QUIZ #4 – MATH 435

MARCH 23RD, 2012

1. Suppose that R is a ring.

(a) Prove that $a(-b) = -(ab)$ for all $a, b \in R$. (1 point)

(b) Suppose R is a commutative ring for which $ab = ac$ implies $b = c$ whenever $a \neq 0$. Prove that R is an integral domain. (1 point)

Solution:

(a) $a(-b) + ab = a(-b + b) = a0 = 0$. Thus $a(-b) = -(ab)$.

(b) Suppose $ab = 0$, and that $a \neq 0$ and $b \neq 0$. Thus $ab = 0 = a0$ which implies that $b = 0$, a contradiction. Thus either $a = 0$ or $b = 0$ as desired.

2. Consider the set S of 2×2 matrices of the form $\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}$ with entries x, y, z in $\mathbb{Z}$. You may take it as given that this is a subring of 2×2 matrices with coefficients in $\mathbb{R}$.

(a) Determine whether this ring is commutative or not (justify your answer). (1 point)

(b) Show that this ring is not a division ring but that it does have unity. (1 point)

(c) Find 4 different elements of S which are invertible. (1 point)

Solution:

(a) This ring is not commutative, note $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but that we do have $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$.

(b) It does have the identity element $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but it is not a division ring since if $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since then $2x = 1$ and $0z = 1$. Neither of those equations have solutions in $\mathbb{Z}$.

(c) Here are my 4 elements:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

The first three elements are their own inverses. The last element has inverse $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$