-component composite, isotropic, two-phase:

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2,\ \mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing,\ \chi_j \in L^2_{\text{per}}(\mathcal{D})$$

for $x \in \mathcal{D}_j$, $\chi_j(x) = \begin{cases} 1 & \text{if } x \in \mathcal{D}_j, \\ 0 & \text{if } x \notin \mathcal{D}_j. \end{cases}$ (characteristic function of $\mathcal{D}_j$)

$$\sigma = \sigma(\sigma_1, \sigma_2) = \sigma_1 \chi_1 I + \sigma_2 \chi_2 I,\ \sigma_1, \sigma_2 \in \mathbb{C}.$$

- Orthogonal $Z(2)$ subspace collection:

$$\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J} = \mathcal{P}_1 \overset{\perp}{\oplus} \mathcal{P}_2.$$

- Orthogonal projections: $\Gamma_0, \Gamma_1, \Gamma_2, \Lambda_j$, $j = 1, 2$, where

$$\Gamma_0 \mathcal{H} = \mathcal{U},\ \Gamma_1 \mathcal{H} = \mathcal{E},\ \Gamma_2 \mathcal{H} = \mathcal{J},\ \Lambda_j \mathcal{H} = \mathcal{P}_j.$$

- For any $F \in \mathcal{H} = \left[L^2_{\text{per}}(\mathcal{D})\right]^d$,

$$\Gamma_0 F = \langle F \rangle,\ \Gamma_1 F = \left[\nabla (\nabla \cdot \nabla)^{-1} \nabla \cdot\right] F,\ \Gamma_2 F = \left[\nabla \times (-\nabla \cdot \nabla)^{-1} \nabla \times\right] F,$$

$$\Lambda_j F = \chi_j F.$$

Quintessential example: conductivity equation

- Explicitly, for any $F \in \mathcal{H} = \left[L^2_{\text{per}}(\mathcal{D})\right]^d$,

$$F(x) = \widehat{F}(0) + \sum_{\mathbf{0} \neq k \in \mathbb{Z}^d} \widehat{F}(k)\, e^{ik\cdot x}, \;\; I = \Gamma_0 + \Gamma_1 + \Gamma_2,$$

$$\Gamma_0 F = \langle F \rangle = \widehat{F}(0),$$

$$\Gamma_1 F = \left[\nabla(\nabla\cdot\nabla)^{-1}\nabla\cdot\right] F = \sum_{0 \neq k \in \mathbb{Z}^d} \frac{k}{|k|^2} k\cdot\widehat{F}(k)\, e^{ik\cdot x},$$

$$\Gamma_2 F = \left[\nabla\times(-\nabla\cdot\nabla)^{-1}\nabla\times\right] F = \sum_{0 \neq k \in \mathbb{Z}^d} \frac{k\times\left(k\times\widehat{F}(k)\right)}{|k|^2} e^{ik\cdot x}$$

$$= \sum_{0 \neq k \in \mathbb{Z}^d} \left(1 - \frac{k}{|k|^2} k\cdot\right) \widehat{F}(k)\, e^{ik\cdot x}.$$

- There are other examples, e.g., Dirichlet-to-Neumann map, which also fits into the abstract theory of composites as an effective operator. See [1]!

The Y-problem and Y-operator

- Hilbert space: $\mathcal{K}$; inner product: $(\cdot, \cdot)$.
- 2-orthogonal decompositions: $\mathcal{K} = \mathcal{E} \overset{\perp}{\oplus} \mathcal{J} = \mathcal{V} \overset{\perp}{\oplus} \mathcal{H}$.
- Y-problem: Given $L \in \mathcal{L}(\mathcal{H})$ and $E_1 \in \mathcal{V}$, find $(E, J) \in \mathcal{E} \times \mathcal{J}$ such that there exists $J_1 \in \mathcal{V}$, $E_2, J_2 \in \mathcal{H}$ satisfying

$$E = E_1 + E_2, \; J = J_1 + J_2, \text{and } J_2 = LE_2.$$

- Y-operator: If there exists an $Y_* \in \mathcal{L}(\mathcal{V})$ such that

$$J_1 = -Y_* E_1$$

for every $E_1 \in \mathcal{V}$, whenever (E, J) is a solution of the Y-problem associated with E_1, then we call Y_* a Y-operator.

- Orthogonal projections: Γ_j, Π_j, $j = 1, 2$, where

$$\Gamma_1 \mathcal{K} = \mathcal{E},\ \Gamma_2 \mathcal{K} = \mathcal{J},\ \Pi_1 \mathcal{K} = \mathcal{V},\ \Pi_2 \mathcal{K} = \mathcal{H}.$$

- i.e., Y-problem: Given $L \in \mathcal{L}(\mathcal{H})$ and $E_1 \in \mathcal{V}$, find $(E, J) \in \mathcal{E} \times \mathcal{J}$ such that $\Pi_2 J = L \Pi_2 E$ and $\Pi_1 E = E_1$.

Proposition (Y-Problem: Existence & Uniqueness)

If $L^{-1} : \mathcal{H} \to \mathcal{H}$ and $\left(\Gamma_2 \Pi_2 L^{-1} \Pi_2 \Gamma_2\right)^{-1} : \mathcal{J} \to \mathcal{J}$ exist then (i) solution is unique; (ii) exists with hypothesis $\Pi_2 L^{-1} \Pi_2 J = L^{-1} \Pi_2 J$:

$$J = -\left(\Gamma_2 \Pi_2 L^{-1} \Pi_2 \Gamma_2\right)^{-1} \Gamma_2 E_1,\ E = E_1 + E_2,\ E_2 = L^{-1} \Pi_2 J, \tag{3}$$

$$Y_* = \Pi_1 \Gamma_2 \left(\Gamma_2 \Pi_2 L^{-1} \Pi_2 \Gamma_2\right)^{-1} \Gamma_2 \Pi_1. \tag{4}$$

Proof.

$$\text{(i) } \Pi_2 J = L \Pi_2 E, \Pi_1 E = E_1 \Rightarrow 0 = \Gamma_2 E = \Gamma_2 \Pi_1 E + \Gamma_2 \Pi_2 E$$

$$= \Gamma_2 E_1 + \Gamma_2 L^{-1} \Pi_2 J = \Gamma_2 E_1 + \Gamma_2 \Pi_2 L^{-1} \Pi_2 \Gamma_2 J;$$

$$\text{(ii) (3) } \Rightarrow \Pi_j E = E_j,\ \Pi_2 J = L \Pi_2 E,\ J = \Gamma_2 J,\ \Gamma_2 E = \Gamma_2 E_1 + \Gamma_2 E_2$$

$$= \Gamma_2 E_1 + \Gamma_2 L^{-1} \Pi_2 J = \Gamma_2 E_1 + \left(\Gamma_2 \Pi_2 L^{-1} \Pi_2 \Gamma_2\right) J = 0 \Rightarrow (E, J) \in \mathcal{E} \times \mathcal{J}$$

Orthogonal Y(2) subspace collection

- Orthogonal $Y(2)$ subspace collection: $\mathcal{K}, \mathcal{E}, \mathcal{J}, \mathcal{V}, \mathcal{H}, \mathcal{P}_j$, $j = 1, 2$, where

$$\mathcal{K} = \mathcal{E} \overset{\perp}{\oplus} \mathcal{J} = \mathcal{V} \overset{\perp}{\oplus} \mathcal{H},\ \mathcal{H} = \mathcal{P}_1 \overset{\perp}{\oplus} \mathcal{P}_2.$$

- Orthogonal projections: $\Lambda_j : \mathcal{H} \to \mathcal{H}$, $j = 1, 2$, where

$$\Lambda_j \mathcal{H} = \mathcal{P}_j.$$

- Two-component composite:

$$L = L(L_1, L_2) = L_1 \Lambda_1 + L_2 \Lambda_2,$$

  where L_1, $L_2 \in \mathcal{L}(\mathcal{H})$.
- Isotropic, two-phase:

$$L = L(L_1, L_2) = L_1 \Lambda_1 + L_2 \Lambda_2,\ L_1, L_2 \in \mathbb{C}.$$

Properties: Homogeneity, Positivity, & Herglotz

- If $\mathcal{V} \cap \mathcal{J} = \emptyset$ then formula (4) implies homogeneity, positivity, & Herglotz properties.
- Homogeneity property:

$$Y_*(cL_1, cL_2) = cY_*(L_1, L_2), \ \forall c \in \mathbb{C} \setminus \{0\}.$$

- Positivity property:

$$Y_*(1,1) = \Pi_1 \Gamma_2 \left(\Gamma_2 \Pi_2 \Gamma_2\right)^{-1} \Gamma_2 \Pi_1 |_{\mathcal{V}} > 0.$$

- Herglotz property: The function $Y_* : \left(\mathbb{C}^+\right)^2 \to \mathcal{L}\left(\mathcal{V}\right)$ is analytic and

$$\operatorname{Im} Y_*\left(L_1, L_2\right) > 0, \ \forall \left(L_1, L_2\right) \in \left(\mathbb{C}^+\right)^2.$$

Hint: Use identity $\operatorname{Im}\left(M^{-1}\right) = -M^{-1}\left(\operatorname{Im} M\right)\left(M^{-1}\right)^*$.

The field recursion method: initial conditions & base case

Field equation recursion method: isotropic, two-phase

(!) Indicates continued fraction formulas

Base case $i = 0$:

- Input – orthogonal $Z(2)$ subspace collection:
  $\mathcal{H}^{(0)}, \mathcal{U}^{(0)}, \mathcal{E}^{(0)}, \mathcal{J}^{(0)}, \mathcal{P}_j^{(0)}, j = 1, 2.$

  (1) Generate – the effective operator $L_*^{(0)}$;

  (2) Generate – orthogonal $Y(2)$ subspace collection:
  $\mathcal{K}^{(0)}, \mathcal{E}^{(0)}, \mathcal{J}^{(0)}, \mathcal{V}^{(0)}, \mathcal{H}^{(1)}, \mathcal{P}_j^{(1)}, j = 1, 2$; the Y-operator $Y_*^{(0)}$;

  (3) Generate – orthogonal $Z(2)$ subspace collection:
  $\mathcal{H}^{(1)}, \mathcal{U}^{(1)}, \mathcal{E}^{(1)}, \mathcal{J}^{(1)}, \mathcal{P}_j^{(1)}, j = 1, 2$; the Z-operator $L_*^{(1)}$.

  (4) Formulas, properties, & relations:

$$L^{(0)} = L, \ (!) \ (L_*^{(0)}, Y_*^{(0)}, L_*^{(1)}).$$

- Output – the new $Z(2)$ subspace collection:
  $\mathcal{H}^{(1)}, \mathcal{U}^{(1)}, \mathcal{E}^{(1)}, \mathcal{J}^{(1)}, \mathcal{P}_j^{(1)}, j = 1, 2.$

The field recursion method: Induction step

Field equation recursion method: isotropic, two-phase

(!) Indicates continued fraction formulas

Induction step i, $i \geq 1$:

- Input – orthogonal $Z(2)$ subspace collections: $\mathcal{H}^{(i)}, \mathcal{U}^{(i)}, \mathcal{E}^{(i)}, \mathcal{J}^{(i)}, \mathcal{P}_j^{(i)}$, $j = 1, 2$.

  (1) Generate – the effective operator $L_*^{(i)}$;
  (2) Generate – orthogonal $Y(2)$ subspace collection: $\mathcal{K}^{(i)}, \mathcal{E}^{(i)}, \mathcal{J}^{(i)}, \mathcal{V}^{(i)}, \mathcal{H}^{(i+1)}, \mathcal{P}_j^{(i+1)}$, $j = 1, 2$; the Y-operator $Y_*^{(i)}$;
  (3) Generate – orthogonal $Z(2)$ subspace collection: $\mathcal{H}^{(i+1)}, \mathcal{U}^{(i+1)}, \mathcal{E}^{(i+1)}, \mathcal{J}^{(i+1)}, \mathcal{P}_j^{(i+1)}$, $j = 1, 2$; the Z-operator $L_*^{(i+1)}$.
  (4) Formulas, properties, & relations:

$$L^{(0)} = L, \ (!) \left(L_*^{(i)}, Y_*^{(i)}, L_*^{(i+1)} \right).$$

- Output – the new $Z(2)$ subspace collection: $\mathcal{H}^{(i+1)}, \mathcal{U}^{(i+1)}, \mathcal{E}^{(i+1)}, \mathcal{J}^{(i+1)}, \mathcal{P}_j^{(i+1)}$, $j = 1, 2$.

- Input – orthogonal $Z(2)$ subspace collection:
  $\mathcal{H}^{(0)}, \mathcal{U}^{(0)}, \mathcal{E}^{(0)}, \mathcal{J}^{(0)}, \mathcal{P}_j^{(0)}, \ j = 1, 2.$

(2) Generate – orthogonal $Y(2)$ subspace collection
$\mathcal{K}^{(0)}, \mathcal{E}^{(0)}, \mathcal{J}^{(0)}, \mathcal{V}^{(0)}, \mathcal{H}^{(1)}, \mathcal{P}_j^{(1)}, \ j = 1, 2$:

$$\begin{aligned} \mathcal{K}^{(0)} &= \mathcal{H}^{(0)} \overset{\perp}{\ominus} \mathcal{U}^{(0)}, \ \mathcal{P}_j^{(1)} = \mathcal{P}_j^{(0)} \cap \mathcal{K}^{(0)}, \ j = 1, 2, \\ \mathcal{H}^{(1)} &= \mathcal{P}_1^{(1)} \overset{\perp}{\oplus} \mathcal{P}_2^{(1)}, \ \mathcal{V}^{(0)} = \mathcal{K}^{(0)} \overset{\perp}{\ominus} \mathcal{H}^{(1)}. \end{aligned}$$

(3) Generate – orthogonal $Z(2)$ subspace collection
$\mathcal{H}^{(1)}, \mathcal{U}^{(1)}, \mathcal{E}^{(1)}, \mathcal{J}^{(1)}, \mathcal{P}_j^{(1)}, \ j = 1, 2$:
$\mathcal{E}^{(1)} = \mathcal{E}^{(0)} \cap \mathcal{H}^{(1)}, \ \mathcal{J}^{(1)} = \mathcal{J}^{(0)} \cap \mathcal{H}^{(1)}, \ \mathcal{U}^{(1)} = \mathcal{H}^{(1)} \overset{\perp}{\ominus} \left[\mathcal{E}^{(1)} \overset{\perp}{\oplus} \mathcal{J}^{(1)} \right]$

(4) Formulas, properties, & relations:

$$\begin{aligned} L_*^{(l)} &= \Gamma_0^{(l)} L \Gamma_0^{(l)} - \Gamma_0^{(l)} L \Gamma_1^{(l)} \left(\Gamma_1^{(l)} L \Gamma_1^{(l)} \right)^{-1} \Gamma_1^{(l)} L \Gamma_0^{(l)} \\ &= \Gamma_0^{(l)} \left[\left(\Gamma_0^{(l)} + \Gamma_2^{(l)} \right) L^{-1} \left(\Gamma_0^{(l)} + \Gamma_2^{(l)} \right) \right]^{-1} \Gamma_0^{(l)}, \ l = 0, 1 \end{aligned}$$

(!) $Y_*^{(0)} = \Pi_1^{(0)} \Gamma_2^{(0)} \left(\Gamma_2^{(0)} \Pi_2^{(0)} L^{-1} \Pi_2^{(0)} \Gamma_2^{(0)} \right)^{-1} \Gamma_2^{(0)} \Pi_1^{(0)} = K^{(0)} L_*^{(1)} \left(K^{(0)} \right)^*,$

$K^{(0)} \in \mathcal{L}\left(\mathcal{U}^{(1)}, \mathcal{V}^{(0)} \right)$ invertible, $K^{(0)} = -\left(\Pi_1^{(0)} \Gamma_1^{(0)} \Pi_1^{(0)} \right)^{-1} \Pi_1^{(0)} \Gamma_1^{(0)} \Pi_2^{(0)},$

(!) $L_*^{(0)} = \Gamma_0^{(0)} L \Gamma_0^{(0)} - \Gamma_0^{(0)} L \Pi_1^{(0)} \left(\Pi_1^{(0)} L \Pi_1^{(0)} + Y_*^{(0)} \right)^{-1} \Pi_1^{(0)} L \Gamma_0^{(0)}.$

Quintessential example revisited: conductivity equation

$\nabla \cdot \sigma \nabla u = 0$, unit cell $\mathcal{D} = [0, 2\pi]^d = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{H}^{(0)} = \left[L^2_{\text{per}}(\mathcal{D})\right]^d$,

$$\mathcal{U}^{(0)} = \left\{ U \in \mathcal{H}^{(0)} | U \equiv C, \text{ for some } C \in \mathbb{C}^d \right\};$$

$$\mathcal{E}^{(0)} = \left\{ E \in \mathcal{H}^{(0)} | \nabla \times E = 0 \text{ and } \langle E \rangle = 0 \right\};$$

$$\mathcal{J}^{(0)} = \left\{ J \in \mathcal{H}^{(0)} | \nabla \cdot J = 0 \text{ and } \langle J \rangle = 0 \right\}.$$

$$\mathcal{H}^{(1)} = \mathcal{U}^{(1)} \overset{\perp}{\oplus} \mathcal{E}^{(1)} \overset{\perp}{\oplus} \mathcal{J}^{(1)} = \mathcal{P}_1^{(1)} \overset{\perp}{\oplus} \mathcal{P}_2^{(1)}, \mathcal{K}^{(0)} = \mathcal{E}^{(0)} \overset{\perp}{\oplus} \mathcal{J}^{(0)} = \mathcal{V}^{(0)} \overset{\perp}{\oplus} \mathcal{H}^{(1)}.$$

$$\mathcal{P}_j^{(1)} = \mathcal{P}_j^{(0)} \cap \mathcal{K}^{(0)} = \left\{ F \in \mathcal{H}^{(0)} : \chi_j F = F \text{ and } \langle F \rangle = 0 \right\}, j = 1, 2.$$

$$\mathcal{H}^{(1)} = \left\{ F \in \mathcal{H}^{(0)} : \langle \chi_j F \rangle = 0, \ j = 1, 2 \right\}.$$

$$\mathcal{V}^{(0)} = \mathcal{K}^{(0)} \overset{\perp}{\ominus} \mathcal{H}^{(1)} = \left\{ F \in \mathcal{H}^{(0)} : \chi_j F|_{\mathcal{D}_j} \equiv C_j \in \mathbb{C}^d, \ j = 1, 2 \text{ and } \langle F \rangle = 0 \right\}$$

$$\mathcal{E}^{(1)} = \left\{ E \in \mathcal{H}^{(1)} : \nabla \times E = 0 \right\}, \mathcal{J}^{(1)} = \left\{ J \in \mathcal{H}^{(1)} : \nabla \cdot J = 0 \right\}.$$

$$\mathcal{U}^{(1)} = \mathcal{H}^{(1)} \overset{\perp}{\ominus} \left[\mathcal{E}^{(1)} \overset{\perp}{\oplus} \mathcal{J}^{(1)} \right] \text{ (more complicated).}$$

- Input – for $i \geq 1$, orthogonal $Z(2)$ subspace collection: $\mathcal{H}^{(i)}, \mathcal{U}^{(i)}, \mathcal{E}^{(i)}, \mathcal{J}^{(i)}, \mathcal{P}_j^{(i)}, j = 1, 2.$

(2) Generate – orthogonal $Y(2)$ subspace collection $\mathcal{K}^{(i)}, \mathcal{E}^{(i)}, \mathcal{J}^{(i)}, \mathcal{V}^{(i)}, \mathcal{H}^{(i+1)}, \mathcal{P}_j^{(i+1)}, j = 1, 2$:

$$\mathcal{K}^{(i)} = \mathcal{H}^{(i)} \overset{\perp}{\ominus} \mathcal{U}^{(i)}, \ \mathcal{P}_j^{(i+1)} = \mathcal{P}_j^{(i)} \cap \mathcal{K}^{(i)}, \ j = 1, 2,$$

$$\mathcal{H}^{(1)} = \mathcal{P}_1^{(i+1)} \overset{\perp}{\oplus} \mathcal{P}_2^{(i+1)}, \ \mathcal{V}^{(i)} = \mathcal{K}^{(i)} \overset{\perp}{\ominus} \mathcal{H}^{(i+1)}.$$

(3) Generate – orthogonal $Z(2)$ subspace collection $\mathcal{H}^{(i+1)}, \mathcal{U}^{(i+1)}, \mathcal{E}^{(i+1)}, \mathcal{J}^{(i+1)}, \mathcal{P}_j^{(i+1)}, j = 1, 2$:

$$\mathcal{E}^{(i+1)} = \mathcal{E}^{(i)} \cap \mathcal{H}^{(i+1)}, \mathcal{J}^{(i+1)} = \mathcal{J}^{(i)} \cap \mathcal{H}^{(i+1)}, \mathcal{U}^{(i+1)} = \mathcal{H}^{(i+1)} \overset{\perp}{\ominus} [\mathcal{E}^{(i+1)} \overset{\perp}{\oplus} \mathcal{J}^{(i+1)}]$$

(4) Formulas, properties, & relations:

$$L_*^{(i+l)} = \Gamma_0^{(i+l)} L \Gamma_0^{(i+l)} - \Gamma_0^{(i+l)} L \Gamma_1^{(i+l)} \left(\Gamma_1^{(i+l)} L \Gamma_1^{(i+l)} \right)^{-1} \Gamma_1^{(i+l)} L \Gamma_0^{(i+l)}$$

$$= \Gamma_0^{(i+l)} \left[\left(\Gamma_0^{(i+l)} + \Gamma_2^{(i+l)} \right) L^{-1} \left(\Gamma_0^{(i+l)} + \Gamma_2^{(i+l)} \right) \right]^{-1} \Gamma_0^{(i+l)}, \ l = 0, 1$$

(!) $$Y_*^{(i)} = \Pi_1^{(i)} \Gamma_2^{(i)} \left(\Gamma_2^{(i)} \Pi_2^{(i)} L^{-1} \Pi_2^{(i)} \Gamma_2^{(i)} \right)^{-1} \Gamma_2^{(i)} \Pi_1^{(i)} = K^{(i)} L_*^{(i+1)} \left(K^{(i)} \right)^*,$$

$K^{(i)} \in \mathcal{L}\left(\mathcal{U}^{(i+1)}, \mathcal{V}^{(i)} \right)$ invertible, $K^{(i)} = -\left(\Pi_1^{(i)} \Gamma_1^{(i)} \Pi_1^{(i)} \right)^{-1} \Pi_1^{(i)} \Gamma_1^{(i)} \Pi_2^{(i)},$

(!) $$L_*^{(i)} = \Gamma_0^{(i)} L \Gamma_0^{(i)} - \Gamma_0^{(i)} L \Pi_1^{(i)} \left(\Pi_1^{(i)} L \Pi_1^{(i)} + Y_*^{(i)} \right)^{-1} \Pi_1^{(i)} L \Gamma_0^{(i)}.$$

Theorem ([2], Cassier-Milton-Welters)

Assume $\mathcal{H}^{(0)}, \mathcal{U}^{(0)}, \mathcal{E}^{(0)}, \mathcal{J}^{(0)}, \mathcal{P}_j^{(0)}$, $j = 1, 2$ *is an orthogonal* $Z(2)$ *subspace collection,* $\mathcal{U}^{(0)}$ *is finite-dimensional, the operator* L *on* $\mathcal{H}^{(0)}$ *is invertible [i.e.,* $L_1, L_2 \in \mathbb{C} \setminus \{0\}$*]. Let* $i \in \mathbb{N} \cup \{0\}$*. If, for each* $k = 0, \ldots, i$*, we have* $\ker L_{11}^{(k)} = \{0\}$ *(*$L_{11}^{(k)}$ *– restriction of* $\Gamma_1^{(k)} L \Gamma_1^{(k)}$ *to* $\mathcal{E}^{(k)}$*) [e.g., if* $(L_1, L_2) \in (\mathbb{C}^+)^2$*] then for all* $k = 0, \ldots, i$*, the* $L_*^{(k)} : \mathcal{U}^{(k)} \to \mathcal{U}^{(k)}$ *is given by the previous representation. Furthermore, if* $\mathcal{V}^{(k)} \cap \mathcal{J}^{(k)} = \{0\}$ *for each* $k = 0, \ldots, i$*, then the* $Y_*^{(k)} : \mathcal{V}^{(k)} \to \mathcal{V}^{(k)}$ *is given by the previous representation with* L *and* $L_*^{(k)}$*. Moreover, if*

$$\mathcal{V}^{(k)} \cap \mathcal{E}^{(k)} = \{0\}, \; \mathcal{P}_j^{(k)} \cap \mathcal{U}^{(k)} = \{0\}, \; j = 1, 2, \; k = 0, \ldots, i-1$$

then for all $k = 0, \ldots, i-1$ *we have*

$$\dim \mathcal{U}^{(0)} = \dim \mathcal{U}^{(k)} = \dim \mathcal{V}^{(k)} = \dim \mathcal{U}^{(i)},$$

& $K^{(k)} : \mathcal{U}^{(k+1)} \to \mathcal{V}^{(k)}$, $Y_*^{(k)}$, $L_*^{(k+1)}$ *are linked by the previous formulas.*

Conclusion

- Conditions on materials (L_1, L_2) such that L on $\mathcal{H}^{(0)}$ and $L_{11}^{(k)}$ on $\mathcal{E}^{(k)}$ are invertible, i.e., $\ker L_{11}^{(k)} = \{0\}$ as $L_{11}^{(k)}$ is a Fredholm operator of index 0, is "easier" to check then $\mathcal{V}^{(k)} \cap \mathcal{J}^{(k)} = \{0\}$ or $\mathcal{P}_j^{(k)} \cap \mathcal{U}^{(k)} = \{0\}$, which is "highly dependent on the physical problem and structure of the composite" (e.g., Remark 47 in [2] & Sec. 19.4 in [3]).
    - If all conditions hold $\Rightarrow$ Continued fraction expansion (!), infinite or terminating.
    - Conditions don't hold requires modification - future work.
- Continued fraction expansions - basis for bounding effective moduli of composites, see [3-6] (multiphase composites, see [4].)
- Future work - Extend all these results, i.e., field recursion method, to multiphase composites in abstract theory of composites.
- Open probs., e.g., [3, ch. 29]: "still an open (& interesting) question as to whether every matrix-valued analytic function $L_*(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the homogeneity, Herglotz, & normalization properties can be associated with a $(3, n)$-subspace collection."

Auxiliary Slides

Quintessential example: conductivity equation

- For any $F \in \mathcal{H} = \left[L^2_{\text{per}}(\mathcal{D})\right]^d$,

$$\Gamma_0 F = \langle F \rangle, \ \Gamma_1 F = \left[\nabla(\nabla\cdot\nabla)^{-1}\nabla\cdot\right] F, \ \Gamma_2 F = \left[\nabla\times(\nabla\times\nabla\times)^{-1}\nabla\times\right] F,$$
$$\Lambda_j F = \chi_j F.$$

- Why? Helmholtz decomposition for periodic functions and

$$\nabla\cdot(\nabla\times) = 0$$

and

$$\begin{aligned} \nabla\times(\nabla\times\nabla\times)^{-1}\nabla\times &= \nabla\times(-\nabla\cdot\nabla)^{-1}\nabla\times \\ \nabla\times\nabla\times &= \nabla(\nabla\cdot) - \nabla\cdot\nabla \\ \nabla\times(\nabla) &= 0. \end{aligned}$$