14.1 Vector Fields

Questions: What is a scalar field?

What is a vector field?

How are they similar?

How are they different?

Ex1. Sketch sample of vectors for the given vector field $\vec{F}$.

Suggestion

* Overlapping vectors are hard to interpret. When possible, choose start points beyond where your last vector ended

* use color!

a.) $\vec{F}(x,y) = x\hat{\imath} - y\hat{\jmath}$

b.) $\vec{F}(x,y) = -2\hat{\jmath}$

14.1 cont

Ex 2. Sketch sample vectors for $\vec{F}(x,y,z) = 2\hat{j} + z\hat{k}$.

Suggestion

* Try to draw vectors with starting points in the xy-, yz- and xz- planes.

Memory Check

Let $f(x,y,z)$ be a scalar field.

Let $\vec{F}(x,y,z)$ be a vector field.

What do you remember from the video?

Operator	Notation	Input	Output
gradient of f			
divergence of $\vec{F}$			
curl of $\vec{F}$			

14.1 Cont

Ex 3. Let $\vec{F}(x,y,z) = xyz\,\hat{\imath} + 2y^2\,\hat{\jmath} - 3x^2z\,\hat{k}$.

Find

a.) div $\vec{F}$

b.) curl $\vec{F}$

c.) grad (div $\vec{F}$)

d.) div (curl $\vec{F}$)

Scalar field:

$f(x,y,z)$

vector field

$\vec{F}(x,y,z) = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$

Gradient of f

$\nabla f = \langle f_x, f_y, f_z \rangle$

Divergence of $\vec{F}$

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$
$$= M_x + N_y + P_z$$

Curl of $\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

determinant

14.2 Line Integrals

Definition of a line integral:

Let C be a curve given parametrically, $x=(t)$, $y=y(t)$, $t\in[a,b]$,
$f(x,y)$ be a scalar valued function,

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Questions: How do we say $\int_C f(x,y)\,ds$?

What are we calculating?

Ex1 Evaluate $\int_C xe^y\,ds$ where C is the line segment from $(-1,2)$ to $(1,1)$.

14.2 Cont

Ex 2 Evaluate $\int_C xz\,dx + (y+z)\,dy + x\,dz$

where C is the curve $x = e^t$, $y = e^{-t}$, $z = e^{2t}$ $0 \le t \le 1$.

14.2 Cont

Ex3 Find the work done by the force field $\vec{F}(x,y,z) = (2x-y)\hat{\imath} + 2z\hat{\jmath} + (y-z)\hat{k}$ when moving a particle along the line segment from $(0,0,0)$ to $(1,4,5)$.

Physics Equations

* mass = volume · density

or for a thin wire,

* mass = length · density
* work = force · distance

Let

$\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$

where $M = M(x,y,z)$
$N = N(x,y,z)$
$P = P(x,y,z)$.

$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$

$$\text{work} = W = \int_C \vec{F}\cdot d\vec{r}$$

C is the parameterized curve over which the particle moves

$$W = \int_{t=a}^{t=b} M\,dx + N\,dy + P\,dz$$

14.3 Independence of Path

Fundemental Theorem of Line Integrals

Let C be a curve given by the parameterization $\vec{r}(t)$, $t \in [a,b]$ such that $\vec{r}(t)$ is differentiable. If $f(\vec{r})$ is continuously differentiable on an open set containing C, then

________________________.

Equivalent Conditions

Let $\vec{F}(\vec{r})$ be continuous on an open, connected set D. The following are equivalent.

a.

b.

c.

Theorem

Let $\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ be continuously differentiable on a open, connected set D.

$$\vec{F} \text{ is conservative} \iff \nabla \times \vec{F} = \vec{0}.$$

In 3 variables,

$\nabla \times \vec{F} = \vec{0}$ iff

$M_y = N_x$

$M_z = P_x$

$N_z = P_y$

In 2 variables

$\nabla \times \vec{F} = \vec{0}$ iff

$M_y = N_x$

Questions

1. What does it mean to be independent of path?

2. Why does D need to be open & simply connected?

3. If $\vec{F}$ is conserative, what is it conserving?

4. Why isn't the Theorem (left) grouped with the equivalent conditions?

14.3

Ex1. Determine whether the given field is conservative. If so, find f so that $\vec{F} = \nabla f$.

a) $\vec{F}(x,y) = \left(x + \frac{1}{(x+y)^2}\right)\hat{\imath} + \left(3 + \frac{1}{(x+y)^2}\right)\hat{\jmath}$

b.) $\vec{F}(x,y) = 4y^2\cos(xy^2)\hat{\imath} + 8x\cos(xy^2)\hat{\jmath}$

14.3 Cont

Ex 2 Use $\vec{F}(x,y) = \left(x + \frac{1}{(x+y)^2}\right)\hat{\imath} + \left(3 + \frac{1}{(x+y)^2}\right)\hat{\jmath}$

a.) What is the largest open, connected set on which $\vec{F}(x,y)$ is continuous.

b.) Evaluate $\vec{F}(x,y)$ using the Fundamental Theorem of Line Integrals.

c.) How would you evaluate b.) with out the fundamental theorem?

14.3 Cont

Ex 3 Show that this integral is independent of path:

$$\int_{(0,0,0)}^{(\pi,\pi,0)} (\cos x + 2yz)\,dx + (\sin y + 2xz)\,dy + (z + 2xy)\,dz$$

Then evaluate it.