QUIZ #3 – MATH 435

MARCH 12TH, 2012

1. Consider the following elements of S_4:

- $\sigma = (123)$
- $\tau = (12)(34)$

Compute $\sigma\tau\sigma^{-1}$ and also compute $\tau\sigma\tau^{-1}$ (by compute, we mean disjoint cycle form, or as a cycle). What do you notice about the *shape* of the outputs? (2 points)

Solution:

$$\sigma\tau\sigma^{-1} = (123)(12)(34)(321) = (14)(23)$$
$$\tau\sigma\tau^{-1} = (12)(34)(123)(43)(21) = (142)$$

I notice that $\sigma\tau\sigma^{-1}$ has the same shape as τ and $\tau\sigma\tau^{-1}$ has the same shape as σ.

2. State the class equation, but *do NOT* prove it. (1 point)

Solution:

$$|G| = |Z(G)| + \sum_{\substack{x \in G \\ |\operatorname{Orb}_G(x)| \geq 2}} |G|/|\operatorname{Stab}_G(x)|$$

where the Orbit of x is under G's action on itself by conjugation.

3. Prove the orbit stabilizer theorem. In other words, suppose that G is a finite group acting on a set S. Show that for any $x \in S$ that

$$|G| = |\operatorname{Orb}_G(x)| \cdot |\operatorname{Stab}_G(x)|. \qquad \text{(2 points)}$$

Solution:
We already know that

$$|G| = (\# \text{ of cosets of } \operatorname{Stab}_G(x)) \cdot |\operatorname{Stab}_G(x)|$$

by Lagrange's theorem. Therefore it is sufficient to prove that

$$(\# \text{ of cosets of } \operatorname{Stab}_G(x)) = |\operatorname{Orb}_G(x)|.$$

We will give a bijection between these two sets which will accomplish this. Define a function

$$\Phi : \{\text{left cosets of } \operatorname{Stab}_G(x)\} \to \operatorname{Orb}_G(x)$$

by the rule $\Phi(a \operatorname{Stab}_G(x)) = a.x$.

For simplicity, we write $H = \operatorname{Stab}_G(x)$ and so our Φ becomes $\Phi(aH) = a.x$.

We need to show that Φ is well defined, surjective, and injective. Note it is *NOT* a homomorphism since $\operatorname{Orb}_G(x)$ is almost certainly not a group.

We first show that Φ is well defined. So suppose that $aH = bH$. We need to show that

$$\Phi(aH) = a.x = b.x = \Phi(bH).$$

Since $aH = bH$, we know that $a \in bH$ so that $a = bh$ for some $h \in H = \mathrm{Stab}_G(x)$. Note that $h.x = x$ by our choice of h. Then

$$a.x = (bh).x = b.(h.x) = b.x$$

which proves that Φ is well defined.

Now we prove that Φ is surjective. Suppose that $y = g.x \in \mathrm{Orb}_G(x)$. Then $\Phi(gH) = g.x = y$ and we conclude that Φ is surjective.

Finally, we prove that Φ is injective. Suppose that $\Phi(aH) = \Phi(bH)$. Thus $a.x = b.x$ and so

$$x = e.x = (a^{-1}a).x = a^{-1}.(a.x) = a^{-1}.(b.x) = (a^{-1}b).x$$

Thus $a^{-1}b \in \mathrm{Stab}_G(x) = H$ by definition. Therefore $a^{-1}bH = H$ and so $bH = aH$ which proves that Φ is injective as desired.