12.2 Partial derivatives

$z = f(x,y)$

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$$

What do you do first?

f_{xyx} work "inside to outside" for this notation.

1. f_x
2. $(f_x)_y = f_{xy}$
3. $(f_{xy})_x = f_{xyx}$

Question - What would the corresponding $\frac{\partial \_\_\_}{\partial \_\_\_}$ notation be for f_{xyx} ?

Ex1 Find f_x & f_y for $f(x,y) = \ln(x^2 - y^2)$.

12.2 Cont

Ex 2 Find the four second order partial derivatives for $f(x,y) = 2x^3 \cos 4y$.

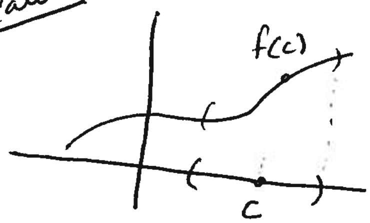
Ex 3 Imagine you "are on" the surface $z = \frac{5\sqrt{16 - x^2}}{4}$ at point $\left(2, 3, \frac{5\sqrt{3}}{2}\right)$. Find the slope of the tangent line to this point that lies in the $x=2$ plane. Repeat for the tangent line that lies in the $y=3$ plane.

12.3 Limits & Continuity.

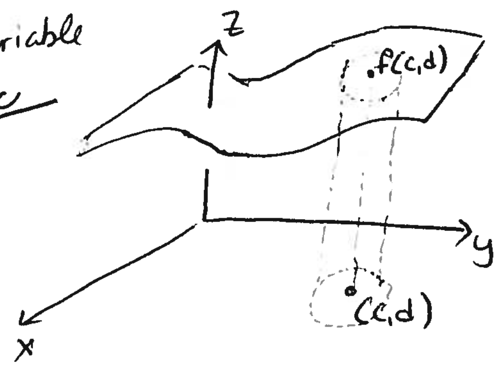
Video shows pictures of discontinuities/limits not existing.

Picture of when a limit exists.

Calc I



Multivariable Calc



Does $\lim_{x \to c} f(x)$ exist?

Look at interval around c. (Approach c from right & left)

Does $\lim_{(x,y) \to (c,d)} f(x,y)$ exist?

Look at disk surrounding it. (Approach from many different directions)

Strategies for Showing a limit exists

1. Plug in the numbers & it exists.
2. Algebraic manipulation
   - factoring & cancelling
   - rewriting trig functions
3. Change to polar coordinates

   $r^2 = x^2 + y^2$

   $x = r\cos\theta$

   $y = r\sin\theta$

   If $(x,y) \to 0$, then $r \to 0$

Strategies for showing a limit does not exist

1. Show the limit (when approaching along any path) goes to $\pm\infty$.
2. Show the limits when approaching along different paths are different.

12.3 Cont

Ex 1 Find the limit or justify that it does not exist.

a.) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 + y^2}$

b.) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$

c. $\lim\limits_{(x,y)\to(0,0)} xy\,\dfrac{x^2-y^2}{x^2+y^2}$

12.3 Cont

Ex 2 Describe the largest set S on which it is correct to say that f is continuous.

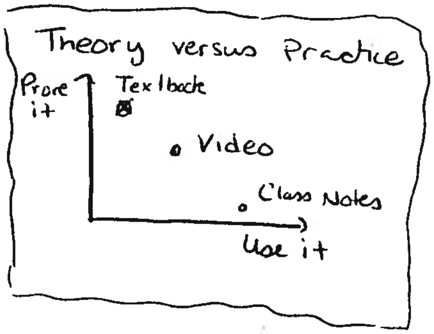
> A function $f(x,y)$ is continuous at (a,b) if $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$.

a.) $f(x,y) = \dfrac{1}{\sqrt{1+x+y}}$

b.) $f(x,y) = \ln(4 - x^2 - y^2 - z^2)$

Ex3 Sketch the idicated set. Describe the boundary of the set. State whether it is open, closed or neither.

12.4 Differentiability



Let $z = f(x,y)$ be a function & (a,b) be a point in the domain.

Gradient of f at point (a,b): $\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$

Tangent plane to z through (a,b): $z = f(a,b) - \nabla f_{(a,b)} \cdot \langle x-a, y-b \rangle$

Questions

1.) Does the gradient always exist?

2.) Does the tangent plane to a point always exist?

3.) What does the gradient have to do with differentiability?

12.4 Cont

Ex3 Find the equation of the tangent "hyperplane" to $w = f(x,y,z) = 2y\cos(2\pi x) + 4x\cos(\pi y) + xz$ at the point $(1, \frac{1}{2}, 3)$.

12.4 Cont

Ex1 Find the gradient ∇f of $f(x,y) = 4xe^{9xy}$

Ex2 Find the gradient of $f(x,y) = \frac{x^2}{y}$ at the point $(2,-1)$.

Then find the equation of the tangent plane at this point.

12.5 Directional Derivatives

Let $z = f(x,y)$ be a function

(a,b) a point in the domain

& $\hat{u}$ be a unit vector.

Theorem

If ______________________________, then f has a directional derivative at (a,b) in the direction of $\hat{u}$:

$$D_{\vec{u}} f(a,b) = \hat{u} \cdot f(a,b)$$

4-space version

$w = f(x,y,z)$, (a,b,c) - pt, $\hat{u} = u_x\hat{\imath} + u_y\hat{\jmath} + u_z\hat{k}$,

$$D_{\vec{u}} f(a,b,c) = \hat{u} \cdot f(a,b,c)$$

Ex1 Find the directional derivative of $f(x,y) = e^{-xy}$ at the point $p = (1,-1)$ in the direction of $\vec{u} = -i + \sqrt{3}j$.

12.5 Cont

Theorem

At pt (a,b), the function $z = f(x,y)$

increases most rapidly in the direction ______________

at a rate ______________.

It decreases most rapidly in the direction ______________

at a rate ______________.

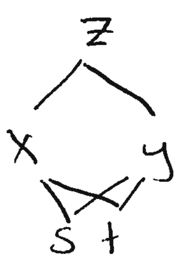
Ex 2. Find a unit vector in the direction in which $f(x) = 4xyz^2$ decreases most rapidly at point $(2,-1,1)$. What is the rate of change in this direction?

12.5 Cont

Ex 3 Find the directional derivative of $f(x,y) = e^{-x} \cos y$ at $(0, \frac{\pi}{3})$ in the direction toward the origin.

12.6 The Chain Rule

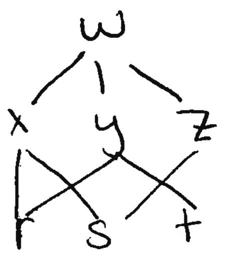
Theorem Let $x = x(s,t)$ & $y = y(s,t)$ have first partial derivatives at $(x(s,t),\ y(s,t))$. Then z has first partial derivatives given by:



$$\frac{\partial z}{\partial s} = \frac{\partial z}{}\ \frac{}{} + \frac{\partial z}{}\ \frac{}{}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{}\ \frac{}{} + \frac{}{}\ \frac{}{\partial t}$$

Imagine the picture was:



① What is

$$\frac{dw}{dr} =$$

② What other first partial derivatives are there?

12.6 Cont

Ex 1 Find $\frac{dw}{dt}$ using the chain rule

a) $w = xy + yz + xz$, $x = t^2$, $y = 1 - t^2$, $z = 1 - t$.
(Express answer in terms of t.)

b) $w = x^2 - y\ln x$, $x = \frac{s}{t}$, $y^2 = s^2 t$. (Express the answer in terms of s & t.)

12.6 Cont

Ex 3 If $w = x^2y + z^2$, $x = \rho\cos\theta\sin\phi$, find
$y = \rho\sin\theta\sin\phi$,
$z = \rho\cos\phi$,

$$\left.\frac{dw}{d\theta}\right|_{\rho=2,\ \theta=\pi,\ \phi=\frac{\pi}{2}}$$

Ex 4 Airplanes A & B depart from point P at the same time. A flies due east & B flies N 50° E. At a certain instance, A is 200 miles from P flying 450 mph & B is 150 miles from P flying 400 mph. How fast are they separating at that instant?

12.7 Tangent Planes

Def Let $F(x,y,z) = k$ be a surface and $P_0 = (x_0, y_0, z_0)$ be a point on F. If it is

1. ____________________________

2. ____________________________

then the tangent Plane to F at P_0 exists.

It is the plane that is pepedicular to ____________

and passes through __________.

Theorem

For a surface $F(x,y,z) = k$, the equation of the tangent plane at (x_0, y_0, z_0) is:

$$\nabla F(x,y,z) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 .$$

12.7 Cont

Ex1 Find the equation of the tangent plane to $x^2+y^2-z^2=4$ at $(2,1,1)$

Ex2 Find a point on the surface $z=2x^2+3y^2$ where the tangent plane is parallel to the plane $8x-3y-z=0$.

12.7 cont

Ex3 Use differentials to approximate the change in $z = \tan^{-1} xy$ from $P(2, -0.5)$ to $Q(-2.03, -0.51)$. Then find Δz.

Let $z = f(x,y)$ be a differentiable function.
Let dx, dy be variables.
"total differential of f"
$\hookrightarrow dz = df(x,y)$
$= f_x(x,y)dx + f_y(x,y)dy$
$= \nabla f \cdot \langle dx, dy \rangle$

dz = estimated change

Δz = actual change

Ex4 An object's weight in air is $A = 36$ lbs & its weight in water is 20 lbs, with a possible error in each measurement of 0.02 lbs. Find, by approximating, the maximum possible error in calculating its specific gravity from $S(A,W) = \frac{A}{A-W}$.

12.8 Maxima & Minima

Given a function $f(x,y)$, define the following:

global maximum -

local maximum -

global minimum -

local minimum -

Critical point	How do we find it / evaluate it?
Stationary Point (a,b)	
singular point (a,b)	
boundary point(s) of S	

12.8 Cont

Definitions (from 12.3)

- neighborhood -

1-D

2-D

3-D

Let S be a set and P be a point in S.

- P is an interior point of S if

- P is a boundary point of S if

- the boundary of S is

Second Partials Test

Given a function $f(x,y)$ & a point (a,b) in its domain where $\nabla f(a,b) = \vec{0}$.

If f_{xx}, f_{yy}, f_{xy} are continuous in a neighborhood around (a,b), and

$$D = D(a,b) = (f_{xx}(a,b))(f_{yy}(a,b)) - f_{xy}^2(a,b)$$

then

1. $D>0$ & $f_{xx}(a,b)<0$ $\Rightarrow$ $f(a,b)$ local max
2. $D>0$ & $f_{xx}(a,b)>0$ $\Rightarrow$ $f(a,b)$ local min
3. $D<0 \Rightarrow f(a,b)$ is not an extreme value (saddle point)
4. $D=0 \Rightarrow$ test inconclusive.

Ex1 What is the boundary of $\{(x,y) : 1 < x \leq 4\}$?

12.8 Cont

Ex2 Given $f(x,y) = x^2 + a^2 - 2ax\cos y$; $\quad -\pi < y < \pi$.

Find all critical points. Indicate whether each gives a local/global max or min or is a saddle point.

12.8 Cont

Ex 3 Find the point on the plane $x + 2y + 3z = 12$ that is closest to the origin. What is the minimum distance.

12.9 La grange Multipliers

Given: functions $f=(x,y)$ & $g(x,y)$

want: the points that product max/min values of f

and satisfy $g(x,y)=0$.

The La Grange Method says that all the critical points (all the candidates for what we are looking for) are the solutions to the system of equations:

① $\nabla f = \lambda \nabla g$ and ② $g(x,y)=0$

Ex 1. Find the minimum of $f(x,y) = x^2+4xy+y^2$ subject to the constraint $x-y-6=0$.

12.9 Cont

Ex2

Choose the problem that can be solved using the Lagrange Method. Then Solve it with the method.
Explain why

Problem 1

Find the max & min values for $f(x,y) = x^2 - y^2 - 1$ on $S = \{(x,y) \mid x^2 + y^2 \le 1\}$.

Problem 2

Find the 3-D vector of length 9 with the largest possible Sum of its components.

12.9 Cont

Ex3 Find the point on the plane $x+2y+3z=12$ that is closest to the origin. (Use the Lagrange method.)