EXTRA CREDIT – MATH 435

DUE APRIL 27TH, 2012

Exercise 0.1. First show that there exists a function $f\colon \mathbb{N} \to \mathbb{Q}_{>0}$ that is surjective. Here $\mathbb{Q}_{>0}$ is used to denote the set of positive rational numbers. (1 point)

Hint: We are trying to organize the rational numbers into an infinite list. Consider arranging the positive rational numbers like the diagram below.

$$\begin{array}{ccccc} 1/1 & 1/2 & 1/3 & 1/4 & \dots \\ 2/1 & 2/2 & 2/3 & 2/4 & \dots \\ 3/1 & 3/2 & 3/3 & 3/4 & \dots \\ 4/1 & 4/2 & 4/3 & 4/4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

Exercise 0.2. Show that there is a surjective function $g : \mathbb{N} \to \mathbb{Q}$. (1 point)

Hint: First construct an surjective function from $\mathbb{Z}$ to $\mathbb{Q}$, then find a surjective function from $\mathbb{N}$ to $\mathbb{Z}$. Conclude by composing surjective functions.

We know that

- Between two rational numbers there is an irrational number.
- Between two irrational numbers there is an rational number.

This can give a feeling that there are as many irrationals than rationals. Since both sets have infinitely many numbers, this is true in some sense. However, sets with infinitely many numbers can be quite different and we will argue in what follows that in some sense there are more irrationals than rationals.

Theorem 0.3 (Cantor)**.** *There is NO function $f\colon \mathbb{N} \to [0, 1)$ that is surjective.*

Our next goal is to prove this theorem.

We will prove the theorem by contradiction. Suppose we have such a function f. Write each $f(n)$ as a decimal expansion and arrange them in a list like so,

$$\begin{array}{lcccccccc} f(1) = & 0 & . & d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & \dots \\ f(2) = & 0 & . & d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & \dots \\ f(3) = & 0 & . & d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & \dots \\ f(4) = & 0 & . & d_{41} & d_{42} & d_{43} & d_{44} & d_{45} & \dots \\ \dots & & & & & & & & \end{array}$$

We also assume that each decimal expansion does NOT have an infinite trail of 9s at the end. (Remember $0.64999999999\ldots = 0.65$). Recall that once we make this assumption, the decimal expansion of a number is unique (that is, there is only one way to do it).

Exercise 0.4. Prove Theorem 0.3. Then prove that there is NO surjective function $g : \mathbb{N} \to \mathbb{R}$. (2 points)

Hint: For the first part, draw a diagonal line down and to the right, starting at the digit d_{11}. By considering the digits on that diagonal, explain why you can construct a new decimal number that is

(a) Not on the list.

(b) Inside the interval $(0, 1)$.
(c) Does not have any 9s at all in it (let alone an infinite trail of 9s).

For the second part, suppose there was, then compose with a surjective map $\mathbb{R} \to [0, 1)$ (which you must find).

We are now ready to show that there are "more" irrationals than rationals. But first a bit of notation, $\mathbb{R} \setminus \mathbb{Q}$ is just an expression that means *all the real numbers* that are *not rational*. That is $\mathbb{R} \setminus \mathbb{Q}$ is equal to the set of irrational numbers.

Exercise 0.5. Show that there is no function $f \colon \mathbb{N} \to \mathbb{R} \setminus \mathbb{Q}$ that is surjective. (1 point)

Now, we come to the question of algebraic numbers.

Exercise 0.6. Let $\mathbb{A}$ be the set of algebraic real numbers (here algebraic means algebraic over $\mathbb{Q}$). Prove that there is a surjection $\mathbb{N} \to \mathbb{A}$. (3 points)

Exercise 0.7. Prove that there is NO surjection $\mathbb{N} \to \mathbb{R} \setminus \mathbb{A}$. (1 point)