# Non-Cohen-Macaulay Normal Domains

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# Background–Looking for Cohen-Macaulay Algebras

Let $R$ be a Noetherian local ring. A major question is whether $R$ has a Cohen-Macaulay Algebra, that is, a ring $A$ containing $R$ such that a system of parameters for $R$ becomes a regular sequence on $A$ and $A/m_R A \neq 0$. Standard methods allow us to reduce to the case in which $R$ is a complete local normal domain. We assume that from now on. The interesting cases start in dimension 3.

We also ask whether $R$ has an **almost** Cohen-Macaulay Algebra. We mean that there is a non-zero element $c$ such that the local cohomology $H^i_{m_R}(A)$ is annihilated by $c^{1/n}$ for arbitrarily large $n$.

# Cases

1. Characteristic $p > 0$: $R^+$ is Cohen-Macaulay (Hochster-Huneke, Huneke-Lyubeznik).
2. Characteristic Zero: $R$ cannot have a finite extension that is Cohen-Macaulay (use the trace map).
3. Mixed Characteristic: Does $R$ have an almost Cohen-Macaulay algebra contained in $R^+$?

# An Example

Let $V$ be an unramified DVR of mixed characteristic $p$ and let $S = V[x, y]/(p^3 + x^3 + y^3)$, and let $R$ be the subring $S[pT, xT, yT]$ of $S[T]$. Then a system of parameters is $p, xT, x + pT$ and a relation making it not Cohen-Macaulay is

$$(x + pT)(y^2 T) = (xT)(y^2) + p(yT)^2.$$

This is a blow-up ring and analogous to a Segre product.

# Another Example

The second example was shown to me by Ray Heitmann.

**Version 1.** $p = 3$, $V$ is an unramified $DVR$ of mixed characteristic 3, $B = V[a, z]/(81 + a^4 + z^2)$, and $R$ is the subring $B[3T, aT, zT, zT^2]$ of $B[T]$. This is a normalization of a blow-up.

**Version 2.** $p$ is any prime, $V$ is an unramified $DVR$ of mixed characteristic $p$, $B = V[a, b, z]/(b^4 + a^4 + z^2)$, and $R$ is the subring $B[bT, aT, zT, zT^2]$ of $B[T]$.

The system of parameters is $a, bT, b + aT$ and the relation is $(b + aT)(zT) = z(bT) + a(zT^2)$. In the second version this relation remains after inverting $p$ so cannot be killed in a finite extension.

# Remarks on these examples

According to Ray Heitmann, examples of this type exist whenever the (regular version of) the Briançon-Skoda Theorem does not hold.

In these examples, the relation can be almost killed in an integral extension. The property that these and other examples have that makes this work is that they have a bigradings with the right properties.

A more general framework. Let $x_1(= p), x_2, \ldots, x_d$ be a system of parameters and let

$$ux_i \in (x_1, \ldots, x_{i-1}).$$

We want an element $c$ such that

$$c^{1/p^n} u \in (x_1, \ldots, x_{i-1})R^+$$

for all $n$.

The first property of the bigrading is that if the degree of $u$ is high enough, then $u \in (x_1, \ldots, x_{i-1})R^+$.

To describe the other property of the grading we outline a few ingredients of the construction of almost Cohen-Macaulay algebras.

Let $R$ be a homomorphic image of a power series $S$ ring over an unramified DVR of mixed characteristic. From this we construct a perfect ring of positive characteristic $E$ and a map from the ring of Witt vectors over $E$ to the $p$-adic completion of an integral extension of $R$.

$E$ contains an element $P$ that maps to $p$.

We reduce the question in $R$ to one in $E$.

# What this looks like.

Each element of the kernel of the map from $S$ to $R$ defines a sequence $Z_n$ of elements of $E$.
We take $p = 2$ and list the first few $Z_n$. Suppose the element in the kernel is $A + Bp + Cp^2 + \cdots = \sum A_i p^i$. the first few elements are

1. $Z_0 = A$.
2. $Z_1 = A^{1/2} + B^{1/4}P^{1/2}$.
3. $Z_2 = A^{1/4} + B^{1/8}P^{1/4} + B^{1/16}A^{1/8}P^{3/8} + f(P^{1/16}, A^{1/16}, B^{1/32}, C^{1/32})$.

In general

$$Z_n = A^{1/p^n} + B^{1/p^{n+1}}P^{1/p^n} + f(P^{1/p^m}, A_i^{1/p^m})$$

for some $m$.

# The main question

Let $a_1, \dots a_t$ generate the kernel of the map from $S$ to $R$ and let the corresponding sequences be $Z_{n1}, \dots Z_{nt}$. We translate the problem to $E$, with elements $X_1 = P, \dots X_d$, $U$ and we want $C$ such that $C^{1/p^n}U \in (X_1, \dots, X_{i-1})$.

To do this it suffices to find an $i$ such that $C^{1/p^n}U$ is in $(X_1, \dots, X_{i-1})$ modulo$Z_i 1, \dots, Z_i t$.
We can do it modulo $P^{1/p^i}$ using standard methods in positive characteristic.

In our examples, we can solve successively modulo higher powers of $P$ because the coefficients of $Z_i$ are all homogeneous of the same degree and finish the construction.

Two Questions:

1. Find a general condition that would make the construction outlined above work.
2. Find more challenging examples of non-Cohen-Macaulay normal domains to test the construction.

One possible source of examples –coordinate rings of cones of abelian varietes of dimension at least two.