FONTAINE RINGS AND LOCAL COHOMOLOGY

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1. Introduction

In this paper we use Fontaine rings to study properties of local cohomology for rings of mixed characteristic. Fontaine rings, which we define and discuss in more detail in Section 3, give a method of constructing a ring of positive characteristic from one of mixed characteristic in such a way that, under certain circumstances, the original ring can be reconstructed up to p-adic completion. The idea is to use the Frobenius map on the ring of positive characteristic and then to deduce results on the original ring. We discuss the extent to which this program can be carried out and problems that arise. Essentially, we begin with a certain Noetherian subring of the Fontaine ring of a ring R and look for extensions T of this ring satisfying two properties. The first property is that a given quotient of the ring of Witt vectors on T is almost Cohen-Macaulay. Almost Cohen-Macaulay algebras are algebras for which certain local cohomology groups are almost zero; these concepts will be defined in section 2. We will also explain their connection with some of the homological conjectures. The second property is that relations that are mapped to zero in R are also mapped to zero in this quotient. It is possible to construct an extension satisfying the first property, but in this paper we concentrate on the second one and construct a minimal extension T such that the required relations are mapped to zero. It is not clear whether or not this ring is almost Cohen-Macaulay. In the last section we present an example to show that, at least in some nontrivial cases, the image of the relevant local cohomology modules of the original ring are almost zero in the local cohomology of this extension, giving some evidence that this ring might be almost Cohen-Macaulay.

2. Almost vanishing of local cohomology

Let R_0 be a complete local Noetherian domain of mixed characteristic. We will usually assume in addition that R_0 is normal. We let R_0^+ be the absolute integral closure of R_0; that is, R_0^+ is the integral closure of R_0 in the algebraic closure of its quotient field. Throughout this paper R will denote a ring between R_0 and R_0^+. Although we do not assume that R is Noetherian, R is a union of local Noetherian domains that are integral over R_0. As a result, we can define a system of parameters in R to be a sequence of elements $x_1, \ldots, x_d$ of R that is a system of parameters in a local Noetherian subring of R containing R_0.

Let R be a ring as above; we next take a valuation v on R with values in the ordered abelian group $\mathbb{R}$ of real numbers, Then v is a function from R to $\mathbb{R} \cup \{\infty\}$ satisfying

(1) $v(rs) = v(r) + v(s)$ for $r, s \in R$.
(2) $v(r+s) \geq \min\{v(r), v(s)\}$ for $r, s \in R$.
(3) $v(r) = \infty$ if and only if $r = 0$.

We will assume also that $v(r) \geq 0$ for $r \in R$ and that $v(r) > 0$ for r in the maximal ideal of R. The existence of such a valuation follows from standard facts on extensions of valuations, see for example Zariski-Samuel [15], Chapter VI.

We say that an R-module M is *almost zero* with respect to v if for all $m \in M$ and for all $\epsilon > 0$, there exists an $r \in R$ with $v(r) < \epsilon$ and $rm = 0$.

We will also use another version of almost zero modules—if c is an element of R, we will say that M is almost zero for c if for every element m of M we have $c^{1/n}m = 0$ for arbitrarily large integers n. If v is a valuation on R and $v(c) < \infty$, this condition implies that M is almost zero with respect to v.

While we will be concerned in this paper with the two classes of almost zero modules defined above, we will also mention that more general definitions can be used.

Definition 1. *A class $\mathcal{C}$ of modules is a class of almost zero modules if it satisfies the following two conditions.*

(1) *If we have a short exact sequence*
$$0 \to M' \to M \to M'' \to 0,$$
then M is in $\mathcal{C}$ if and only if M' and M'' are in $\mathcal{C}$.

(2) *$\mathcal{C}$ is closed under direct limits.*

It is easy to see that both of the classes we have defined satisfy these conditions. Given a class $\mathcal{C}$ satisfying these conditions we say that a module M is almost zero with respect to $\mathcal{C}$ if $M \in \mathcal{C}$. For the remainder of this section we will assume that we have fixed such a class $\mathcal{C}$ and "almost zero" will mean with respect to $\mathcal{C}$.

We now recall some facts about local cohomology and define almost Cohen-Macaulay rings.

Let $x_1, \dots, x_d$ be a system of parameters for R, and let $H^i_{(x)}(R)$ denote the local cohomology of R with support in $(x_1, \dots, x_d)$. More precisely, $H^i_{(x)}(R)$ is the cohomology in degree i of the Čech complex
$$0 \to R \to \prod_i R_{x_i} \to \prod_{i<j} R_{x_i x_j} \to \cdots \to R_{x_1 x_2 \cdots x_d} \to 0,$$
where R has degree 0 and $R_{x_1 x_2 \cdots x_d}$ has degree d.

Definition 2. *An R-algebra A is* almost Cohen-Macaulay *if*

(1) *$H^i_{(x)}(A)$ is almost zero for $i = 0, \dots, d-1$.*

(2) *$A/(x_1, \dots, x_d)A$ is not almost zero.*

An alternative definition of almost Cohen-Macaulay can be obtained by defining a sequence $x_1, \dots, x_d$ to be *almost regular* if $\{r | r x_i \in (x_1, \dots, x_{i-1})\}/(x_1, \dots, x_{i-1})$ is almost zero for $i = 1, \dots, d$ and defining R to be almost Cohen-Macaulay if a system of parameters is almost regular (together with condition (2)). Standard methods show that this definition implies the former one. In this paper we will only use the weaker property defined above.

The importance of the existence of Cohen-Macaulay algebras has been known for many years—see for instance Hochster [7]. Recently Heitmann [5] has shown that the weaker condition of having an almost Cohen-Macaulay algebra in the sense defined here has some of the same consequences; in particular, he showed that the Direct Summand conjecture in dimension 3 follows from the existence of an almost Cohen-Macaulay algebra. He also showed that R_0^+ is an almost Cohen-Macaulay algebra in dimension 3; he was using a form of the second definition, although in dimension 3 it does not really matter which one you use. We next show that in any dimension the existence of an almost Cohen-Macaulay algebra in the weaker sense we are using here implies the Monomial Conjecture, which is equivalent to the Direct Summand Conjecture (see for instance Hochster [6] for a discussion of these conjectures).

The Monomial Conjecture states that if $x_1, \dots, x_d$ is a system of parameters for a local ring R_0, then
$$x_1^t x_2^t \cdots x_d^t \notin (x_1^{t+1}, \dots, x_d^{t+1})$$
for any $t \geq 0$. We assume that there is an almost Cohen-Macaulay algebra A over R_0. Mel Hochster ([6]) has proven that it suffices to consider the case in which $x_1 = p$ in the mixed characteristic case. We can also assume that R_0 is complete. Using these facts, we let Q be a regular subring of R_0 containing $x_1 (= p), x_2, \dots, x_d$ as a regular system of parameters. Let F be a minimal free resolution of $Q/(x_1^t x_2^t \cdots x_d^t, x_1^{t+1}, x_2^{t+1}, \dots, x_d^{t+1})$ over Q. This resolution can be chosen so that $F_0 = Q$, $F_1 = Q^{d+1}$ with the map d_1^F from F_1 to F_0 given by the row of matrix $(x_1^t x_2^t \cdots x_d^t \quad x_1^{t+1} \quad x_2^{t+1} \quad \dots \quad x_d^{t+1})$ and $F_2 = Q^{d+\binom{d}{2}}$, where the map d_2^F from F_2 to F_1 is given by the matrix
$$\begin{pmatrix} x_1 & x_2 & \cdots & x_d & 0 & \cdots & 0 \\ -x_2 \cdots x_d & 0 & \cdots & 0 & & \text{Koszul} & \\ 0 & -x_1 x_3 \cdots x_d & \cdots & 0 & & \text{relations} & \\ \vdots & \vdots & & \vdots & & \text{on } x_1^{t+1}, \dots, x_d^{t+1} & \\ 0 & 0 & \cdots & -x_1 \cdots x_{d-1} & & & \end{pmatrix}.$$

Since Q is a regular local ring of dimension d and $Q/(x_1^t x_2^t \cdots x_d^t, x_1^{t+1}, \ldots, x_d^{t+1})$ has finite length, F is a complex of length d. Let this complex tensored with A be denoted F^A. Since F becomes exact when any of the x_i is inverted, the same holds for F^A. Let

$$C = 0 \to R \to \prod_i R_{x_i} \to \prod_{i<j} R_{x_i x_j} \to \cdots \to R_{x_1 x_2 \cdots x_d} \to 0$$

be the complex defining the local cohomology of R with support in $(x_1, \ldots, x_d)$. In this complex we let R have degree 0 and $R_{x_1 \cdots x_d}$ have degree d. We next consider the total tensor product complex $F^A \otimes C$ over R. In this tensor product we give $F_i \otimes C^j$ degree $i - j$.

Lemma 1. (1) *The complex $F^A \otimes C$ is quasi-isomorphic to F^A.*
(2) *The cohomology of $F^A \otimes C$ is almost zero in positive degrees (and hence the same holds for F^A).*

Proof. The two parts of the proof follow from the two spectral sequences associated with the tensor product $F^A \otimes C$. For the first statement, we consider the spectral sequence obtained by first taking the homology of $F^A \otimes C^i$ for fixed i. For $i = 0$, $C^0 = R$, so $F^A \otimes C^0 \cong F^A$ and this gives F^A. For $i > 0$, $F^A \otimes C^i$ is a product of localizations of F^A obtained by inverting products of the x_i, so $F^A \otimes C^i$ is exact. Thus the projection of $F^A \otimes C$ onto $F^A \otimes C^0 = F^A$ is a quasi-isomorphism, proving (1).

For the second statement, we consider the homology of $F_i^A \otimes C$ for fixed i. Since F_i^A is a finitely generated free module over A, $F_i^A \otimes C$ is a finite direct sum of copies of $A \otimes C$, and since A is almost Cohen-Macaulay, the homology of $A \otimes C$ is almost zero in degrees less than d. To prove (2), it will suffice to show that the homology of $F_i^A \otimes C$ is almost zero at $F_i^A \otimes C^j$ whenever $i - j > 0$. The only value of j for which the homology of $F_i^A \otimes C^j$ might not be almost zero is $j = d$. If $i - j > 0$, then $i > j$, and if $j = d$, then we must have $i > d$. However, the complex F^A has length d, so this means that $F_i^A \otimes C^j = 0$. This completes the proof of the second statement.

It follows from the lemma in particular that $H_1(F^A)$ is almost zero. We now consider the projection of $F_1^A \cong A^{d+1}$ onto the first factor; this gives a map to A. The image of the kernel of d_1^F under this map is the set of $a \in A$ for which there is an n-tuple $a, a_1, \ldots, a_d$ with

$$a x_1^t x_2^t \cdots x_d^t + a_1 x_1^{t+1} + \cdots + a_d x_d^{t+1} = 0.$$

In other words, the image is $\{a \in A | a x_1^t x_2^t \cdots x_d^t \in (x_1^{t+1}, \ldots, x_d^{t+1})\}$. We denote this ideal $\mathfrak{a}$. Thus the projection of F_1^A onto A induces a map from $\mathrm{Ker} d_1^F$ onto $\mathfrak{a}$. The image of d_2^F maps to the ideal $I = (x_1, \ldots, x_d)$. Hence this defines a map from $H_1(F^A)$ onto $\mathfrak{a}/I$. Since $H_1(F^A)$ is almost zero, it follows that $\mathfrak{a}/I$ is almost zero. On the other hand, one of the conditions for A to be almost Cohen-Macaulay is that $A/(x_1, \ldots, x_d)$ is not almost zero. Hence 1 is not in $\mathfrak{a}$, so $x_1^t x_2^t \cdots x_d^t$ is not in $(x_1^{t+1}, \ldots, x_d^{t+1})$, and the Monomial Conjecture holds.

3. Fontaine Rings and Witt Vectors

Let Q be a ring of mixed characteristic p. The *Fontaine ring* of Q, denoted $E(Q)$, is defined to be the inverse limit over n of Q_n as n ranges over the ordered set of nonnegative integers, where each Q_n is Q/pQ and the map from Q_{n+1} to Q_n is the Frobenius map for all n. The notation and terminology that we use is taken from Gabber and Ramero [4], section 8.2 (in the most recent version); however, we use the notation $E(Q)$ rather than $E(Q)^+$ to avoid confusion with the absolute integral closure. An element of $E(Q)$ can be represented by a sequence $(q_0, q_1, \ldots) = (q_i)$, where each q_i is an element of Q taken modulo pQ and $q_i^p \equiv q_{i-1}$ modulo pQ for $i > 0$. $E(Q)$ is a perfect ring of characteristic p.

Let R_0 be a complete local normal domain of mixed characteristic as above; we assume in addition that the residue field k of R_0 is perfect. Let D_0 be an unramified DVR contained in R_0 with the same residue field as R_0 such that there exists a homomorphism from a power series ring $S_0 = D_0[[y_2, y_3, \ldots, y_t]]$ onto R_0. Such a homomorphism exists by the Cohen structure theorem (see for example Matsumura [10], Section 29). Let x_i be the image of y_i for each i, and assume that the elements are chosen in such a way that $p, x_2, \ldots, x_d$ form a system of parameters for R_0.

We let $S = \cup_n S_0[p^{1/p^n}, y_2^{1/p^n}, \ldots, y_t^{1/p^n}]$, adjoining p^nth roots of p and the y_i for each i. Similarly, adjoin p^nth roots of the elements p and x_i of R_0 to form a ring $R = \cup_n R_0[p^{1/p^n}, x_2^{1/p^n}, \ldots, x_t^{1/p^n}]$. We can then extend the map from S_0 to R_0 to give a surjective homomorphism from S to R. Since pth roots are unique only up to a root of unity, there is a choice in adjoining the roots; we choose the roots so that $(p^{1/p^{n+1}})^p = p^{1/p^n}$ for each n and similarly for the x_i and y_i. We can choose the surjection from S to R so that it takes p^{1/p^n} to p^{1/p^n} and y_i^{1/p^n} to x_i^{1/p^n} for each i and n.

We now take the Fontaine rings $E(R)$ and $E(S)$. We let P denote $(p, p^{1/p}, \ldots)$, $Y_i = (y_i, y_i^{1/p}, \ldots)$, and $X_i = (x_i, x_i^{1/p}, \ldots)$ for our given choices of p^nth roots.

Next, we take the rings of Witt vectors $W(E(S))$ and $W(E(R))$ of the rings we have defined. We refer to Bourbaki [2] and Serre [13] for general facts about Witt vectors and Gabber and Ramero [4] for connections with Fontaine rings.

Let $\hat{R}$ be the p-adic completion of R; that is, the inverse limit over n of R/p^nR. We have a map $\tilde{\phi}_R$, or simply $\tilde{\phi}$, from $E(R)$ to $\hat{R}$ defined by letting

$$\tilde{\phi}_R((r_i)) = \lim_{n\to\infty} r_i^{p^i}.$$

It is shown in [4] that this sequence converges. The map $\tilde{\phi}$ preserves multiplication but not addition. We have $\tilde{\phi}(P) = p$ and $\tilde{\phi}(X_i) = x_i$ for each i. The map $\tilde{\phi}$ induces a ring homomorphism ϕ from $E(R)$ to R/pR, and it extends to a ring homomorphism ψ from $W(E(R))$ to $\hat{R}$ that also takes P to p and X_i to x_i, where elements e of $E(R)$ are identified with the corresponding elements $(e, 0, 0, \ldots)$ in the ring of Witt vectors. The reduction of ψ modulo p is ϕ. We refer again to Gabber and Ramero [4], Section 8.2 for details.

As outlined in the introduction, the aim of this paper is to investigate the possibility of using this construction to construct almost Cohen-Macaulay algebras in mixed characteristic. We first review the situation for rings of positive characteristic.

4. The case of positive characteristic

We assume now that R is a Noetherian integral domain of positive characteristic p and define the perfect closure of R, denoted R^∞, to be the ring obtained from R by adjoining all p^nth roots of elements of R. Alternatively, R^∞ can be defined as the direct limit of R_n for $n \geq 0$, where $R_n = R$ for all n and the map from R_n to R_{n+1} is the Frobenius map.

Theorem 1. *Let R be a complete Noetherian local domain of positive characteristic, and let $x_1, \ldots, x_d$ be a system of parameters for R. Then there is a nonzero element c in R such that $c^{1/p^n}\eta = 0$ for all $\eta \in H^i_{(x_j)}(R^\infty)$ for $i = 0, \ldots, d-1$ and for all integers $n \geq 0$.*

Proof. We will use the fact that there is a nonzero element $c \in R$ which annihilates the local cohomology $H^i_{(x_j)}(R)$ for $i = 0, \ldots, d-1$; this can be found in Roberts [11] or Hochster and Huneke [8], section 3. We claim, in fact, that this element will satisfy the statement in the theorem.

To see this, let i be an integer with $0 \leq i \leq d-1$, and let η be an element of $H^i_{(x_j)}(R^\infty)$. Let F denote the map induced by Frobenius map on $H^i_{(x_j)}(R^\infty)$. Since every element of R^∞ has its p^kth power in R for large enough k, there is some k for which $F^k(\eta) \in H^i_{(x_i)}(R)$. Let m be an integer such that $m + k \geq n$. Then c^{1/p^n} is a multiple of $c^{1/p^{m+k}}$, so it suffices to show that $c^{1/p^{m+k}}\eta = 0$. We have

$$F^{m+k}(c^{1/p^{m+k}}\eta) = cF^{m+k}(\eta) = cF^m(F^k(\eta)) = 0,$$

since $F^m(F^k(\eta))$ is in $H^i_{(x_j)}(R)$. Since R^∞ is perfect, F is an isomorphism on R^∞ so induces an isomorphism on $H^i_{(x_j)}(R^\infty)$. Hence the fact that $F^{m+k}(c^{1/p^{m+k}}\eta) = 0$ implies that $c^{1/p^{m+k}}\eta = 0$. $\square$

The above theorem implies that R^∞ is an almost Cohen-Macaulay algebra for R in either of the senses introduced in Section 2.

As outlined in the introduction, the aim of this paper is to attempt to apply this kind of argument to a ring constructed via the Fontaine ring and to deduce results in mixed characteristic by using the ring of Witt vectors. In fact, for any ring R_0 it appears that one can indeed find an almost Cohen-Macaulay ring by this method. The procedure is to first find a system of generators $\{x_i\}$ for R_0 for which the Fontaine ring $E(R)$ of the ring R defined by adjoining p^nth roots as above contains a Noetherian ring E_0 with an ideal J_0 with the following properties.

(1) P is a non-zero-divisor in E_0/J_0
(2) $P, X_2, \ldots, X_d$ form a system of parameters for E_0/J_0.

If we then let E be the perfect closure of E_0 and let J be the radical of the ideal generated by J_0 in E, then we can define a ring $T(J)$ by adjoining all elements of the form j/P^n to E for all $j \in J$ and for all integers $n \geq 0$. The ring $W(T(I))/(P-p)$ is then an almost Cohen-Macaulay ring. However, there is in general no way of mapping R_0 into this ring, so it does not give an almost Cohen-Macaulay algebra for R_0, and we do not pursue this idea further here. In the next section we describe a somewhat more complicated construction for which the required map from R_0 does exist. In the final section we give a nontrivial example R_0 for which we can verify that the image of the nontrivial local cohomology of R_0 is almost zero in the local cohomology of this algebra. However, we do not know whether this holds in general.

5. An algebra defined from the Fontaine ring

We now return to the situation in which R is obtained by adjoining p^nth roots of certain elements to a complete local normal domain of mixed characteristic R_0. More precisely, we have a power series ring $S_0 = V[[y_2, \ldots, y_t]]$ over a DVR with perfect residue field k together with a surjective map from S_0 to R_0 that takes y_i to x_i for each i. We assume also that $p, x_2, \ldots, x_d$ is a system of parameters for R_0. R is then obtained from R_0 by adjoining p^nth roots of p and the x_i, and S is obtained similarly from S_0.

We then take the Fontaine rings $E(S)$ and $E(R)$ together with their rings of Witt vectors $W(E(S))$ aand $W(E(R))$. We would now like to pass to $E(W(R))/(P-p)$ and show that it is an almost Cohen-Macaulay algebra for R_0. However, there are two problems. First, $E(W(R))$ is not the perfect closure of a Noetherian ring, so we cannot directly apply the method of the previous section. We address this question first.

The ring $E(S)$ contains a power series ring in $P, Y_2, \ldots, Y_t$ over k and we define E_0 to be the subring of $E(R)$ which is the image of this power series ring under the induced map from $E(S)$ to $E(R)$. Then E_0 is a Noetherian ring of positive characteristic, and we let E_0^∞ denote its perfect closure, a subring of $E(R)$.

The second problem is that the kernel of the map ψ from $W(E(R))$ to $\hat{R}$ does not go to zero in $W(E(R))/(P-p)$, so that there is no map induced from R_0. One way around this problem is to embed R into a larger ring $C(R) = \{s \in R_p |\ s^{p^n} \in R \text{ for some } n\}$. It is shown in Roberts [12] that the kernel of the map from $W(E(C(R)))$ to $\widehat{C(R)}$ is generated by $P-p$. What we do here is find a smaller ring T containing E_0^∞ for which there exists a map from R_0 to $W(T)/(P-p)$.

We will identify $E(R)$ as a subset of the ring $W(E(R))$ of Witt vectors by associating $e \in E(R)$ with the Teichmüller element $(e, 0, 0 \ldots)$ of $W(E(R))$. Under this identification, let W_0 be the completion of the subring of $W(E(R))$ generated by $P, X_2, \ldots, X_t$ over the discrete valuation ring V. Let I_0 be the kernel of the map from W_0 to $\hat{R}$ induced by ψ; we note that W_0 maps onto R_0 under this map. Then if T is a ring containing E_0 such that every element in I_0 maps to $(P-p)W(T)$ under the inclusion from $W(E)$ to $W(T)$, we will have an induced map from R_0 to $W(T)/(P-p)W(T)$.

Since $P - p = P(1 - p(1/P))$ and the ring of Witt vectors $W(E_0^\infty)$ is p-adically complete, $P-p$ is a unit in $W((E_0^\infty)_P)$, where $(E_0^\infty)_P$ denotes the ring obtained from E_0^∞ by inverting P. Hence for any element $a = (a_0, a_1, \ldots, a_n, \ldots)$ of W_0 we can find a unique element $z = (z_0, z_1, \ldots)$ of $W((E_0^\infty)_P)$ such that $a = (P-p)z$. We define T_0 to be the subring of $(E_0^\infty)_P$ generated by all such z_i for all elements a that belong to the ideal I_0, and we define T to be the perfect closure of T_0.

We now go into more detail as to how the ring T_0 can be computed. Let $a = (a_0, a_1, \ldots, a_n, \ldots)$ be an element of I_0. Then we need to find $z = (z_0, z_1, \ldots, z_n, \ldots,)$ such that $a = (P-p)z = Pz - pz$.

Using the standard formulas for Witt vectors (see for instance Bourbaki [2] we have

$$Pz = (Pz_0, P^p z_1, \ldots, P^{p^n} z_n, \ldots)$$

and

$$pz = (0, z_0^p, z_1^p, \ldots, z_{n-1}^p, \ldots).$$

Thus, again using the rules for computation in the ring of Witt vectors, to find the z_n we must solve recursively the following formulas for z_n:

$$P^{p^n} z_0^{p^n} + pP^{p^n} z_1^{p^{n-1}} + \cdots + p^n P^{p^n} z_n =$$
$$a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n + pz_0^{p^n} + p^2 z_1^{p^{n-1}} + \cdots + p^n z_{n-1}^p.$$

We remarked above that the z_i are in $(E_0^\infty)_P$; in fact, it is clear that when we solve these equations the values will be in $(E_0^\infty)_P$. More precisely, it can be shown that z_n will be of the form $f(P, X_i)/P^{(n+1)p^n}$, where $f(P, X_i)$ is in E_0.

We also note that it is enough to take the ring generated by the z_i where the (a_i) run over a set of generators for I_0. To see this we must show, for instance, that if a and b are in I_0 and $a/(P-p) = (u_i)$, $b/(P-p) = (v_i)$, and $(a+b)/(P-p) = (z_i)$, then the z_i are in the ring generated over E_0 by the u_i and v_i. This follows from the fact that $(a+b)/(P-p) = a/(P-p) + b/(P-p)$ and that the entries in a sum of Witt vectors are polynomials in the entries of the summands. Similarly, one shows that if $a \in I_0$ and $e \in E_0$, and if $a/(P-p) = (u_i)$, $e/(P-p) = (v_i)$, and $(ea)/(P-p) = (z_i)$, then the z_i are in the ring generated over E_0 by the u_i and v_i.

As stated above, we let T be the perfect closure of T_0. It is not clear whether $W(T)/(P-p)$ is an almost Cohen-Macaulay algebra or not. We show next that if T is any extension of E_0^∞ such that the local cohomology of T is almost zero in the sense described below, then the local cohomology of $W(T)/(P-p)$ will be almost zero as well.

Theorem 2. *Let T be a ring containing E_0^∞, and suppose that P is not a zero-divisor in T and that there exists an element $c \in T$ such that c^{1/p^n} annihilates $H^i_{(P,X_2,\ldots,X_d)}(T)$ for $i = 0, \ldots, d-1$ for all $n \geq 0$. Let c_1 be the image of c in $W(T)$ or $W(T)/(P-p)$. Then c_1^{1/p^n} annihilates $H^i_{(p,X_2,\ldots,X_d)}(W(T)/(P-p))$ for $i = 0, \ldots, d-1$ for all $n \geq 0$.*

Proof. We note first that since the map from T to $W(T)/(P-p)$ preserves multiplication, c^{1/p^n} will map to a p^nth root of c_1. We will use the term "almost zero" to describe the property of being annihilated by c^{1/p^n} for all n for T-modules or by c_1^{1/p^n} for all n for $W(T)$-modules.

We first use the long exact sequences of local cohomology associated to the short exact sequences

$$0 \to W(T)/pW(T) \to W(T)/p^nW(T) \to W(T)/p^{n-1}W(T) \to 0$$

and induction to show that the local cohomology of $W(T)/p^nW(T)$ with support in $(P, X_2, \ldots, X_d)$ is almost zero in degrees $0, \ldots, d-1$; the case $n = 1$ is the hypothesis. Note that local cohomology of $W(T)/p^nW(T)$ with support in $(P, X_2, \ldots, X_d)$ is the same as local cohomology with support in $(p, P, X_2, \ldots, X_d)$ since $W(T)/p^nW(T)$ is annihilated by a power of p. We then consider the long exact sequence associated to

$$0 \to W(T) \xrightarrow{p^n} W(T) \to W(T)/p^nW(T) \to 0.$$

This long exact sequence produces, for each i, an exact sequence

$$H^{i-1}_{(p,P,X_2,\ldots,X_d)}(W(T)/p^nW(T)) \to H^i_{(p,P,X_2,\ldots,X_d)}(W(T)) \xrightarrow{p^n} H^i_{(p,P,X_2,\ldots,X_d)}(W(T)).$$

From this we deduce that the submodule of $H^i_{(p,P,X_2,\ldots,X_d)}(W(T))$ annihilated by p^n is almost zero for $i = 0, \ldots, d$. Since $H^i_{(p,P,X_2,\ldots,X_d)}(W(T))$ is the union of these submodules, $H^i_{(p,P,X_2,\ldots,X_d)}(W(T))$ is almost zero for $i = 0, \ldots, d$.

Finally, since P is not a zero-divisor in T, $P-p$ is not a zero-divisor in $W(T)$. We then use the long exact sequence coming from the short exact sequence

$$0 \to W(T) \xrightarrow{P-p} W(T) \to W(T)/(P-p) \to 0$$

to conclude that $H^i_{(p,X_2,\ldots,X_d)}(W(T)/(P-p))$ is almost zero for $i=0,\ldots,d-1$. □

We note that if we have a map from R_0 to $W(T)/(P-p)$ that takes x_i to X_i for all i, then the local cohomology modules $H^i_{(p,X_2,\ldots,X_d)}(W(T)/(P-p))$ are the same as the local cohomology modules $H^i_{(p,x_2,\ldots,x_d)}(W(T)/(P-p))$.

6. An Example

We now give an example of how this construction works in practice. The example is a non-Cohen-Macaulay normal domain R_0 of dimension three, and we show that the image of the local cohomology of R_0 in the algebra described in the previous section is almost zero. I would like to thank Anurag Singh for bringing this example to my attention and explaining many of its properties.

Let p be a prime number greater than 3, and let V_0 be a complete DVR with maximal ideal generated by p. Let R_0 be the power series ring $V_0[[x,y,u,v,w]]$ modulo the ideal generated by, first, the 2 by 2 minors of the matrix

$$\begin{pmatrix} p & x & y \\ u & v & w \end{pmatrix},$$

and, second, the elements

$$p^3+x^3+y^3, p^2u+x^2v+y^2w, pu^2+xv^2+yw^2, u^3+v^3+w^3.$$

The ring R_0 has the following properties.

(1) R_0 is a 3-dimensional normal domain.
(2) A system of parameters for R_0 is $p, v, x+u$.
(3) R_0 is not Cohen-Macaulay, and its local cohomology in degree 2 is generated by the element coming from the relation

$$(x+u)(yw) = xyw + uyw = vy^2 + pw^2.$$

We will not prove these facts here, but we remark that the corresponding facts in the analogous situation in which p is replaced by another variable over a field can be deduced from the fact that the ring is a completion of a Segre product (see for example [9]) and our case can deduced from that case.

We now compute what happens when we adjoin p^nth roots of the generators of R_0. We first adjoin p^nth roots of p for each n to form a (non-discrete) valuation ring V. Let $\pi_n, x_n, y_n, u_n, v_n, w_n$ be elements of R_0^+ such that $\pi_n^p = \pi_{n-1}$ for all n and similarly for the other variables. By choosing v_n and w_n appropriately we can ensure also that the 2 by 2 minors of the matrix

$$\begin{pmatrix} \pi_n & x_n & y_n \\ u_n & v_n & w_n \end{pmatrix} \qquad (*)$$

are zero.

We claim that the only other relations on these elements are the $3p^n+1$ polynomials

$$\pi_n^{3p^n}+x_n^{3p^n}+y_n^{3p^n}, \pi_n^{3p^n-1}u+x_n^{3p^n-1}v+y_n^{3p^n-1}w, \ldots, u_n^{3p^n}+v_n^{3p^n}+w_n^{3p^n}. \qquad (**)$$

We first note that these elements are zero; the first one is the first of the original cubic relations, and the others can be shown to be zero using the determinantal relations among $\pi_n, x_n, y_n, u_n, v_n, w_n$. To see that they generate the ideal of relations, it suffices to show that the ideal generated by the elements $(*)$ and $(**)$ in the power series ring $V_n[[x_n, y_n, u_n, v_n, w_n]]$ is prime, where V_n is a discrete valuation ring with maximal ideal generated by π_n. Let U_n denote the quotient of the power series ring by the determinantal ideal $(*)$ and localize U_n by inverting one of $\pi_n, x_n, y_n, u_n, v_n$, or w_n. The ideal generated by the above polynomials in this localization is generated by either $\pi_n^{3p^n}+x_n^{3p^n}+y_n^{3p^n}$ or $u_n^{3p^n}+v_n^{3p^n}+w_n^{3p^n}$. We assume that it is generated by $\pi_n^{3p^n}+x_n^{3p^n}+y_n^{3p^n}$; the other case is similar. The localization of the determinantal ring is regular, and $\pi_n^{3p^n}+x_n^{3p^n}$ is a product of distinct prime elements. It follows that the element $\pi_n^{3p^n}+x_n^{3p^n}+y_n^{3p^n} = y_n^{3p^n}+(\pi_n^{3p^n}+x_n^{3p^n})$ is prime. Thus the ideal is prime after localization at any of the six generators, and to finish the proof it suffices to know that the depth is at least two, which can be carried out by reduction to the case of a Segre product as outlined above.

Thus the quotient obtained by dividing by these polynomials is an integral domain and is isomorphic to the extension obtained by adjoining the p^nth roots of the generators of our ring. Unlike the case of R_0, this extension is not normal.

We now investigate what happens when we take the Fontaine ring of R. We have elements in $E(R)$ corresponding to p and the variables in R_0 that we will denote, as above, by capitals: P, X, Y, U, V, W. As in the previous section, they generate a Noetherian subring E_0 (up to completion), and we let E_0^∞ be its perfect closure. We have relations given by the determinants of the matrix

$$\begin{pmatrix} P^{1/p^n} & X^{1/p^n} & Y^{1/p^n} \\ U^{1/p^n} & V^{1/p^n} & W^{1/p^n} \end{pmatrix}$$

for each n, and we claim that these generate the relations among these elements in E_0^∞. Since E_0^∞ is the perfect closure of E_0, it suffices to show that the kernel of the map induced from $k[[P, X, Y, U, V, W]]$ to $E(R)$ is generated by $PV - XU, PW - YU$, and $XW - YV$. Let $f(P, X, Y, U, V, W)$ be an element of this kernel. If we represent f by $(f_0, f_1, \ldots)$ in $E(R)$, the component f_i in degree i is given by the same power series f with the coefficients and variables replaced by their p^ith roots modulo p in R. From the above description of the relations between these elements in R, we deduce that this component is in the ideal generated by $\pi_i v_i - x_i u_i, \pi_i w_i - y_i u_i$, and $x_i w_i - y_i v_i$ and the relations $(**)$ modulo p. The relations $(**)$ are contained in the ideal generated by p, x, y, u, v, w, so f_i is in the ideal generated by $\pi_i v_i - x_i u_i, \pi_i w_i - y_i u_i$, and $x_i w_i - y_i v_i$ and p, x, y, u, v, w. Write

$$f_i = a_i(\pi_i v_i - x_i u_i) + b_i(\pi_i w_i - y_i u_i) + c_i(x_i w_i - y_i v_i) + d_i,$$

where a_i, b_i, and c_i are in R and d_i is in the ideal of R generated by p, x, y, u, v, w.

If we knew that the d_i were zero and that the a_i satisfied $a_i^p = a_{i-1}$ in R/pR and similarly for the b_i and c_i, we could conclude that f was in the ideal generated by $PV - XU, PW - YU$, and $XW - YV$. By noting that $f_i = f_j^{j-i}$ for all $j \geq i$ and using the equation

$$f_j = a_j(\pi_j v_j - x_j u_j) + b_j(\pi_j w_j - y_j u_j) + c_j(x_j w_j - y_j v_j) + d_j$$

we can deduce that f_i is in the ideal generated by $\pi_i v_i - x_i u_i, \pi_i w_i - y_i u_i$, and $x_i w_i - y_i v_i$ modulo the ideal generated by the p^{j-i}th power of (x, y, u, v, w) (modulo p) for all $j \geq i$, so f_i is in the ideal generated by $\pi_i v_i - x_i u_i, \pi_i w_i - y_i u_i$, and $x_i w_i - y_i v_i$. Finally, any relation between these three generators of this ideal can be lifted to a relation between the corresponding generators in degree $i + 1$ (using that the relations are given by the rows of $(*)$), so we can adjust a_i, b_i, and c_i step by step to make them compatible and conclude that f is in the ideal generated by $PV - XU, PW - YU$, and $XW - YV$.

Thus the quotient has dimension 4 and is a determinantal ring. The elements $P, V, X + U$ are not part of a system of parameters; in fact, they generate an ideal of height 2.

We now let I_0 denote the kernel of the map from W_0 to $\hat{R}$ as in Section 5. Let T be the perfect closure of the extension of E_0 as defined there. We claim that there is an element of local cohomology of T_0 with support in $(P, V, X + U)$ that maps to the generator of the local cohomology of E_0 defined above and that this element is annihilated by arbitrarily small powers of every one of the generators of the ring E_0. The first statement is clear; the element is defined by the relation

$$(X + U)(YW) = XYW + UYW = VY^2 + PW^2.$$

To see that this element of local cohomology is annihilated by small powers of the generators we need to go back and compute some relations in the ring T. We note first that I_0 is generated by the elements $P^3 + X^3 + Y^3, P^2U + X^2V + Y^2W, PU^2 + XV^2 + YW^2$, and $U^3 + V^3 + W^3$, all computed in $W(E(R))$ (as well as $P - p$). As stated in the previous section, if a denotes one of these elements and we let z be the element with $a = (P - p)z$, then we have

$$z_n = f(P, X_i)/P^{(n+1)p^n},$$

where $f(P, X_i)$ is in E_0. We will compute more precisely what the z_n look like, but first we prove a simple lemma on Witt vectors over graded rings.

Lemma 2. *Let A be a graded ring, and let $f(x_t)$ be a polynomial with coefficients in $\mathbb{Z}$ of degree k with entries in A. Let $\tau(x_t)$ denote the element $(x_t, 0, 0, \ldots)$ of $W(A)$ for each t, and let $f(\tau(x_t)) = (a_0, a_1, \ldots)$ in $W(A)$. Then a_i is homogeneous of degree kp^i for each i.*

Proof. We prove this by induction on i. For $i = 0$ we have that $a_0 = f(x_t)$, which has degree $k = kp^0$, so the lemma is true in this case.

Now let $i > 0$, and assume that the lemma holds for all j with $0 \leq j < i$. We have

$$a_0^{p^i} + pa_1^{p^{i-1}} + \cdots + p^j a_j^{p^{i-j}} + \cdots + p^i a_i = f(x_t^{p^i}).$$

Since $f(x_t)$ is homogeneous of degree k, $f(x_t^{p^i})$ is homogeneous of degree kp^i. Also, for each $j < i$, a_j is homogeneous of degree kp^j by induction, so $a_j^{p^{i-j}}$ is homogeneous of degree kp^i. Hence a_i is also homogeneous of degree kp^i. □

In our example we use two gradings, one in which P, X, Y have degree 1 and U, V, W have degree 0, and one in which P, X, Y have degree 0 and U, V, W have degree 1. This gives a bidegree to each of the generators of I_0. The lemma implies, for example, that when $PU^2 + XV^2 + YW^2$ is expanded as the Witt vector $(a_0, a_1, \ldots,)$, a_i will have degree p^i in the first grading and $2p^i$ in the second grading.

The next Lemma gives a description of the quotient when divided by $P - p$.

Lemma 3. *Let A be a graded ring as above, and let $(a_0, a_1, \ldots)$ be an element of $W(A)$ such that a_i is homogeneous of degree kp^i for each i. Let $(a_0, a_1, \ldots) = (P - p)(z_0, z_1, \ldots)$. Then for each $i \geq 0$ we can write*

$$z_i = \frac{a_0^{p^i} + \sum P^{n_{ij}} b_{ij}}{P^{(i+1)p^i}}$$

where each n_{ij} is a positive integer and b_{ij} is a homogeneous element of degree kp^i.

Proof. Again we prove this by induction on i. We can write the equation defining the z_i as

$$(a_0, a_1, a_2 \ldots) = (Pz_0, P^p z_1, P^{p^2} z_2, \ldots,) - (0, z_0^p, z_1^p, \ldots).$$

Thus for $i = 0$ we have $z_0 = a_0/P$, and since $P = P^{(0+1)p^0}$, this is in the correct form (here all the other terms are zero).

We now assume that the result holds for $j < i$ and prove that it holds for i. The defining equation for z_i is

$$a_0^{p^i} + pa_1^{p^{i-1}} + \cdots + p^j a_j^{p^{i-j}} + \cdots + p^i a_i = (Pz_0)^{p^i} + p(P^p z_1)^{p^{i-1}} + \cdots + p^j (P^{p^j} z_j)^{p^{i-j}} + \cdots + p^i (P^{p^i} z_i)$$
$$-p(z_0^p)^{p^{i-1}} - \cdots - p^j (z_{j-1}^p)^{p^{i-j}} - \cdots - p^i (z_{i-1}^p).$$

Hence z_i is a combination of the other terms in the above expression divided by $p^i P^{p^i}$. The factor p^i will divide the other terms in this expression after the formulas for the z_j for $j < i$ are substituted from the general theory of Witt vectors; the factor we have to consider is P^{p^i}. Thus to complete the proof we must show that each term in the above equation other than $p^i(P^{p^i} z_i)$ is a sum of terms that can be written in the form $P^n a / P^{ip^i}$ with a homogeneous of degree kp^i and that the only term for which $n = 0$ is $a_0^{p^i}$.

Each of the terms $p^j a_j^{p^{i-j}}$ is homogeneous of degree kp^i and we can take $n = ip^i$, so these terms clearly satisfy the required condition.

We next consider an element of the form $p^j (P^{p^j} z_j)^{p^{i-j}}$. By induction, z_j is a sum of terms $P^{k_m} b_m$ divided by $P^{(j+1)p^j}$ with b_m homogeneous of degree kp^j and exactly one $k_m = 0$, for which $b_m = a_0^{p^j}$. When this sum is multiplied by P^{p^j} and raised to the p^{i-j}th power we obtain a sum of integer multiples of terms of the form

$$\frac{(P^{p^j})^{p^{i-j}} \prod_n P^{r_n k_{m_n}} b_{m_n}^{r_n}}{(P^{(j+1)p^j})^{p^{i-j}}}. \qquad (***)$$

In this product the sum of the r_n is p^{j-i}, the k_{m_n} are positive except for one term (coming from $a_0^{p^j}$) which we compute below, and the b_{m_n} are homogeneous of degree kp^j. It follows that the product of the $b_{m_n}^{r_n}$ is homogeneous of degree $(\sum r_n)kp^j = p^{i-j}kp^j = kp^i$. Denoting this product b, and letting $k = \sum r_n k_{m_n}$, we can write this term in the form

$$\frac{P^{p^i+k}b}{P^{(j+1)p^i}}.$$

Since the denominator of this term is $(P^{(j+1)p^j})^{p^{i-j}} = P^{(j+1)p^i}$, if $j < i-1$, then any term of this form can be written in the desired form with positive power of P. If $j = i$, there is only one term where the power is zero. That term has to come from a product of b_{m_n} in expression $(***)$ in which every k_{m_n} is zero, and the only such term is $(a_0^{p^i})^p = a_0^{p^{i+1}}$. Hence z_i has the stated form, so this completes the proof.

$\square$

We will now show that for η equal to each of the variables P, X, Y, U, V, W and for any $n \geq 1$, we have $\eta^{1/p^n}YW \in (P,V)T$. Choose such an n, and let m be a positive integer such that $4/p^m < 1/p^n$. Let a be one of the four generators of I_0 as above, and let (z_i) be the the Witt vector $a/(P-p)$. We consider z_i for $i = p^k - 1$. By Lemma 3 we know that z_i is the quotient whose numerator is a polynomial in P with constant term $a_0^{p^i}$ and coefficients homogeneous of the degree of a_0 times p^i and whose denominator is $P^{(i+1)p^i} = P^{p^k p^i} = P^{p^{k+i}}$. We now take the p^{k+i}th root of this element. It is now of the form β/P, where β is a polynomial in P with fractional exponents with constant term a_0^{1/p^k}. The coefficients are homogeneous of the degree of a_0 divided by p^k. We have $\beta = P(\beta/P) \in (P,V)T$, so β is a multiple of P in T so is in the ideal (P,V).

Thus we are reduced to showing that if we set the elements described in the previous paragraph to zero for each of the four generators of I_0 and for k ranging between 0 and m, we can show that $\eta^{1/p^n}YW$ is in the ideal (P,V). To illustrate the method, we outline one step in detail. Letting $k = 1$, the procedure above applied to the element $PU^2 + XV^2 + YW^2$ gives an element $Y^{1/p}W^{2/p} - (-P^{1/p}U^{2/p} - X^{1/p}V^{2/p} + P^a\gamma)$, where a is a positive rational number and γ is a polynomial each of whose terms has degree in P, X, Y at least $1/p$ and degree in U, V, W at least $2/p$. Thus if we substitute the expression in parentheses for $Y^{1/p}W^{2/p}$ we will decrease the total degree of $\eta^{1/p^n}YW$ in Y and W and add terms that are multiples of P^a. We will show that if the degree in Y and W of a monomial of the same bidegree as $\eta^{1/p^n}YW$ is small enough, then the monomial is in (P,V). Thus this process will eventually increase the (rational) power of P that divides our element, so that it will eventually be a multiple of P. It is important that we are fixing a bound m on the exponents that we use, so that this is in fact a finite process and will eventually terminate.

Let $m = P^aU^bX^cV^dY^eW^f$ be a monomial satisfying the inequalities on the degrees satisfied by $\eta^{1/p^n}YW$; that is, the degrees in P, X, Y and U, V, W are at least 1 and the total degree is at least $2 + 1/p^n$. If $a + b \geq 1$, since $a + c + e \geq 1$, we can use the relations $X^{1/p^k}U^{1/p^k} = P^{1/p^k}V^{1/p^k}$ and $Y^{1/p^k}U^{1/p^k} = P^{1/p^k}W^{1/p^k}$ to raise the exponent of P to 1, so the term is a multiple of P. Similarly, if $c + d \geq 1$ we can show that the term is a multiple of V. If both are less than one, then $e + f > 1/p^n$, since the total degree is at least $2 + 1/p^n$. Since $4/p^m < 1/p^n$, we can find nonnegative integers i and j with $i/p^m \leq e$, $j/p^m \leq f$, and $i + j = 3$. We then change the leading term by one of the generators as in the previous paragraph and replace this term by terms that either lower the degree of the leading coefficient in Y and W or are multiples of a higher power of P. We can continue this until the result is a multiple of P, so that the whole element is in $(P,V)T$.

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