WORKSHEET # 7
QUATERNIONS

The set of *quaternions* are the set of all formal $\mathbb{R}$-linear combinations of “symbols” i, j, k

$$a + bi + cj + dk \quad \text{for } a, b, c, d \in \mathbb{R}.$$

We give them the following addition rule:

$$(a + bi + cj + dk) + (a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k$$

Multiplication is induced by the following rules ($\lambda \in \mathbb{R}$)

$$i^2 = j^2 = k^2 = -1$$

$$\begin{array}{lll} ij = k, & jk = i, & ki = j \\ ji = -k & kj = -i & ik = -j \\ \lambda i = i\lambda & \lambda j = j\lambda & \lambda k = k\lambda \end{array}$$

combined with the distributive rule.

1. Use the above rules to carefully write down the product

$$(a + bi + cj + dk)(a' + b'i + c'j + d'k)$$

as an element of the quaternions.

Solution:

$$\begin{array}{rl} & (a + bi + cj + dk)(a' + b'i + c'j + d'k) \\ = & aa' + ab'i + ac'j + ad'k + ba'i - bb' + bc'k - bd'j \\ & ca'j - cb'k - cc' + cd'i + da'k + db'j - dc'i - dd' \\ = & (aa' - bb' - cc' - dd') + (ab' + a'b + cd' - dc')i \\ & (ac' - bd' + ca' + db')j + (ad' + bc' - cb' + da')k \end{array}$$

2. Prove that the quaternions form a ring.

Solution: It is obvious that the quaternions form a group under addition. We just showed that they were closed under multiplication. We have defined them to satisfy the distributive property.

The only thing left is to show the associative property. I will only prove this for the elements i, j, k (this is sufficient by the distributive property).

$$i(ij) = i(k) = ik = -j = (ii)j$$
$$i(ik) = i(-j) = -ij = -k = (ii)k$$
$$i(ji) = i(-k) = -ik = j = ki = (ij)i$$
$$i(jj) = -i = kj = (ij)j$$
$$i(jk) = i(i) = (-1) = (k)k = (ij)k$$
$$i(ki) = ij = k = -ji = (ik)i$$
$$i(kk) = -i = -jk = (ik)k$$
$$j(ii) = -j = -ki = (ji)i$$
$$j(ij) = jk = i = -kj = (ji)j$$
$$j(ik) = j(-j) = 1 = -kk = (ji)k$$
$$j(ji) = j(-k) = -jk = -i = (jj)i$$
$$j(jk) = ji = -k = (jj)k$$
$$j(ki) = jj = -1 = ii = (jk)i$$
$$j(kj) = j(-i) = -ji = k = ij = (jk)j$$
$$j(kk) = -j = ik = (jk)k$$
$$k(ii) = -k = ji = (ki)i$$
$$k(ij) = kk = -1 = jj = (ki)j$$
$$k(ik) = k(-j) = -kj = i = jk = (ki)k$$
$$k(ji) = k(-k) = 1 = (-i)i = (kj)i$$
$$k(jj) = k(-1) = -k = -ij = (kj)j$$
$$k(jk) = ki = j = -ik = (kj)k$$
$$k(ki) = kj = -i = (kk)i$$
$$k(kj) = k(-i) = -ki = -j = (kk)j$$

3. Prove that the quaternions form a division ring.

Solution: Given $a + bi + cj + dk \neq 0$, consider

$$\frac{a}{a^2+b^2+c^2+d^2} - \frac{b}{a^2+b^2+c^2+d^2}i - \frac{c}{a^2+b^2+c^2+d^2} - \frac{d}{a^2+b^2+c^2+d^2}k = \frac{1}{a^2+b^2+c^2+d^2}(a - bi - cj - dk)$$

Notice that this makes sense as long as $a^2 + b^2 + c^2 + d^2 \neq 0$, but that certainly happens as long as a, b, c, d are not all zero (at the same time).

Then it is easy to verify that the product

$$(a + bi + cj + dk)\left(\frac{1}{a^2 - b^2 - c^2 - d^2}(a - bi - cj - dk)\right)$$

is equal to 1.

4. Find noncommutative subring of the quaternions that is *not* a division ring.

Solution: Consider the subset of the quaternions.

$$S = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z}\}$$

It is easy to see that this is a subring of the quaternions (and of course is non-commutative since $ij = -k \neq k = ji$. Notice that this set has the same multiplicative identity $1 + 0i + 0j + 0k$ as the ordinary quaternions, but it is not a division ring since the multiplicative inverse of 2 in the ordinary quaternions is $1/2$, which does not live in this subset.

5. Prove that the equation $x^2 = -1$ has an *infinite* number of solutions in the quaternions.

Solution: Choose *any* real number b such that $-1 \leq b \leq 1$ (there are infinitely many possibilities). Then choose $c = \sqrt{1 - b^2}$. Notice that $b^2 + c^2 = 1$ and that there are infinitely many such b, c pairs (points on the circle). Now, observe that

$$(bi + cj)(bi + cj) = b^2i^2 + bcij + cbji + c^2j^2 = -b^2 + bck - bck - c^2 = -(b^2 + c^2) = -1$$