10.4 Parametric Equations

Vocabulary

parametric equation

closed curve

simple curve

Ex 1 Eliminate the parameter t, graph the curve & tell if it it is simple & closed.

$x = t^2 + 1$ $\qquad$ $y = t - 1$ $\qquad$ $-2 \le t \le 2$

10.4 (cont)

Ex 2 Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ without eliminating the parameter.

$x = \cot t - 3$ $\quad y = -2\csc t$, $0 < t < \pi$

If $x = f(t)$ & $y = g(t)$ and $f'(t)$ & $g'(t)$ exist, are continuous, and $f'(t) \neq 0$ on $\alpha < t < \beta$,

then on this interval,

1. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

2. $$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dt}\right)'}{\frac{dx}{dt}}$$

Length of a Curve

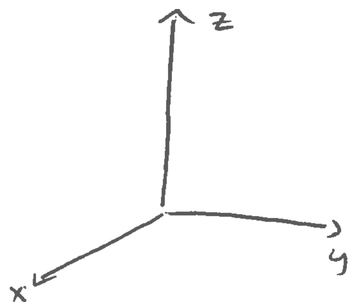
$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Question This is a "calc" version of what famous theorem?

Ex 3 Find the length of the curve given by

$x = t^3 \qquad y = 6t^2 \qquad 1 \leq t \leq 4$

11.1 Cartesian Coordinates in 3-Space



How we depict "3-D" on a piece of paper.

Ex 1a. Plot

a.) the ______ $(1, 3, -4)$

b.) the ______

$3x - 2y + z = 6$

c.) the ______

$X = 4$

d.) the ______ with center $(2, 4, 7)$ and radius 1.

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11.1 Continued

Ex 2 Describe the graph $xz = 0$ in 3-space

Question In 2-D I can draw axes & a point and estimate what point it is. Does this work in 3-D?

Formula

Let (x_1, y_1, z_1) & (x_2, y_2, z_2) be points

distance between them: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

mid point: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$

sphere with center (h, k, l) and radius r: $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

11.1 Continued

Ex 3) Calculus hero is at $(5, 4, -2)$ and there is treasure at $(1, -3, -1)$. Calulus hero's lasso is 10 units long. Can he/she use the tool to reach it?

Rewrite as a "bland" problem & solve.

Ex 4) Calchero's nemesis lands halfway between Calc hero & the treasure. At what point is the nemesis. (Use info from Ex 3).

11.1 Cont

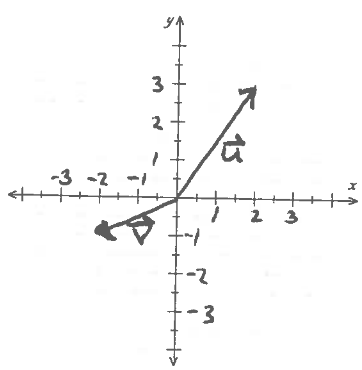
Ex 5) Find the center & radius of the sphere

$$x^2 + y^2 + z^2 - 2x + 18z = -57$$

11.2 Vectors

Vocabulary

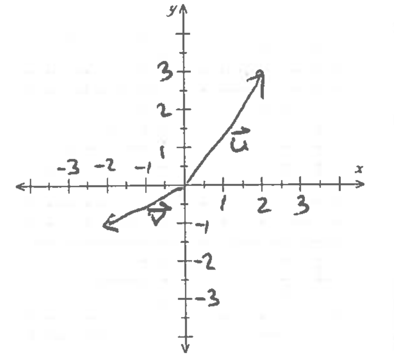
Ex 1. Vectors $\vec{u}$ & $\vec{v}$ are shown.



a) Describe $\vec{u}$ & $\vec{v}$ algebraically.

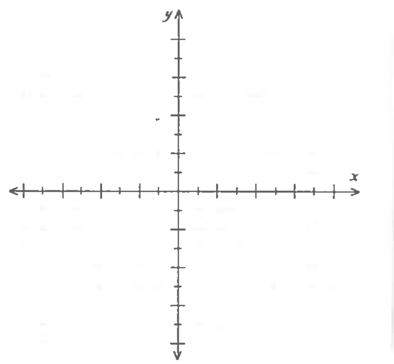
b.) Find $\|\vec{u}\|$ & $\|\vec{v}\|$

Ex2. Use the same $\vec{u}$ and $\vec{v}$ as before

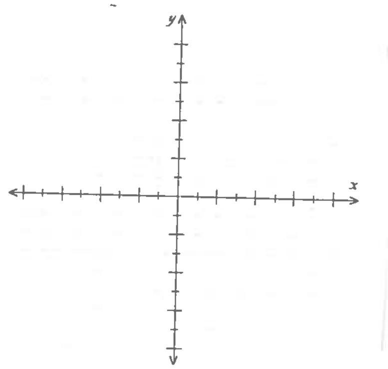


Calculate

a. $2\vec{u} + \vec{v}$ geometrically & algebraically.



b. $\vec{u} - \vec{v}$ geometrically & algebraically



Ex 3 The water from a fire hose exerts a force of 150 lbs on the person holding the hose. The nozzle weighs 10 lbs. What is the magnitude and direction of the force exerted by the person holding the hose?

11.2 Cont

Ex 4 An airplane flies 400 mph in still air. How Should the airplane be headed & how fast will it be flying with respect to the ground if it flies against a 20 mph wind blowing N 40° W

Think about how the solution would have been easier if the wind was blowing in the direction N 30° E.

11.3 The Dot Product

Definition: Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$ with angle θ between them, their dot product is.

(1.) $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ ← more geometric

(2.) $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ ← more algebraic

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Ex 1, Let $\vec{u} = \langle 3, -1, 1 \rangle$, $\vec{v} = \langle 2, 1, 0 \rangle$. Find

a.) $\|a\|$ $b \cdot a$

b. the angle between $\vec{a}$ & $\vec{b}$

c. the angle between $\hat{a}$ & $\hat{b}$

The video & text give many useful properties. My favorites are

$\|\vec{u}\|^2 = u \cdot u$

$\hat{u} \cdot \hat{v} = \cos\theta$

11.3 Cont

Ex2 Let $\vec{u}$ & $\vec{v}$ be vectors & θ be the angle between them. Write three more statements that are different, but equivalent to the first one.

1.) $\vec{u}$ & $\vec{v}$ are orthogonal

2.)

3.)

4.)

Question What's a normal vector?

Ex3 Show $\langle -1, 2, 0 \rangle$ is parallel to the plane $6x+3y+z=2$.

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11.3 Continued

	Ingredients	Result
a.)	two vectors $\vec{u}$ & $\vec{v}$	the angle between them $\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$
b)	two vectors $\vec{u}$ & $\vec{v}$	the projection of $\vec{u}$ onto $\vec{v}$ $pr_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$
c.)	equation of a plane $Ax + By + Cz + D = 0$	equation of a normal vector to the plane, $\langle A, B, C \rangle$
d.)	point (x_0, y_0, z_0) on a plane & a normal vector $\langle A, B, C \rangle$ to the same plane.	equation of the plane $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$
e.)	a plane $Ax + By + Cz = D$ and a point not on the plane, (x_0, y_0, z_0)	Shortest distance from the point to the plane $L = \frac{\lvert Ax_0 + By_0 + Cz_0 - D\rvert}{\sqrt{A^2 + B^2 + C^2}}$

11.3 Continued

Ex 4) How would you transform the following problems to be likes ones on the last page. Then solve.

a.) the planes $2x + 3y - 5z = 2$ and $2x + 3y - 5z = 9$ are parallel. What is the distance between them?

b.) Find the equation of the plane through $(-1, 2, -3)$ and parallel to the plane $2x + 4z - z = 6$.

11.4 The Cross Product

$\vec{u} = \langle u_1, u_2, u_3 \rangle \qquad \vec{v} = \langle v_1, v_2, v_3 \rangle$

The Cross Product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

generalizes ____________

result is a ____________

only for 3-D $\vec{u}$ & $\vec{v}$

$\vec{u} \times \vec{v} = 0 \iff$ ____________

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

The Dot Product

$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

generalizes ____________

result is a ____________

Defined for 1-D, 2-D, 3-D

$\vec{u} \cdot \vec{v} = 0 \iff$ ____________

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

11.4 Cont

Theorem if $\vec{u}, \vec{v}$ & $\vec{w}$ are vectors in 3-space & $k \in \mathbb{R}$, then

1. $u \times v = -(v \times u)$
2. $u \times (v+w) = (u \times v) + (u \times w)$
3. $k(u \times v) = (ku) \times v = u \times (kv)$
4. $u \times 0 = 0 \times u = u \times u = 0$
5. $(u \times v) \cdot w = u \cdot (v \times w)$
6. $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

Ex 1. If $\vec{a} = \langle 3, 4, 1 \rangle$, $\vec{b} = \langle -2, 0, 5 \rangle$, $\vec{c} = \langle 2, -1, 3 \rangle$

Find.

a.) $(\vec{a} + \vec{b}) \times \vec{c}$

b) $\vec{a} \times \vec{b} + \vec{b} \times \vec{a}$

Ingredients	Result
3 points (non collinear)	equation of a plane ① use the 3 points to find 2 vectors in the plane ② the cross product of the two vectors $\Longrightarrow$ normal vector to plane ③ pt + normal vector $\Rightarrow$ eqn. of plane
vectors $\vec{u}$ & $\vec{v}$	area of parallelogram spanned by $\vec{u}$ & $\vec{v}$. $A = \lVert \vec{u} \times \vec{v} \rVert$

Ex 2) Find all vectors $\perp$ to both $\vec{a} = -2\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$

$\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$.

11.4 Cont

Ex3 Find the area of a $\triangle$ with $(1,2,3)$, $(3,1,5)$ and $(4,5,6)$ as vertices

Ex4 Find the eqn. of the plane through $(2,-1,4)$ that is $\perp$ to both the planes $x-3y+2z=7$ and $2x-2y-z=-3$.

11.5 Vector-valued functions & curvilinear motion

Let f, g, & h be real-valued functions. A vector valued function of a real variable t is:

$$F(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$
$$= \langle f(t), g(t), h(t) \rangle.$$

Note: input: ________ output: ________

Processes that behave similarly to how they did for real-valued functions 1. 2. 3.	$\vec{r}(t)$ is $\vec{v}(t)$ $\vec{a}(t)$ what's the difference between $\vec{v}(t)$ & the speed?

11.5 Cont

Ex1. Find the limit if it exists

$$\lim_{t \to -2} \left[\frac{2t^2 - 10t - 28}{t+2} \hat{\imath} - \frac{7t^3}{t-3} \hat{\jmath} \right]$$

Ex 2 State the domain of the vector-valued function

$$r(t) = \ln(t-1)\,\hat{\imath} + \sqrt{20-t}\,\hat{\jmath}.$$

Ex 3. Find functions $\vec{v}(t)$ & $\vec{a}(t)$ given

$\vec{r}(t) = t^6\hat{\imath} + (6t^2 - 5)^6\hat{\jmath} + t\,k.$ What is the speed at $t=1$?

Ex 4

Ex 4 Find the length of the curve with given vector equation $r(t) = t\cos t\,\hat{\imath} + t\sin t\,\hat{\jmath} + \sqrt{2}\,t\,k$ $\quad 0 \le t \le 2$

11.6 Lines & Tangent Lines

Comment 1 We could talk about planes in 3-space (linear equations) before we could talk about lines in 3-space (vector equations).

Comment 2 Idea behind describing a line in 2-space is the same as in 3-space.

2-Space

Point $(0,b)$ + slope m $\Longrightarrow$ equation

$\uparrow$ y-intercept; (direction line is going in)

$$y = mx + b$$

3 space

Point (x_0, y_0, z_0), but it's easier to describe with a position vector $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

+ direction vector $\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \langle a, b, c \rangle$

$\Longrightarrow$ equation

$$\vec{r} = \vec{r}_0 + \vec{v}t = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$$

Parametric Equation of a line

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

Symmetric Equations of a line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

11.6 Cont

Ingredients	Result
a.) point & a direction vector $\vec{v}$	line through the point and in the direction of the vector $\vec{r} = \vec{r_0} + \vec{v}t$ ↑ point expressed as position vector
b.) two points	(1.) find the direction vector from one point to the other. (2.) use a.) to find the line through both points
c) two planes	(1) find normal vectors of planes (2) Use cross product to find the vector $\perp$ to both normal vectors. This vector has to lie in both planes. Thus it's in the direction of the line of intersection (3) find a pt on one line & return to a.)

11.6 Cont

Ex1 Find the parametric equations & the symmetric equations (if possible) for the line through $(-1,3,6)$ that is parallel to the vector $\langle 2,0,5\rangle$.

Ex2 Find the symmetric equations of the line through $(2,-4,5)$ that is parallel to the plane $3x+y-2z=5$ and perpendicular to the line $\frac{x+8}{2}=\frac{y-5}{3}=\frac{z-1}{-1}$.

11.6 Cont

Ex3 Find the equation of the plane containing the line $x=3,\ y=1+t,\ z=2t$ and parallel to the intersection of the planes $2x-y+z=0$ & $y+z+1=0$.

Comments on Tangent Lines

Combines 11.5 material (derivatives of vector-valued functions) & line material from this section. See example in the video.

11.8 Surfaces in 3-Space

Use the video notes for pictures!

Ex 1. Name & sketch the graph.

a. $z^2 + x^2 = 9$

b. $y^2 + z^2 - 4x^2 + 4 = 0$

c. $9x^2 + 25y^2 + 9z^2 = 225$

d. $y^2 - x^2 + z = 0$

11.8 cont

Ex 2 Name & sketch the graph

a.) $2x^2 - 6z^2 = 0$

b.)

Ex 3 What surface results when the curve $z = 2y$ in the yz-plane is revolved around the z-axis? Sketch it & find an equation for it.

11.9 Cylindrical & Spherical Coordinates

Rectangular

$r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$

$z = z$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Cylindrical

$\rho = \sqrt{x^2+y^2+z^2}$

$\tan\theta = \frac{y}{x}$

$\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\theta$

Spherical

Ex1 Change $\left(4, \frac{\pi}{3}, \frac{3\pi}{4}\right)$ from spherical to Cartesian coordinates.

Question Can you tell just by looking at a point, for example $\left(4, \frac{\pi}{3}, \frac{3\pi}{4}\right)$ what coordinate system it belongs to?

11.9 Cont

Ex2 Sketch the graph of the given cylindrical or spherical equation.

a.) $r = 2\sin 2\theta$

b. $\rho = \sec \varphi$

Ex3 Change the equation to the indicated coordinate system.

a. $\rho \sin \varphi = 2$ to cylindrical

b. $r^2 + 6z^2 = 7$ to spherical

12.1 Functions of two or more variables.

Describe the input & output for functions in this section.

$$F(\boxed{}) = \boxed{}$$

Ex1 Let $f(x,y,z) = \sqrt{x\cos y} + z^2$. Find $f(2, \frac{\pi}{3}, -1)$.

Ex2. Describe the domain of $f(x,y,z) = z \cdot \ln(xy)$.

12.1 Continued

Ex 3 Sketch the level curves ~~q~~ $z = k$, where $k = 0, 1, 2, 3$ for $f(x,y) = 2 - x - y^2$

Ex 4. Sketch the graph of $f(x,y) = x^2 + y^2 - 4$.