Exact relations for Green's functions in linear PDE and boundary field equalities: a generalization of conservation laws

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Conductivity

$$\underbrace{\mathbf{j}(\mathbf{x})}_{\mathbf{J}(\mathbf{x})} = \underbrace{\boldsymbol{\sigma}(\mathbf{x})}_{\mathbf{L}(\mathbf{x})}\underbrace{\mathbf{e}(\mathbf{x})}_{\mathbf{E}(\mathbf{x})}, \qquad \nabla\cdot\mathbf{j} = 0, \qquad \mathbf{e} = -\nabla V = 0,$$

Linear elasticity

$$\underbrace{\boldsymbol{\sigma}(\mathbf{x})}_{\mathbf{J}(\mathbf{x})} = \underbrace{\boldsymbol{\mathcal{C}}(\mathbf{x})}_{\mathbf{L}(\mathbf{x})}\underbrace{\boldsymbol{\epsilon}(\mathbf{x})}_{\mathbf{E}(\mathbf{x})}, \quad \nabla\cdot\boldsymbol{\sigma} = 0, \quad \boldsymbol{\epsilon} = [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]/2,$$

Magnetoelectricity

$$\underbrace{\begin{pmatrix}\mathbf{d}\\ \mathbf{b}\end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix}\boldsymbol{\varepsilon} & \boldsymbol{\beta}\\ \boldsymbol{\beta}^T & \boldsymbol{\mu}\end{pmatrix}}_{\mathbf{L}(\mathbf{x})}\underbrace{\begin{pmatrix}\mathbf{e}\\ \mathbf{h}\end{pmatrix}}_{\mathbf{E}(\mathbf{x})}, \quad \nabla\cdot\mathbf{d} = \nabla\cdot\mathbf{b} = 0, \quad \mathbf{e} = -\nabla V, \quad \mathbf{h} = -\nabla\psi,$$

Piezoelectricity

$$\underbrace{\begin{pmatrix} \epsilon \\ \mathrm{d} \end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix} \boldsymbol{\mathcal{S}} & \boldsymbol{\mathcal{D}} \\ \boldsymbol{\mathcal{D}}^T & \varepsilon \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \underbrace{\begin{pmatrix} \boldsymbol{\sigma} \\ \mathrm{e} \end{pmatrix}}_{\mathbf{E}(\mathbf{x})},$$

$$\boldsymbol{\epsilon} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2, \quad \nabla \cdot \mathbf{d} = 0, \quad \nabla \cdot \boldsymbol{\sigma} = 0, \quad \mathbf{e} = -\nabla V.$$

Additional coupling with magnetic fields

$$\begin{pmatrix} \boldsymbol{\epsilon}(\mathbf{x}) \\ \mathbf{d}(\mathbf{x}) \\ \mathbf{b}(\mathbf{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{\mathcal{S}}(\mathbf{x}) & \boldsymbol{\mathcal{D}}(\mathbf{x}) & \boldsymbol{\mathcal{Q}}(\mathbf{x}) \\ \boldsymbol{\mathcal{D}}^T(\mathbf{x}) & \boldsymbol{\varepsilon}(\mathbf{x}) & \boldsymbol{\beta}(\mathbf{x}) \\ \boldsymbol{\mathcal{Q}}^T(\mathbf{x}) & \boldsymbol{\beta}^T(\mathbf{x}) & \boldsymbol{\mu}(\mathbf{x}) \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \begin{pmatrix} \boldsymbol{\sigma}(\mathbf{x}) \\ \mathbf{e}(\mathbf{x}) \\ \mathbf{h}(\mathbf{x}) \end{pmatrix},$$

Thermoelasticity

$$\begin{pmatrix} \boldsymbol{\epsilon}(\boldsymbol{x}) \\ \varsigma(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{S}}(\boldsymbol{x}) & \boldsymbol{\alpha}(\boldsymbol{x}) \\ \boldsymbol{\alpha}(\boldsymbol{x}) & c(\boldsymbol{x})/T_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}(\boldsymbol{x}) \\ \theta \end{pmatrix}, \quad \text{with} \quad \begin{array}{c} \nabla \cdot \boldsymbol{\tau} = 0, \\ \boldsymbol{\epsilon} = [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]/2, \end{array}$$

Equivalent to Poroelasticity

$$\begin{pmatrix} \boldsymbol{\epsilon}_s(\boldsymbol{x}) \\ -\zeta(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{S}}(\boldsymbol{x}) & \boldsymbol{\alpha}(\boldsymbol{x}) \\ \boldsymbol{\alpha}(\boldsymbol{x}) & c(\boldsymbol{x}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}_c(\boldsymbol{x}) \\ -p_f \end{pmatrix}, \quad \boldsymbol{\epsilon}_s = [\nabla \boldsymbol{u}_s + (\nabla \boldsymbol{u}_s)^T]/2, \quad \nabla \cdot \boldsymbol{\tau}_c = 0,$$

Fields independent of x_3

Elasticity looks like Piezoelectricity

$$\begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \sqrt{2}\tau_{12} \\ e_1' \\ e_2' \\ \tau_{33} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & \sqrt{2}C_{1112} & \sqrt{2}C_{1123} & -\sqrt{2}C_{1113} & C_{1133} \\ C_{1122} & C_{2222} & \sqrt{2}C_{2212} & \sqrt{2}C_{2223} & -\sqrt{2}C_{2213} & C_{2233} \\ \sqrt{2}C_{1112} & \sqrt{2}C_{2212} & 2C_{1212} & 2C_{2312} & -2C_{1312} & \sqrt{2}C_{3312} \\ \sqrt{2}C_{1123} & \sqrt{2}C_{2223} & 2C_{2312} & 2C_{2323} & -2C_{2313} & \sqrt{2}C_{3323} \\ -\sqrt{2}C_{1113} & -\sqrt{2}C_{2213} & -2C_{1312} & -2C_{2313} & 2C_{1313} & -\sqrt{2}C_{3313} \\ C_{1133} & C_{2233} & \sqrt{2}C_{3312} & \sqrt{2}C_{3323} & -\sqrt{2}C_{3313} & C_{3333} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \sqrt{2}\epsilon_{12} \\ d_1' \\ d_2' \\ \epsilon_{33} \end{pmatrix}$$

$$\mathbf{e}' = \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} \sqrt{2}\tau_{23} \\ -\sqrt{2}\tau_{13} \end{pmatrix}, \quad \mathbf{d}' = \begin{pmatrix} d_1' \\ d_2' \end{pmatrix} = \begin{pmatrix} \sqrt{2}\epsilon_{23} \\ -\sqrt{2}\epsilon_{13} \end{pmatrix}$$

$$\frac{\partial e_1'}{\partial x_2} - \frac{\partial e_2'}{\partial x_1} = 0, \quad \frac{\partial d_1'}{\partial x_1} + \frac{\partial d_2'}{\partial x_2} = 0. \qquad \epsilon_{33} = \text{constant}$$

The Cherkaev-Gibiansky Trick:

$$\mathbf{d}(\mathbf{x}) = \varepsilon(\mathbf{x})\mathbf{e}(\mathbf{x}), \qquad \nabla \cdot \mathbf{j} = 0, \qquad \mathbf{e} = -\nabla V = 0,$$

$$\mathbf{d} = \mathbf{d}' + i\mathbf{d}'', \quad \mathbf{e} = \mathbf{e}' + i\mathbf{e}'', \quad V = V' + iV'', \quad \varepsilon = \varepsilon' + i\varepsilon''$$

$$\varepsilon'' > 0$$

$$\underbrace{\begin{pmatrix} \mathbf{e}'' \\ \mathbf{d}'' \end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix} [\varepsilon'']^{-1} & [\varepsilon'']^{-1}\varepsilon' \\ \varepsilon'[\varepsilon'']^{-1} & \varepsilon'' + \varepsilon'[\varepsilon'']^{-1}\varepsilon' \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \underbrace{\begin{pmatrix} -\mathbf{d}' \\ \mathbf{e}' \end{pmatrix}}_{\mathbf{E}(\mathbf{x})},$$

$$\mathbf{L}(\mathbf{x}) > 0$$

Can easily be extended to other non-self-adjoint equations:

$$\boldsymbol{j}(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}(\boldsymbol{x}), \quad \nabla \cdot \boldsymbol{j}(\boldsymbol{x}) = 0, \quad \boldsymbol{e}(\boldsymbol{x}) = \nabla\phi(\boldsymbol{x}),$$

$$\boldsymbol{j}'(\boldsymbol{x}) = \boldsymbol{\sigma}^T(\boldsymbol{x})\boldsymbol{e}'(\boldsymbol{x}), \quad \nabla \cdot \boldsymbol{j}'(\boldsymbol{x}) = 0, \quad \boldsymbol{e}'(\boldsymbol{x}) = \nabla\phi'(\boldsymbol{x}),$$

$$\boldsymbol{\sigma}_s(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{\sigma}^T(\boldsymbol{x}), \quad \boldsymbol{\sigma}_a(\boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{x}) - \boldsymbol{\sigma}^T(\boldsymbol{x}),$$

$$\boldsymbol{j}_s = (\boldsymbol{j} + \boldsymbol{j}')/2, \quad \boldsymbol{j}_a = (\boldsymbol{j} - \boldsymbol{j}')/2, \quad \boldsymbol{e}_s = (\boldsymbol{e} + \boldsymbol{e}')/2, \quad \boldsymbol{e}_a = (\boldsymbol{e} - \boldsymbol{e}')/2,$$

$$\begin{pmatrix} -\boldsymbol{j}_s \\ \boldsymbol{j}_a \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}_s & -\boldsymbol{\sigma}_a \\ \boldsymbol{\sigma}_a & \boldsymbol{\sigma}_s \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_s \\ \boldsymbol{e}_a \end{pmatrix},$$

Saddle shaped

$$\begin{pmatrix} \boldsymbol{e}_s \\ \boldsymbol{j}_a \end{pmatrix} = \boldsymbol{\mathcal{L}} \begin{pmatrix} \boldsymbol{j}_s \\ \boldsymbol{e}_a \end{pmatrix}, \quad \begin{matrix} \nabla \times \boldsymbol{e}_s = 0, \\ \nabla \cdot \boldsymbol{j}_a = 0, \end{matrix} \quad \begin{matrix} \boldsymbol{j}_s = \nabla \times \boldsymbol{\psi}_s, \\ \boldsymbol{e}_a = \nabla\phi_a. \end{matrix} \quad \boldsymbol{\mathcal{L}} = \begin{pmatrix} \boldsymbol{\sigma}_s^{-1} & -\boldsymbol{\sigma}_s^{-1}\boldsymbol{\sigma}_a \\ \boldsymbol{\sigma}_a\boldsymbol{\sigma}_s^{-1} & \boldsymbol{\sigma}_s - \boldsymbol{\sigma}_a\boldsymbol{\sigma}_s^{-1}\boldsymbol{\sigma}_a \end{pmatrix}$$

Quadratic form associated with $\mathcal{L}$ is positive definite:

$$\begin{pmatrix} \mathbf{j}_s \\ \mathbf{e}_a \end{pmatrix} \cdot \mathcal{L} \begin{pmatrix} \mathbf{j}_s \\ \mathbf{e}_a \end{pmatrix} = (\mathbf{j}_s - \sigma_a^T \mathbf{e}_a) \cdot \sigma_s^{-1} (\mathbf{j}_s - \sigma_a \mathbf{e}_a) + \mathbf{e}_a \cdot \sigma_s \mathbf{e}_a$$

Abstract theory of effective tensors

$\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}$, taking values in a N-dimensional tensor space $\mathcal{T}$

$\mathcal{U}$ is the space of constant fields

Γ_0 projects onto $\mathcal{U}$: $\Gamma_0 \mathbf{P} = \langle \mathbf{P} \rangle$
Γ_1 projects onto $\mathcal{E}$
$\Gamma_2 = I - \Gamma_1 - \Gamma_0$ projects onto $\mathcal{J}$

Γ_1 acts locally in Fourier space: If $\mathbf{Q} = \Gamma_1 \mathbf{P}$ then $\widehat{\mathbf{Q}}(\mathbf{k}) = \Gamma_1(\mathbf{k})\widehat{\mathbf{P}}(\mathbf{k})$

e.g. for conductivity

$$\boldsymbol{\Gamma}_1(\boldsymbol{k}) = \frac{\boldsymbol{k} \otimes \boldsymbol{k}}{|\boldsymbol{k}|^2}, \quad \boldsymbol{\Gamma}_2(\boldsymbol{k}) = \boldsymbol{I} - \frac{\boldsymbol{k} \otimes \boldsymbol{k}}{|\boldsymbol{k}|^2}. \quad \text{when } \mathbf{k} \neq 0, \text{ zero when } \mathbf{k} = 0,$$

Given $\mathbf{L} : \mathcal{H} \to \mathcal{H}$ generally satisfying some boundedness and coercivity conditions

Then for every $\boldsymbol{E}_0 \in \mathcal{U}$ solve

$$\boldsymbol{J} = \boldsymbol{L}\boldsymbol{E} \qquad \boldsymbol{J} \in \mathcal{U} \oplus \mathcal{J}, \quad \boldsymbol{E} \in \mathcal{U} \oplus \mathcal{E}, \qquad \boldsymbol{\Gamma}_0 \boldsymbol{E} = \boldsymbol{E}_0$$

compute

$$\boldsymbol{J}_0 = \boldsymbol{\Gamma}_0 \boldsymbol{J},$$

as it depends linearly on $\boldsymbol{E}_0$ we may write

$$\boldsymbol{J}_0 = \boldsymbol{L}_* \boldsymbol{E}_0 \quad \text{which defines } \mathbf{L}_*$$

A related Gamma operator.

Given a reference tensor $\mathbf{L}_0 > 0$

$$\boldsymbol{E}' = \boldsymbol{\Gamma} \boldsymbol{P} \text{ if and only if } \boldsymbol{E}' \in \mathcal{E} \text{ and } \boldsymbol{P} - \boldsymbol{L}_0 \boldsymbol{E}' \in \mathcal{U} \oplus \mathcal{J}.$$

In Fourier space:

$$\boldsymbol{\Gamma}_1(\boldsymbol{k}) \widehat{\boldsymbol{E}}'(\boldsymbol{k}) = \widehat{\boldsymbol{E}}'(\boldsymbol{k}) \text{ and } \boldsymbol{\Gamma}_1(\boldsymbol{k})(\widehat{\boldsymbol{P}}(\boldsymbol{k}) - \boldsymbol{L}_0 \widehat{\boldsymbol{E}}'(\boldsymbol{k})) = 0 \text{ for all } \boldsymbol{k} \neq 0,$$

Hence:

$$\boldsymbol{\Gamma}(\boldsymbol{k}) = \boldsymbol{\Gamma}_1(\boldsymbol{k})[\boldsymbol{\Gamma}_1(\boldsymbol{k}) \boldsymbol{L}_0 \boldsymbol{\Gamma}_1(\boldsymbol{k})]^{-1} \boldsymbol{\Gamma}_1(\boldsymbol{k}),$$

Inverse is on the space $\mathcal{E}_{\mathbf{k}}$ onto which $\Gamma_1(\mathbf{k})$ projects

Conductivity: $\Gamma(\mathbf{k}) = \frac{\mathbf{k} \otimes \mathbf{k}}{\mathbf{k} \cdot \mathbf{L}_0 \mathbf{k}}$ for $\mathbf{k} \neq 0$

Use: Solution for the fields and effective tensor,

$$\boldsymbol{P}(\boldsymbol{x}) = (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{x}) - \boldsymbol{L}_0\boldsymbol{E}(\boldsymbol{x}).$$

$$\boldsymbol{\Gamma}\boldsymbol{P} = \langle \boldsymbol{E} \rangle - \boldsymbol{E},$$

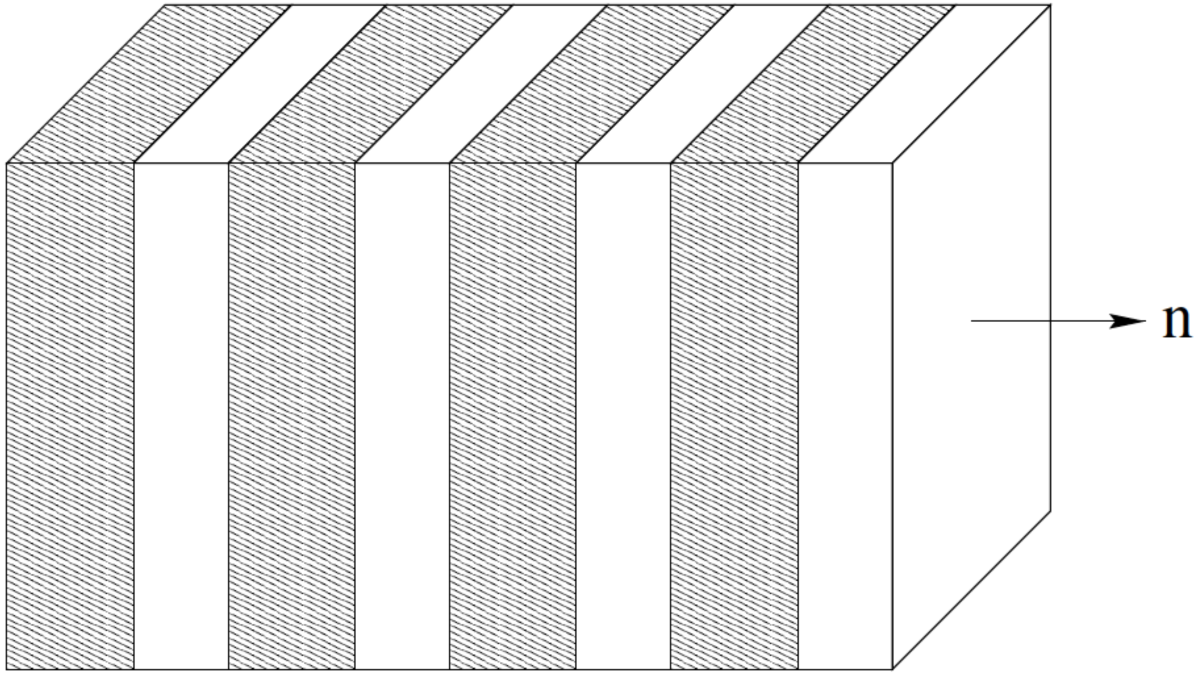
$$[\boldsymbol{I} + (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{\Gamma}]\boldsymbol{P} = (\boldsymbol{L} - \boldsymbol{L}_0)\langle \boldsymbol{E} \rangle.$$

$$\boldsymbol{P} = [\boldsymbol{I} + (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{\Gamma}]^{-1}(\boldsymbol{L} - \boldsymbol{L}_0)\langle \boldsymbol{E} \rangle.$$

$$\boldsymbol{L}_* = \boldsymbol{L}_0 + \langle [\boldsymbol{I} + (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{\Gamma}]^{-1}(\boldsymbol{L} - \boldsymbol{L}_0) \rangle$$

Does not depend on $\mathbf{L}_0$

Action of Gamma especially simple in a laminate geometry

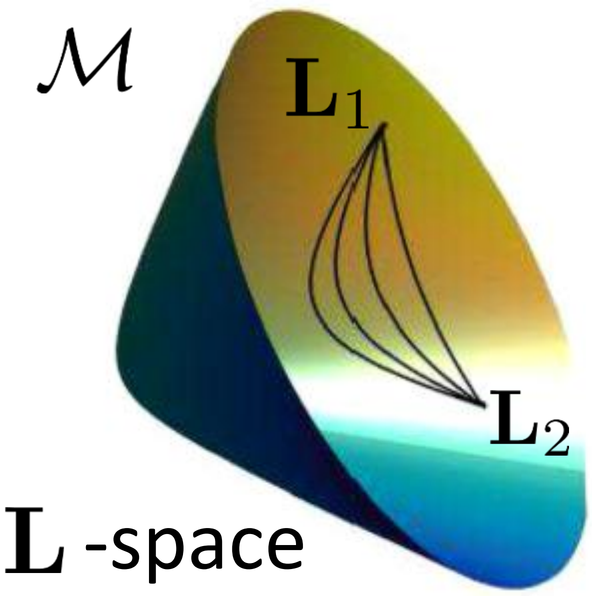


For all fields $\mathbf{P}(\mathbf{x})$ that only depend on $\mathbf{x} \cdot \mathbf{n}$

$$\Gamma\mathbf{P} = \Gamma(\mathbf{n})(\mathbf{P} - \langle\mathbf{P}\rangle)$$

Classic example of an exact relation: Keller-Mendelson-Dykhne relation for 2-dimensional conductivity

$$\det \boldsymbol{\sigma}_* = c \qquad \text{when } \det \boldsymbol{\sigma}(\mathbf{x}) = c \ \text{ for all } \ \mathbf{x}.$$



Picture courtesy of Y. Grabovsky

The manifold
$$\mathcal{M} = \{\boldsymbol{\sigma} : \det \boldsymbol{\sigma} = c\}$$
is stable under homogenization.

Goal of the theory of exact relations: identify manifolds of tensors, $\mathcal{M}$ that are Stable under homogenization

Given periodic $\mathbf{L}(\mathbf{x})$ with $\mathbf{L}(\mathbf{x}) \in \mathcal{M}\ \forall \mathbf{x}$ then $\mathbf{L}_* \in \mathcal{M}$

Classic example of an exact link: Keller-Matheron-Mendelson reciprocal relation for 2-dimensional conductivity

Consider two conductivity problems with tensor fields $\boldsymbol{\sigma}(\mathbf{x})$ and $\widetilde{\boldsymbol{\sigma}}(\mathbf{x})$ related via

$$\widetilde{\boldsymbol{\sigma}}(\mathbf{x}) = [\mathbf{R}_\perp^T \boldsymbol{\sigma}(\mathbf{x}) \mathbf{R}_\perp]^{-1}, \quad \text{with} \quad \mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then the effective conductivities are related in the same way: $\widetilde{\boldsymbol{\sigma}}_* = [\mathbf{R}_\perp^T \boldsymbol{\sigma}_* \mathbf{R}_\perp]^{-1}$.

Treat it as a trivial "coupled field problem" with no couplings!

$$\begin{pmatrix} \mathbf{j}(\mathbf{x}) \\ \widetilde{\mathbf{j}}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}(\mathbf{x}) & 0 \\ 0 & \widetilde{\boldsymbol{\sigma}}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} -\nabla V(\mathbf{x}) \\ -\nabla \widetilde{V}(\mathbf{x}) \end{pmatrix}$$

Then we can take $\mathcal{M}$ to consist of all matrices of the form $\mathbf{L} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & [\mathbf{R}_\perp^T \boldsymbol{\sigma} \mathbf{R}_\perp]^{-1} \end{pmatrix}$

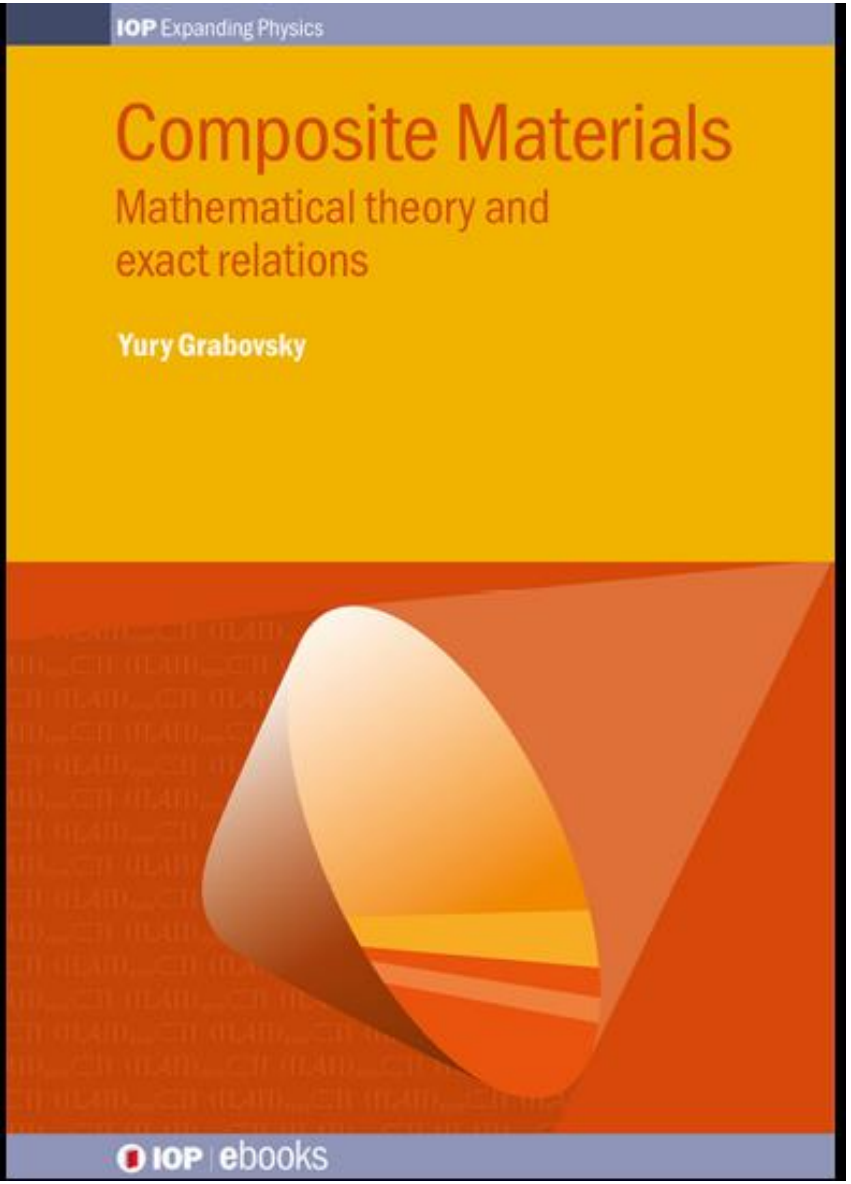
Many Scientists discovered exact relations one at a time:

Benveniste (Piezoelectricity)
Bergman (Hall-effect)
Berryman (Poroelasticity)
Chen (Coupled equations, Elasticity)
Cherkaev (Plate equations)
Cribb (Thermoelasticity)
Dvorak (Piezoelectricity)
Dykhne (Conductivity, Hall Effect)
Gassman (Poroelasticity)
Hashin (Elasticity)
He (Elasticity)
Helsing (Elasticity)
Hill (Elasticity)
Keller (Conductivity)

Levin (Thermoelasticity)
Lurie (Plate equations, Elasticity)
Matheron (Conductivity)
Milgrom (Coupled equations)
Milton (Complex conductivity, Hall effect, elasticity)
Movchan (elasticity)
Murat (Null-Lagrangians)
Shklovskii (Hall effect)
Shtrikman (Coupled equations)
Straley (Coupled Equations)
Strelniker (Hall effect)
Rosen (Thermoelasticity)
Schulgasser (Piezoelectricity)
Tartar (Null-Lagrangians)

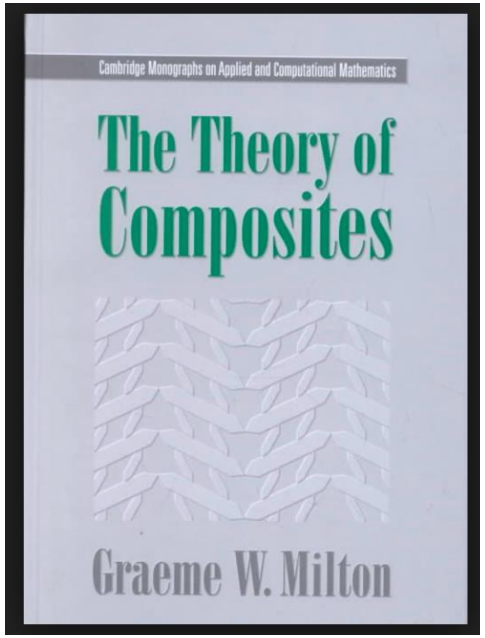
Yury Grabovsky and coworkers discovered hundreds,
(many intersections of more fundamental ones)

Theory of exact relations for composites reviewed in the books:



Grabovsky 2016

Milton 2002



Relevant Chapters:

3. **Duality transformations in two-dimensional media**
4. **Translations and equivalent media**
5. **Some microstructure-independent exact relations**
6. **Exact relations for coupled equations**
9. **Laminate materials**
12. **Reformulating the problem of finding effective tensors**
14. **Series expansions for the fields and effective tensors**
17. **The general theory of exact relations and links between effective tensors**

First major breakthrough: Grabovsky (1998)

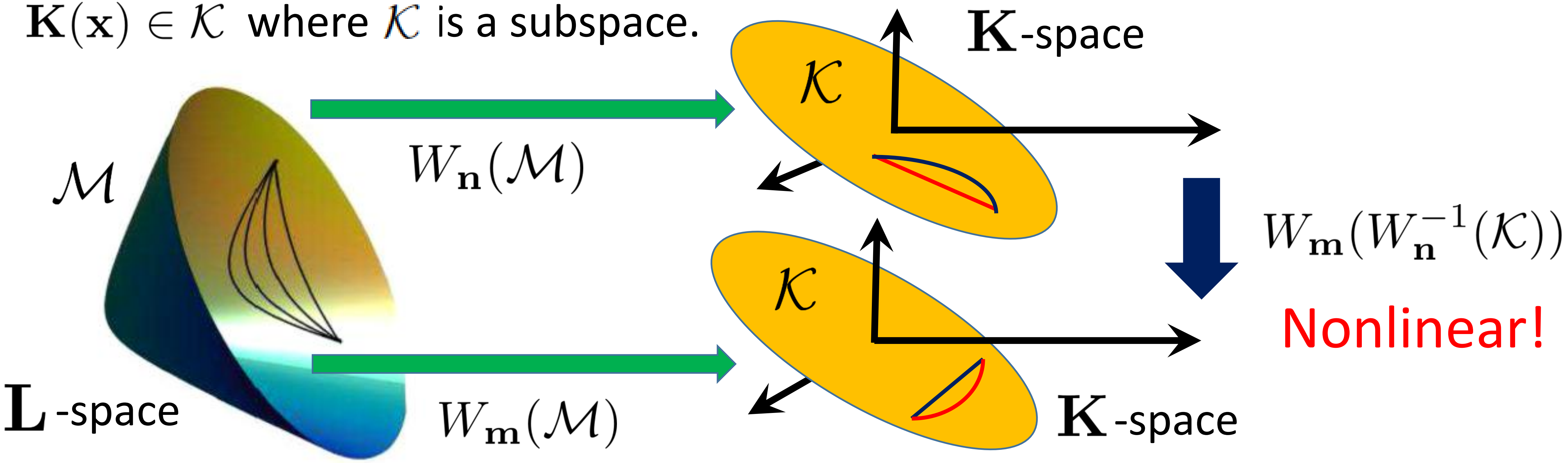
As an exact relation holds for all geometries it must certainly hold for laminate geometries

The transformation (Milton, 1990; Zhikov 1991)

$$W_{\boldsymbol{n}}(\boldsymbol{L}) = [\boldsymbol{I} + (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{\Gamma}(\boldsymbol{n})]^{-1}(\boldsymbol{L} - \boldsymbol{L}_0) = \boldsymbol{K}, \quad \mathbf{L}_0 \in \mathcal{M}$$

converts lamination in direction $\boldsymbol{n}$ to a linear average: $W_{\mathbf{n}}(\mathbf{L}_*) = \langle W_{\mathbf{n}}(\mathbf{L}) \rangle$

Therefore in $\boldsymbol{K}$-space an exact relation must be a linear relation, $\mathbf{K}_* \in \mathcal{K}$, when $\mathbf{K}(\mathbf{x}) \in \mathcal{K}$ where $\mathcal{K}$ is a subspace.



Expansion of the non-linear transformation. Set $\boldsymbol{A}(\boldsymbol{m}) = \boldsymbol{\Gamma}(\boldsymbol{n}) - \boldsymbol{\Gamma}(\boldsymbol{m})$.

$$\begin{aligned} W_{\boldsymbol{m}}(W_{\boldsymbol{n}}^{-1}(\epsilon\boldsymbol{K})) &= \epsilon\boldsymbol{K}\{\boldsymbol{I} - [\boldsymbol{\Gamma}(\boldsymbol{n}) - \boldsymbol{\Gamma}(\boldsymbol{m})]\epsilon\boldsymbol{K}\}^{-1} \\ &= \epsilon\boldsymbol{K} + \epsilon^2\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} + \epsilon^3\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} \\ &\quad + \epsilon^4\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} + \cdots, \end{aligned}$$

So $\mathcal{K}$ independent of $\boldsymbol{n}$ and

$\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} \in \mathcal{K}$ for all $\boldsymbol{m}$ and for all $\boldsymbol{K} \in \mathcal{K}$. (Necessary Condition)

Then all terms in the series lie in $\mathcal{K}$

The search for candidate exact relations becomes a search for subspaces $\mathcal{K}$ satisfying this algebraic constraint.

Example: Two-dimensional conductivity

Take $\mathbf{L}_0 = \sigma_0 \mathbf{I}$. Then

$$\mathbf{A}(\mathbf{m}) = \frac{\mathbf{n}\mathbf{n}^T}{(\sigma_0 \mathbf{n} \cdot \mathbf{n})} - \frac{\mathbf{m}\mathbf{m}^T}{(\sigma_0 \mathbf{m} \cdot \mathbf{m})}$$

is trace-free and symmetric. We can take $\mathcal{K}$ as the space of 2×2 symmetric trace-free matrices.

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} b & c \\ c & -b \end{pmatrix} = \begin{pmatrix} ab & ac \\ -ac & ab \end{pmatrix}$$

But with 3 matrices:

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} b & c \\ c & -b \end{pmatrix} \begin{pmatrix} d & e \\ e & -d \end{pmatrix} = \begin{pmatrix} abd + ace & abe - acd \\ abe - acd & -abd - ace \end{pmatrix}$$

Then $\mathcal{M} = W_{\mathbf{n}}^{-1}(\mathcal{K})$ consists of 2×2 symmetric matrices with determinant σ_0^2.

Second major breakthrough: (Grabovsky, Milton, Sage 2000)

The transformation $W_n(L)$ and series expansions of Milton and Golden (1990) [that formed the basis of the rapidly converging FFT approach of Eyre and Milton (1999)] provided the essential clues for a condition that guarantees a candidate exact relation holds for all geometries not just **laminate** ones.

$$\mathbf{K}(\mathbf{x}) = W_{\mathbf{M}}(\mathbf{L}(\mathbf{x})) = [\mathbf{I} + (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)$$

$$\mathbf{K}_* = W_{\mathbf{M}}(\mathbf{L}_*) = [\mathbf{I} + (\mathbf{L}_* - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}_* - \mathbf{L}_0)$$

Series expansion: let $AP = M(P - \langle P \rangle) - \Gamma P$ define A (acts locally in Fourier space)

$$K_* = \langle [I - KA]^{-1} K \rangle = \sum_{j=0}^{\infty} \langle (KA)^j K \rangle$$

$$\begin{aligned} A(k) &= M - \Gamma(k) && \text{for } k \neq 0, \\ &= 0 && \text{for } k = 0. \end{aligned}$$

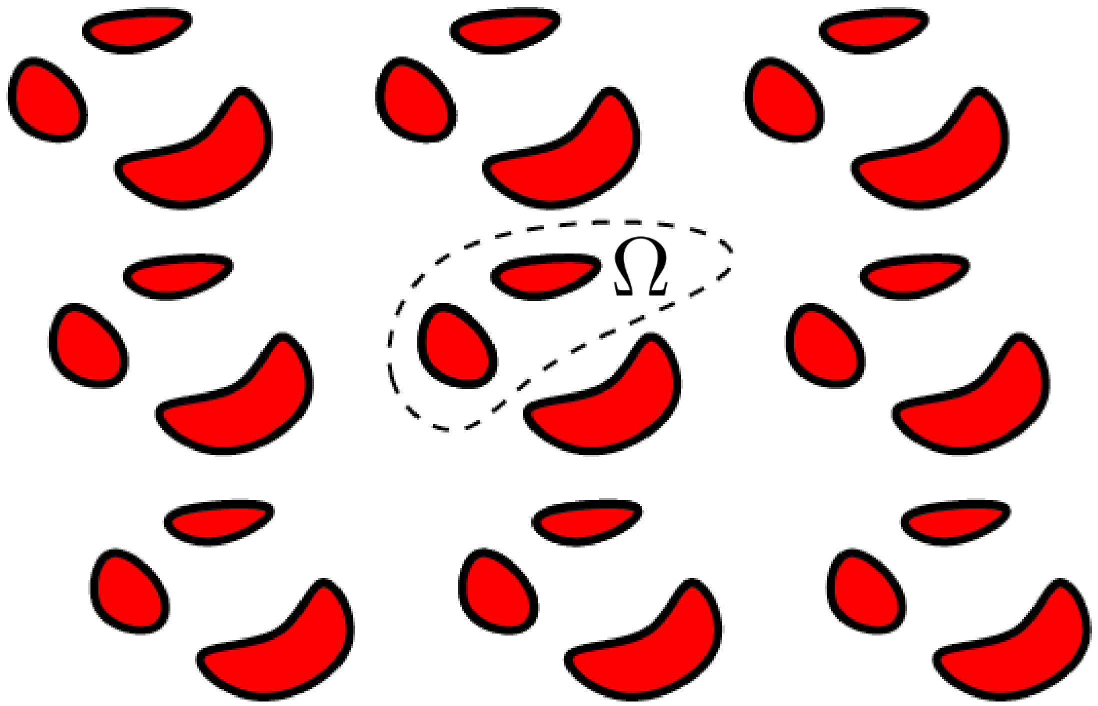
$$\boxed{K_1[M - \Gamma(n)]K_2 \in \overline{\mathcal{K}} \quad \text{and for all } K_1, K_2 \in \overline{\mathcal{K}}, \quad \text{(Sufficient Condition)}}$$

Appropriately defined "polarization fields" within the material also are constrained to take values in $\overline{\mathcal{K}}$

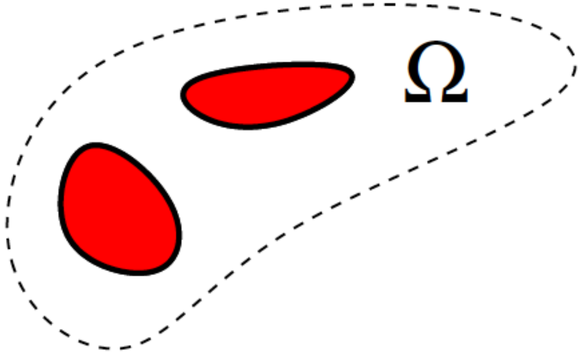
$\mathcal{K}$ can be taken to consist of all symmetric matrices in $\overline{\mathcal{K}}$

If the series does not converge, use analytic continuation

Third Major Breakthrough (Milton and Onofrei, 2019)



Suppose we have a periodic composite for which an exact relation holds,
And hence the "polarization field" takes values in $\overline{\mathcal{K}}$ at each $\boldsymbol{x}$ in Ω



The region Ω marked by the "dashed lines" does not know it is in a periodic medium, but the boundary conditions on the potentials or fluxes on this dashed boundary must be such to force the "polarization field" inside Ω to take values in $\overline{\mathcal{K}}$ and this gives us additional Information about the boundary fields.

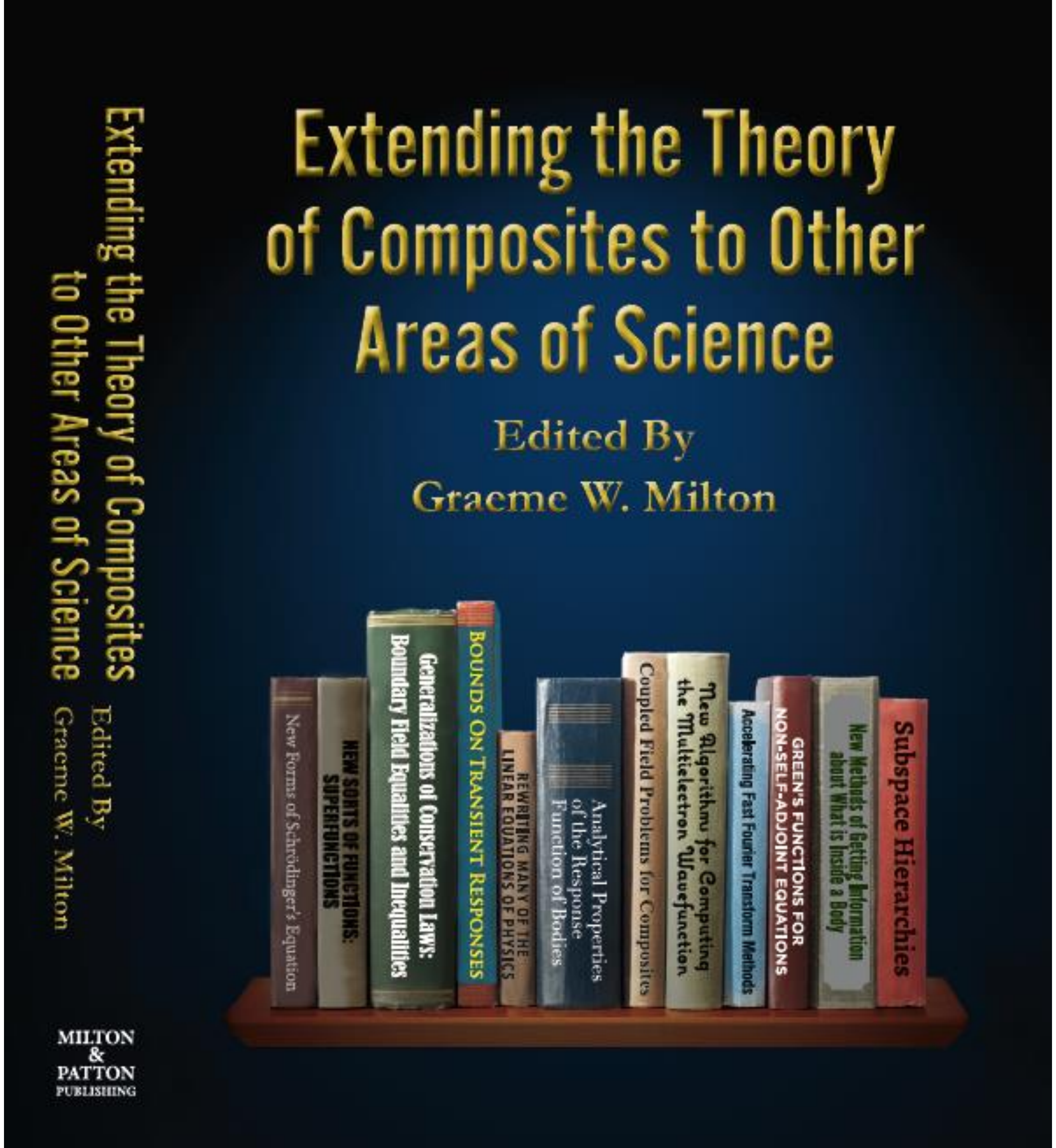
Aim: identify these boundary conditions, and find the associated exact identities (boundary field equalities) satisfied by the "Dirichlet-to-Neumann map".

A new perspective on **conservation laws**:
Boundary field equalities and inequalities

$$\text{If } \nabla \cdot \mathbf{Q} = 0 \text{ in } \Omega \text{ then } \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{Q} = 0$$

$$\text{If } \nabla \cdot \mathbf{Q} \geq 0 \text{ in } \Omega \text{ then } \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{Q} \geq 0$$

Requires information about what is happening inside Ω namely that $\nabla \cdot \mathbf{Q} = 0$ or $\nabla \cdot \mathbf{Q} \geq 0$ in Ω .



4 Chapters with:
Maxence Cassier
Ornella Mattei
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Aaron Welters

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Available on my website

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Only $80.

Reviewed by Pradeep Sharma
(Applied Mech. Reviews)
and Yury Grabovsky
(SIAM Book Reviews)

Are there other boundary field equalities or inequalities that use partial information about what is inside the body?

Simple example, our theory much more powerful

$$\begin{pmatrix}\mathbf{j}(\mathbf{x})\\ \widetilde{\mathbf{j}}(\mathbf{x})\end{pmatrix}=\begin{pmatrix}a(\mathbf{x})\mathbf{I} & c(\mathbf{x})\mathbf{I}\\ c(\mathbf{x})\mathbf{I} & b(\mathbf{x})\mathbf{I}\end{pmatrix}\begin{pmatrix}-\nabla V(\mathbf{x})\\ -\nabla\widetilde{V}(\mathbf{x})\end{pmatrix},\quad \nabla\cdot\mathbf{j}=0,\quad \nabla\cdot\widetilde{\mathbf{j}}$$

$$\mathbf{M}(\mathbf{x})=\begin{pmatrix}a(\mathbf{x})\mathbf{I} & c(\mathbf{x})\mathbf{I}\\ c(\mathbf{x})\mathbf{I} & b(\mathbf{x})\mathbf{I}\end{pmatrix},\quad \beta\mathbf{I}\geq\mathbf{M}(\mathbf{x})\geq\alpha\mathbf{I}\text{ for some }\beta>\alpha>0$$

Following the ideas of Straley, Milgrom and Shtrikman suppose there is a matrix $\mathbf{W}$ such that

$$\mathbf{W}\mathbf{M}(\mathbf{x})\mathbf{W}^T=\begin{pmatrix}a'(\mathbf{x})\mathbf{I} & 0\\ 0 & b'(\mathbf{x})\mathbf{I}\end{pmatrix}$$

$$\boxed{\begin{pmatrix}V(\mathbf{x})\\ \widetilde{V}(\mathbf{x})\end{pmatrix}=\mathbf{W}^T\begin{pmatrix}f(\mathbf{x})\\ 0\end{pmatrix}\text{ for all }\mathbf{x}\in\partial\Omega \quad\Rightarrow\quad W_{21}[\mathbf{n}\cdot\mathbf{j}(\mathbf{x})]+W_{22}[\mathbf{n}\cdot\widetilde{\mathbf{j}}(\mathbf{x})]=0\quad\text{for all }\mathbf{x}\in\partial\Omega}$$

Another simple example: in two dimensions suppose

$$c(\mathbf{x}) = 0, \quad b(\mathbf{x}) = \alpha^2/a(\mathbf{x})$$

Following ideas of Keller, Dykhne, Matheron and Mendelson, we have the boundary field equality

$$\boxed{\mathbf{n}\cdot\widetilde{\mathbf{j}}(\mathbf{x}) = \alpha\mathbf{t}\cdot\nabla V(\mathbf{x}) \text{ when } \mathbf{t}\cdot\nabla\widetilde{V}(\mathbf{x}) = -\alpha^{-1}\mathbf{n}\cdot\mathbf{j}(\mathbf{x})}$$

$\mathbf{n}$ normal to $\partial\Omega$, $\mathbf{t}$ tangential to $\partial\Omega$,

It's due to the fact that the equations are satisfied with

$$\widetilde{V}(\mathbf{x}) = -\alpha^{-1}\mathbf{R}_\perp\mathbf{j}(\mathbf{x}), \quad \widetilde{\mathbf{j}}(\mathbf{x}) = -\alpha\mathbf{R}_\perp\nabla V(\mathbf{x}),$$

where

$$\mathbf{R}_\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Key point:

These new boundary field equalities that in some sense generalize the divergence theorem, do not result from "integration by parts" but rather from algebraic properties tied with the operator $\mathbf{\Gamma}$ that is associated with the differential constraints satisfied by the fields on the left and right of the constitutive law.

There are "hidden identities" that go beyond integration by parts and still allow one to deduce exact identities satisfied by the fields at the boundary of a region Ω

Generalized viewpont of boundary field inequalities One can eliminate $\mathbf{L}(\mathbf{x})$ from the the constitutive law $\mathbf{J}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{E}(\mathbf{x})$ and just view the constaint on $\mathbf{L}(\mathbf{x})$ that $\mathbf{L}(\mathbf{x}) \in \mathcal{M}$ as a constraint on the field pairs $(\mathbf{J}(\mathbf{x}), \mathbf{E}(\mathbf{x}))$ that is independent of $\mathbf{x}$.

For instance, if

- $\mathbf{E}$ consists of potential gradients,
- $\mathbf{J}$ consists of divergenge free fields (fluxes) that themselves may be expressed as curl's of additional potentials

Then collecting all potentials together as some grand potential $\mathbf{U}$,
The field constraints imply

$$\nabla \mathbf{U}(\mathbf{x}) \in \mathcal{A} \text{ for all } \mathbf{x} \in \Omega$$

where $\mathcal{A}$ is some non-linear manifold (determined by $\mathcal{M}$).

Then with appropriate nonlocal boundary conditions on the surface potential $\mathbf{U}(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$ we obtain the constraint that

$$\nabla \mathbf{U}(\mathbf{x}) \in \mathcal{C} \text{ for all } \mathbf{x} \in \Omega$$

for some appropriately defined subspace $\mathcal{C}$, and this in turns constrains the tangential derivatives of $\mathbf{U}$ at $\partial\Omega$: these are the boundary field equalities.

Note that if $\mathbf{N}$ is perpendicular to $\mathcal{C}$ then

$$0 = \mathrm{Tr}[\mathbf{N}(\nabla \mathbf{U}(\mathbf{x})] = \nabla \cdot (\mathbf{U}(\mathbf{x})\mathbf{N}^T)$$

So there are additional divergence free fields and additional associated boundary field equalities.

Formulation

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} \sum_{\beta=1}^{m} L_{i\alpha j\beta}(\mathbf{x}) \frac{\partial u_\beta(\mathbf{x})}{\partial x_j} \right) = f_\alpha(\mathbf{x}), \quad \alpha = 1, 2, \ldots, m,$$

Rewrite as

$$J_{i\alpha}(\mathbf{x}) = \sum_{j=1}^{d} \sum_{\beta=1}^{m} L_{i\alpha j\beta}(\mathbf{x}) E_{j\beta}(\mathbf{x}) - h_{i\alpha}(\mathbf{x}), \quad E_{j\beta}(\mathbf{x}) = \frac{\partial u_\beta(\mathbf{x})}{\partial x_j}, \quad \sum_{i=1}^{d} \frac{\partial J_{i\alpha}(\mathbf{x})}{\partial x_i} = 0,$$

with

$$\sum_{i=1}^{d} \frac{\partial h_{i\alpha}(\mathbf{x})}{\partial x_i} = f_\alpha(\mathbf{x})$$

Can extend the formulation to plate equations,
wave equations at constant frequency in lossy media, etc.

Abstract and more general formulation

Given a space $\mathcal{H} = \mathcal{E} \oplus \mathcal{J}$

of square integrable fields taking values in a tensor space $\mathcal{T}$

Then for $\mathbf{h} \in \mathcal{H}$ solve

$$\mathbf{J}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{E}(\mathbf{x}) - \mathbf{h}(\mathbf{x}), \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{E} \in \mathcal{E},$$

or with $\mathbf{h} = -(\mathbf{L} - \mathbf{L}_0)\mathbf{s}$

$$\mathbf{J} = \mathbf{L}\mathbf{E} + (\mathbf{L} - \mathbf{L}_0)\mathbf{s}, \text{ with } \mathbf{J} \in \mathcal{J}, \quad \mathbf{E} \in \mathcal{E}, \quad \mathbf{s} \in \mathcal{H}.$$

Wave Equations in Lossy Media:

$$\underbrace{\begin{pmatrix} -i\mathbf{v} \\ -i\nabla\cdot\mathbf{v} \end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix} -(\omega\boldsymbol{\rho})^{-1} & 0 \\ 0 & \omega/\kappa \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla P \\ P \end{pmatrix}}_{\mathbf{E}(\mathbf{x})},$$

Acoustics

$$\underbrace{\begin{pmatrix} -i\mathbf{h} \\ i\nabla\times\mathbf{h} \end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix} -[\omega\boldsymbol{\mu}]^{-1} & 0 \\ 0 & \omega\boldsymbol{\varepsilon} \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla\times\mathbf{e} \\ \mathbf{e} \end{pmatrix}}_{\mathbf{E}(\mathbf{x})},$$

Electromagnetism

$$\underbrace{\begin{pmatrix} -\boldsymbol{\sigma}/\omega \\ -\nabla\cdot\boldsymbol{\sigma}/\omega \end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix} -\boldsymbol{\mathcal{C}}/\omega & 0 \\ 0 & \omega\boldsymbol{\rho} \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \underbrace{\begin{pmatrix} [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]/2 \\ \mathbf{u} \end{pmatrix}}_{\mathbf{E}(\mathbf{x})},$$

Elastodynamics

$$\underbrace{\begin{pmatrix} i\mathrm{M} \\ \nabla\cdot(\nabla\cdot\mathrm{M}) \end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix} -\boldsymbol{\mathcal{D}}(\mathbf{x})/\omega & 0 \\ 0 & h(\mathbf{x})\omega\rho(\mathbf{x}) \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla\nabla v \\ iv \end{pmatrix}}_{\mathbf{E}(\mathbf{x})}.$$

Dynamic Plate Equation

Define the "polarization field"

$$\mathbf{P}(\mathbf{x}) = \mathbf{J}(\mathbf{x}) - \mathbf{L}_0\mathbf{E}(\mathbf{x}) = [\mathbf{L}(\mathbf{x}) - \mathbf{L}_0]\mathbf{E}(\mathbf{x}) - \mathbf{h}(\mathbf{x})$$

$$\begin{aligned}
\mathbf{P} &= -[\mathbf{I} + (\mathbf{L} - \mathbf{L}_0)\mathbf{\Gamma}]^{-1}\mathbf{h} \\
&= -[\mathbf{I} + (\mathbf{L} - \mathbf{L}_0)\mathbf{M} + (\mathbf{L} - \mathbf{L}_0)(\mathbf{\Gamma} - \mathbf{M})]^{-1}\mathbf{h} \\
&= [\mathbf{I} - [\mathbf{I} + (\mathbf{L} - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L} - \mathbf{L}_0)(\mathbf{M} - \mathbf{\Gamma})]^{-1}[\mathbf{I} + (\mathbf{L} - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L} - \mathbf{L}_0)\mathbf{s} \\
&= (\mathbf{I} - \mathbf{K}\mathbf{\Psi})^{-1}\mathbf{K}\mathbf{s},
\end{aligned}$$

$$\mathbf{\Psi} = \mathbf{M} - \mathbf{\Gamma} \qquad \mathbf{h} = -(\mathbf{L} - \mathbf{L}_0)\mathbf{s},$$

Extended polarization fields:

$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ a basis of $\mathcal{T}$. For given q functions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_q \in \mathcal{H}$ consider the following linear map $\mathbb{S} : \mathcal{T} \to \mathcal{H}$ defined by

$\mathbb{S}\mathbf{e}_i = \mathbf{s}_i$, for all $i \in \{1, \ldots, q\}$ Define $\mathbb{P}\mathbf{e}_i = \mathbf{P}_i$, *for all* $i \in \{1, \ldots, q\}$

Define $\mathcal{S}$ as a subspace of $L(\mathcal{T})$ such that $\mathbf{A}\mathcal{S} \subset \mathcal{K}$ for all $\mathbf{A} \in \mathcal{K}$.

Suppose $\mathcal{K}$ is such that

$$\mathcal{K}\mathcal{A}\mathcal{K} \doteq \{\mathbf{B}_1\mathbf{A}\mathbf{B}_2, \text{ for } \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}, \mathbf{A} \in \mathcal{A}\} \subset \mathcal{K}.$$

$\mathcal{A}$ is the subspace spanned by the $\mathbf{\Psi}(\mathbf{k})$ as $\mathbf{k}$ varies

Main Theorem:

If:

$\mathbf{L}(\mathbf{x}) \in \mathcal{M}$ for all $\mathbf{x}$

$\mathbf{L}$ *satisfies appropriate boundedness and coercivity conditions*

$\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_q \in \mathcal{H}$ *are such that* $\mathbb{S}(\mathbf{x}) \in \mathcal{S}$ *for all* $\mathbf{x}$.

Then: $\mathbb{P}(\mathbf{x}) \in \mathcal{K}$ *for all* $\mathbf{x}$.

Green's function

$\mathbf{E}$ depends linearly on $\mathbf{h}$ and defines the (modified) infinite body Green's function in the inhomogeneous medium.

$$\mathbf{E}(\mathbf{x}) = \int_{\mathbb{R}^d} \mathbf{G}(\mathbf{x}, \mathbf{x}')\mathbf{h}(\mathbf{x}')\, d\mathbf{x}',$$

Consider a point $\mathbf{x}^0$ and take $\mathbf{h}(\mathbf{x})$ to be proportional to a Dirac delta function localized at $\mathbf{x} = \mathbf{x}^0$:

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}^0 \delta(\mathbf{x} - \mathbf{x}^0), \quad \text{with } \mathbf{h}^0 = -(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0,$$

$$\boxed{\mathbf{P}(\mathbf{x}) = (\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0 \delta(\mathbf{x} - \mathbf{x}^0) - (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{G}(\mathbf{x}, \mathbf{x}^0)(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0}$$

So $\mathbf{P}(\mathbf{x}) = \mathbf{T}(\mathbf{x}, \mathbf{x}_0)\mathbf{s}^0$ with

$$\mathbf{T}(\mathbf{x}, \mathbf{x}^0) = (\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\delta(\mathbf{x} - \mathbf{x}^0) - (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{G}(\mathbf{x}, \mathbf{x}^0)(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0).$$

$$\mathbf{T} = (\mathbf{L} - \mathbf{L}_0) - (\mathbf{L} - \mathbf{L}_0)\mathbf{G}(\mathbf{L} - \mathbf{L}_0) = (\mathbf{I} - \mathbf{K}\boldsymbol{\Psi})^{-1}\mathbf{K},$$

$$\mathbf{K}(\mathbf{x}) = W_{\mathbf{M}}(\mathbf{L}(\mathbf{x})) = [\mathbf{I} + (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)$$

$\boldsymbol{\Psi} = \mathbf{M} - \boldsymbol{\Gamma}$ takes values in $\mathcal{A}$

Expand:

$$\mathbf{T}(\mathbf{x},\mathbf{x}^0) = \delta(\mathbf{x}-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0) + \mathbf{K}(\mathbf{x})\widehat{\boldsymbol{\Psi}}(\mathbf{x}-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)$$

$$+ \int_{R^d} \mathbf{K}(\mathbf{x})\widehat{\boldsymbol{\Psi}}(\mathbf{x}-\mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\boldsymbol{\Psi}}(\mathbf{y}_1-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\, d\mathbf{y}_1$$

$$+ \int_{R^d}\int_{R^d} \mathbf{K}(\mathbf{x})\widehat{\boldsymbol{\Psi}}(\mathbf{x}-\mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\boldsymbol{\Psi}}(\mathbf{y}_1-\mathbf{y}_2)\mathbf{K}(\mathbf{y}_2)\widehat{\boldsymbol{\Psi}}(\mathbf{y}_2-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\, d\mathbf{y}_1\, d\mathbf{y}_2 + \dots,$$

Upshot :

$\mathbf{T}(\mathbf{x},\mathbf{x}_0)$ takes values in $\overline{\mathcal{K}}$ when $\mathbf{L}(\mathbf{x})$ takes values in $\mathcal{M}$.

Links between Green's functions of different physical problems (in inhomogeneous media)

$$\mathbf{J}^{(i)}(\mathbf{x}) = \mathbf{L}^{(i)}(\mathbf{x})\mathbf{E}^{(i)}(\mathbf{x}) - \mathbf{h}^{(i)}(\mathbf{x}), \text{ with } \mathbf{J}^{(i)} \in \mathcal{J}^{(i)}, \quad \mathbf{E}^{(i)} \in \mathcal{E}^{(i)}, \quad \mathbf{h}^{(i)} \in \mathcal{H}^{(i)},$$

Embed the problems:

$$\underbrace{\begin{pmatrix} \mathbf{J}^{(1)}(\mathbf{x}) \\ \mathbf{J}^{(2)}(\mathbf{x}) \\ \vdots \\ \mathbf{J}^{(m)}(\mathbf{x}) \end{pmatrix}}_{\mathbf{J}(\mathbf{x})} = \underbrace{\begin{pmatrix} \mathbf{L}^{(1)}(\mathbf{x}) & 0 & \dots & 0 \\ 0 & \mathbf{L}^{(2)}(\mathbf{x}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{L}^{(m)}(\mathbf{x}) \end{pmatrix}}_{\mathbf{L}(\mathbf{x})} \underbrace{\begin{pmatrix} \mathbf{E}^{(1)}(\mathbf{x}) \\ \mathbf{E}^{(2)}(\mathbf{x}) \\ \vdots \\ \mathbf{E}^{(m)}(\mathbf{x}) \end{pmatrix}}_{\mathbf{E}(\mathbf{x})} - \underbrace{\begin{pmatrix} \mathbf{h}^{(1)}(\mathbf{x}) \\ \mathbf{h}^{(2)}(\mathbf{x}) \\ \vdots \\ \mathbf{h}^{(m)}(\mathbf{x}) \end{pmatrix}}_{\mathbf{h}(\mathbf{x})},$$

$\mathbf{J}(\mathbf{x})$, $\mathbf{E}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ take values in $\mathcal{T} = \mathcal{T}^{(1)} \oplus \mathcal{T}^{(2)} \oplus \dots \oplus \mathcal{T}^{(m)}$,

$$\mathbf{E} \in \mathcal{E}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{h} \in \mathcal{H},$$

$$\mathbf{G}(\mathbf{x},\mathbf{x}^0) = \begin{pmatrix} \mathbf{G}^{(1)}(\mathbf{x},\mathbf{x}^0) & 0 & \dots & 0 \\ 0 & \mathbf{G}^{(2)}(\mathbf{x},\mathbf{x}^0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{G}^{(m)}(\mathbf{x},\mathbf{x}^0) \end{pmatrix},$$

$$\mathbf{L}_0 = \begin{pmatrix} \mathbf{L}_0^{(1)} & 0 & \dots & 0 \\ 0 & \mathbf{L}_0^{(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{L}_0^{(m)} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{M}^{(1)} & 0 & \dots & 0 \\ 0 & \mathbf{M}^{(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{M}^{(m)} \end{pmatrix}.$$

$$\Psi(\mathbf{k}) = \begin{pmatrix} \Psi^{(1)}(\mathbf{k}) & 0 & \dots & 0 \\ 0 & \Psi^{(2)}(\mathbf{k}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Psi^{(m)}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} \mathbf{M}^{(1)} - \mathbf{\Gamma}^{(1)}(\mathbf{k}) & 0 & \dots & 0 \\ 0 & \mathbf{M}^{(2)} - \mathbf{\Gamma}^{(2)}(\mathbf{k}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{M}^{(m)} - \mathbf{\Gamma}^{(m)}(\mathbf{k}) \end{pmatrix}$$

$$\mathbf{\Gamma}^{(i)}(\mathbf{k}) = [\mathbf{\Gamma}_1^{(i)}(\mathbf{k})\mathbf{L}_0^{(i)}\mathbf{\Gamma}_1^{(i)}(\mathbf{k})]^{-1}\mathbf{\Gamma}_1^{(i)}(\mathbf{k}),$$

$\mathbf{\Gamma}_1^{(i)}(\mathbf{k})$ is the projection onto $\mathcal{E}_{\mathbf{k}}^{(i)}$ and the operator inverse is taken on $\mathcal{E}_{\mathbf{k}}^{(i)}$.

Again seek subspaces $\mathcal{K}$ such that

$$\mathcal{K}\mathcal{A}\mathcal{K} \doteq \{\mathbf{B}_1\mathbf{A}\mathbf{B}_2,\ \text{for } \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K},\ \mathbf{A} \in \mathcal{A}\} \subset \mathcal{K}.$$

$\mathcal{A}$ is the subspace spanned by the $\Psi(\mathbf{k})$ as $\mathbf{k}$ varies

How to get the "boundary field equalities" satisfied by the "Dirichlet to Neumann Map".

The basic idea here (following Thaler and Milton, 2014, where for a body Ω containing 2-phases sharing the same shear modulus, the boundary field equalities give the volume fraction occupied by one phase in the body) is to choose nonlocal boundary conditions that mimic the body Ω embedded in an infinite medium with appropriate sources outside that ensure the appropriately defined polarization field takes values in the subspace $\overline{\mathcal{K}}$

Formally: "Dirichlet-to-Neumann map" (DtN map) $\mathbf{\Lambda}_\Omega$

$$\partial \mathbf{J} = \mathbf{\Lambda}_\Omega(\partial \mathbf{E})$$

With a point source outside Ω the equations in Ω_C are informally

$$\mathbf{J}(\mathbf{x}) = \mathbf{L}_1\mathbf{E}(\mathbf{x}) + (\mathbf{L}_1 - \mathbf{L}_0)\mathbf{s}^0\delta(\mathbf{x}-\mathbf{x}_0), \quad \mathbf{J} \in \mathcal{J}_{\Omega^C}, \quad \mathbf{E} \in \mathcal{E}_{\Omega^C}$$

Consider the solution in all space

$$\mathbf{J}^0(\mathbf{x}) = \mathbf{L}_1\mathbf{E}^0(\mathbf{x}) + (\mathbf{L}_1 - \mathbf{L}_0)\mathbf{s}^0\delta(\mathbf{x}-\mathbf{x}_0), \quad \mathbf{J}^0 \in \mathcal{J}, \quad \mathbf{E}^0 \in \mathcal{E}$$

Subtract

$$\widetilde{\mathbf{J}}(\mathbf{x}) = \mathbf{L}_1\widetilde{\mathbf{E}}(\mathbf{x}), \quad \widetilde{\mathbf{J}} \in \mathcal{J}_{\Omega^C}, \quad \widetilde{\mathbf{E}} \in \mathcal{E}_{\Omega^C}, \quad \widetilde{\mathbf{J}}(\mathbf{x}) = \mathbf{J}(\mathbf{x}) - \mathbf{J}^0(\mathbf{x}), \quad \widetilde{\mathbf{E}}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - \mathbf{E}^0(\mathbf{x}).$$

In terms of exterior DtN map $\mathbf{\Lambda}_{\Omega^C}$: $\mathbf{\Lambda}_{\Omega^C}(\partial\widetilde{\mathbf{E}}) = \partial\widetilde{\mathbf{J}}$

Desired (non-local) boundary condition:

$$\partial\mathbf{J} - \partial\mathbf{J}^0 = \mathbf{\Lambda}_{\Omega^C}(\partial\mathbf{E} - \partial\mathbf{E}^0), \quad \text{i.e., } \partial\mathbf{J} - \mathbf{\Lambda}_{\Omega^C}(\partial\mathbf{E}) = \partial\mathbf{J}^0 - \mathbf{\Lambda}_{\Omega^C}(\partial\mathbf{E}^0)$$

Generally, to reveal the exact relations satisfied by the DtN map one applies not just one boundary condition but a succession of them.

In general,

- $\mathbf{E}(\mathbf{x})$, $\mathbf{J}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$ take values in some N-dimensional tensor space $\mathcal{T}$
- $\mathbf{L}(\mathbf{x})$, $\mathbf{L}_0$, $\mathbf{K}(\mathbf{x})$ take values in $L(\mathcal{T})$ the N^2-dimensional space of linear maps $\mathcal{T} \to \mathcal{T}$
- $\mathcal{K}$ and $\overline{\mathcal{K}}$ are therefore subspaces of $L(\mathcal{T})$

Hence it does not really make sense to say $\mathbf{P}(\mathbf{x})$ takes values in $\overline{\mathcal{K}}$.

- Rather one should take a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N^2}$ for $\mathcal{L}(\mathcal{T})$.
- Consider polarization fields $\mathbf{P}_1(\mathbf{x}), \mathbf{P}_2(\mathbf{x}), \ldots, \mathbf{P}_{N^2}(\mathbf{x})$ associated with N^2 experiments, with appropriate sources outside Ω.

Define $\mathbb{P}(\mathbf{x})$: $\mathcal{T} \to \mathcal{T}$ via $\mathbb{P}(\mathbf{x})\mathbf{e}_i = \mathbf{P}_i(\mathbf{x})$

It is $\mathbb{P}(\mathbf{x})$ that takes values in $\overline{\mathcal{K}}$ for appropriately chosen sources outside Ω

Applications:

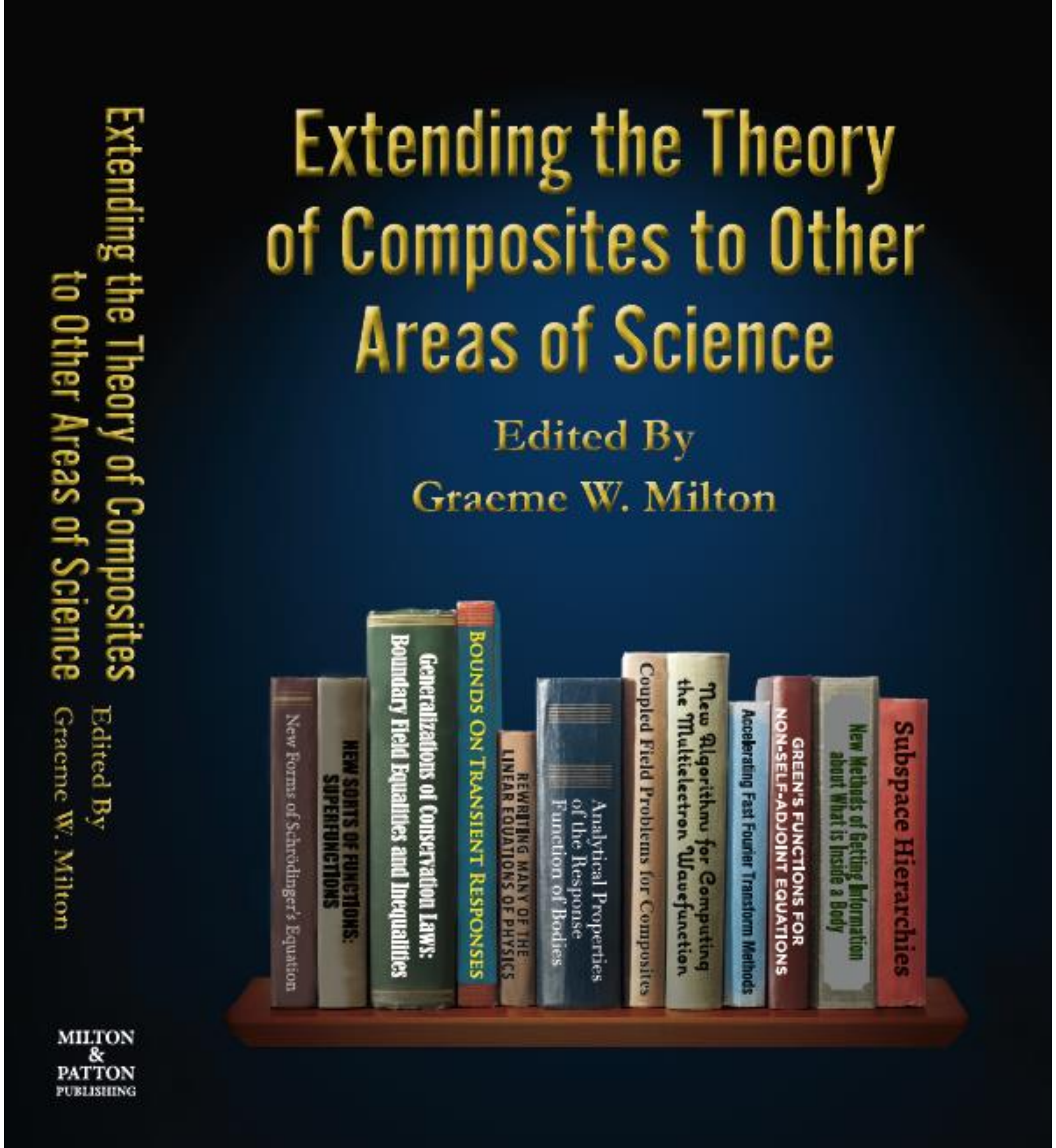
Serves as a benchmark for testing numerical code

Sometimes the exact relations involve the volume fractions of the phases. Then they can be used in an inverse way to determine these volume fractions.

Thank-you for listening

For more details see

Milton, G.W. & Onofrei, D. **Exact relations for Green's functions in linear PDE and boundary field equalities: a generalization of**



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