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Local cohomology modules of a smooth $\mathbb{Z}$-algebra have finitely many associated primes

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Abstract Let R be a commutative Noetherian ring that is a smooth $\mathbb{Z}$-algebra. For each ideal $\mathfrak{a}$ of R and integer k, we prove that the local cohomology module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals. This settles a crucial outstanding case of a conjecture of Lyubeznik asserting this finiteness for local cohomology modules of all regular rings.

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1 Introduction

A question of Huneke [5, Problem 4] asks whether local cohomology modules of Noetherian rings have finitely many associated prime ideals. The answer is negative in general: the first counterexample was given by Singh [13, Sect. 4], and further counterexamples were obtained by Katzman [7] and Singh and Swanson [14].

However, there are several affirmative answers: by work of Huneke and Sharp [6], for regular rings R of prime characteristic; by work of Lyubeznik, for regular local and affine rings of characteristic zero [8], and for unramified regular local rings of mixed characteristic [10]; for a partial result in the case of ramified regular local rings, see Núñez-Betancourt [12]. These results support Lyubeznik's conjecture, [8, Remark 3.7]:

Conjecture 1.1 If R is a regular ring, then each local cohomology module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals.

While the counterexamples from [7] and [14] are for rings containing a field, the local cohomology module with infinitely many associated primes from [13] has the form $H^k_{\mathfrak{a}}(R)$ where R is a hypersurface over the integers; in this example, $H^k_I(R)$ has nonzero p-torsion for each prime integer p. A major stumbling block in making progress with Lyubeznik's conjecture for rings not containing a field was the possibility of p-torsion for infinitely many prime integers p. The key point in this paper is to show that for a smooth $\mathbb{Z}$-algebra R, the p-torsion of each local cohomology module $H^k_{\mathfrak{a}}(R)$ can be controlled; this allows us to settle an important case of Lyubeznik's conjecture:

Theorem 1.2 *Let R be a smooth $\mathbb{Z}$-algebra, $\mathfrak{a}$ an ideal of R, and k a non-negative integer. Then the set of associated primes of the local cohomology module $H^k_{\mathfrak{a}}(R)$ is finite.*

Our proof uses $\mathscr{D}$-modules over $\mathbb{Z}$, $\mathbb{F}_p$, and $\mathbb{Q}$, along with the theory of $\mathscr{F}$-modules developed in [9]. The relevant results are reviewed in Sect. 2. A crucial step in the proof is to relate the integer torsion in a local cohomology module to the integer torsion in a Koszul cohomology module; since the latter is finitely generated, it has p-torsion for at most finitely many p. The proof of the main theorem occupies Sect. 3.

Our techniques work somewhat more generally: in Sect. 4 we indicate the changes that need to be made to tackle the case where R is a smooth algebra over a Dedekind domain, all of whose residue fields at nonzero prime ideals are of characteristic p. Our techniques are also sufficient to give a new and much simpler proof of the case of an unramified regular local ring of mixed characteristic, originally obtained by Lyubeznik in [10].

2 $\mathscr{D}$-modules and $\mathscr{F}$-modules

2.1 $\mathscr{D}$-modules

Let R be a commutative ring. *Differential operators* on R are defined inductively as follows: for each $r \in R$, the multiplication by r map $\widetilde{r}\colon R \longrightarrow R$ is a differential operator of order 0; for each positive integer n, the differential operators of order less than or equal to n are those additive maps $\delta\colon R \longrightarrow R$ for which the commutator

$$[\widetilde{r}, \delta] = \widetilde{r} \circ \delta - \delta \circ \widetilde{r}$$

is a differential operator of order less than or equal to $n-1$. If δ and δ' are differential operators of order at most m and n respectively, then $\delta \circ \delta'$ is a differential operator of order at most $m+n$. Thus, the differential operators on R form a subring $\mathscr{D}(R)$ of $\mathrm{End}_{\mathbb{Z}}(R)$.

When R is an algebra over a commutative ring A, we define $\mathscr{D}(R, A)$ to be the subring of $\mathscr{D}(R)$ consisting of differential operators that are A-linear. Note that $\mathscr{D}(R, \mathbb{Z}) = \mathscr{D}(R)$; if R is an algebra over a perfect field $\mathbb{F}$ of prime characteristic, then $\mathscr{D}(R, \mathbb{F}) = \mathscr{D}(R)$, see, for example, [9, Example 5.1 (c)].

By a $\mathscr{D}(R, A)$-module, we mean a *left* $\mathscr{D}(R, A)$-module. Since $\mathscr{D}(R, A) \subseteq \mathrm{End}_A(R)$, the ring R has a natural $\mathscr{D}(R, A)$-module structure. Using the quotient rule, localizations of R also carry a natural $\mathscr{D}(R, A)$-structure. Let $\mathfrak{a}$ be an ideal of R. The Čech complex on a generating set for $\mathfrak{a}$ is a complex of $\mathscr{D}(R, A)$-modules; it then follows that each local cohomology module $H^k_{\mathfrak{a}}(R)$ is a $\mathscr{D}(R, A)$-module.

More generally, if M is a $\mathscr{D}(R, A)$-module, then each local cohomology module $H^k_{\mathfrak{a}}(M)$ is also a $\mathscr{D}(R, A)$-module, see [8, Examples 2.1 (iv)] or [9, Example 5.1 (b)].

If R is a polynomial or formal power series ring in variables $x_1, \dots, x_d$ over a commutative ring A, then $\frac{1}{t_i!}\frac{\partial^{t_i}}{\partial x_i^{t_i}}$ can be viewed as a differential operator on R even if the integer $t_i!$ is not invertible. In each of these cases,

$\mathscr{D}(R, A)$ is the free R-module with basis

$$\frac{1}{t_1!}\frac{\partial^{t_1}}{\partial x_1^{t_1}} \cdots \frac{1}{t_d!}\frac{\partial^{t_d}}{\partial x_d^{t_d}} \quad \text{for } (t_1, \dots, t_d) \in \mathbb{N}^d,$$

see [3, Théorème 16.11.2]. If B is an A-algebra, it follows that

$$\mathscr{D}(R, A) \otimes_A B \cong \mathscr{D}(R \otimes_A B, B).$$

Specifically, for each element $a \in A$, one has

$$\mathscr{D}(R, A)/a\mathscr{D}(R, A) \cong \mathscr{D}(R/aR, A/aA). \tag{2.1}$$

To obtain analogous results for any smooth A-algebra, we use an alternative description of $\mathscr{D}(R, A)$ from [3, 16.8]: consider the left $R \otimes_A R$-module structure on $\operatorname{End}_A(R)$ under which $r \otimes s$ acts on δ to give the endomorphism $\widetilde{r} \circ \delta \circ \widetilde{s}$ where, as before, $\widetilde{r}$ denotes the multiplication by r map. Set $\Delta_{R/A}$ to be the kernel of the ring homomorphism $R \otimes_A R \longrightarrow R$ with $r \otimes s \longmapsto rs$. The ideal $\Delta_{R/A}$ is generated by elements of the form $r \otimes 1 - 1 \otimes r$. Since

$$(r \otimes 1 - 1 \otimes r)(\delta) = [\widetilde{r}, \delta],$$

it follows that an element δ of $\operatorname{End}_A(R)$ is a differential operator of order at most n precisely if it is annihilated by $\Delta_{R/A}^{n+1}$. By [3, Proposition 16.8], the A-linear differential operators on R of order at most n correspond to

$$\operatorname{Hom}_{R \otimes_A R}\big((R \otimes_A R)/\Delta_R^{n+1}, \operatorname{End}_A(R)\big) \cong \operatorname{Hom}_R\big(P^n_{R/A}, R\big),$$

where

$$P^n_{R/A} = (R \otimes_A R)/\Delta_R^{n+1},$$

viewed as a left R-module via $r \longmapsto r \otimes 1$.

A ring R is said to be *smooth* over A if R is a finitely presented and flat A-algebra, such that for each prime ideal $\mathfrak{p}$ of A, the fiber $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is geometrically regular over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. In this situation, we have:

Lemma 2.1 *If R is a smooth A-algebra, then for each A-algebra B one has*

$$\mathscr{D}(R, A) \otimes_A B \cong \mathscr{D}(R \otimes_A B, B).$$

Proof Since R is A-smooth, the R-module $P^n_{R/A}$ is locally free of finite rank by [3, Proposition 16.10.2]. It follows that

$$\operatorname{Hom}_R\big(P^n_{R/A}, R\big) \otimes_A B \cong \operatorname{Hom}_{R_B}\big(P^n_{R/A} \otimes_A B, R_B\big), \tag{2.2}$$

where $R_B = R \otimes_A B$. Since R is flat over A, one also has $\Delta_{R/A} \otimes_A B \cong \Delta_{R_B/B}$. Tensoring the exact sequence

$$0 \longrightarrow \Delta_{R/A}^{n+1} \longrightarrow R \otimes_A R \longrightarrow P_{R/A}^n \longrightarrow 0$$

with B, one obtains the first row of the commutative diagram

$$\begin{array}{ccccccccc} & & \Delta_{R/A}^{n+1} \otimes_A B & \longrightarrow & R_B \otimes_B R_B & \longrightarrow & P_{R/A}^n \otimes_A B & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \Delta_{R_B/B}^{n+1} & \longrightarrow & R_B \otimes_B R_B & \longrightarrow & P_{R_B/B}^n & \longrightarrow & 0. \end{array}$$

The vertical map on the left is surjective, which gives $P_{R/A}^n \otimes_A B \cong P_{R_B/B}^n$. Combining this with (2.2), we get the desired isomorphism

$$\operatorname{Hom}_R\big(P_{R/A}^n, R\big) \otimes_A B \cong \operatorname{Hom}_{R_B}\big(P_{R_B/B}^n, R_B\big). \qquad \square$$

2.2 $\mathscr{F}$-modules

We next review some aspects of the theory of $\mathscr{F}$-modules, developed by Lyubeznik in [9]. Let R be an F-finite regular ring of prime characteristic p. For each positive integer e, define $R^{(e)}$ to be the R-bimodule that agrees with R as a left R-module, and that has the right R-action

$$r'r = r^{p^e} r' \quad \text{for } r \in R \text{ and } r' \in R^{(e)}.$$

For an R-module M, define $F(M) = R^{(1)} \otimes_R M$; we view this as an R-module via the left R-module structure on $R^{(1)}$.

An *$\mathscr{F}$-module* is an R-module $\mathscr{M}$ with an R-module isomorphism $\theta\colon \mathscr{M} \longrightarrow F(\mathscr{M})$. The ring R has a natural $\mathscr{F}$-module structure, and so does each local cohomology module $H_{\mathfrak{a}}^k(R)$, see [9, Example 1.2]. An $\mathscr{F}$-module carries a natural $\mathscr{D}(R)$-module structure by [9, pages 115–116]. When the $\mathscr{F}$-module $\mathscr{M}$ is the ring R, a localization of R, or a local cohomology module $H_{\mathfrak{a}}^k(R)$, the usual $\mathscr{D}(R)$-module structure on $\mathscr{M}$ agrees with the one induced via the $\mathscr{F}$-module structure; see [9, Example 5.2 (c)].

A *generating morphism* for an $\mathscr{F}$-module $\mathscr{M}$ is an R-module map $\beta\colon M \longrightarrow F(M)$ such that $\mathscr{M}$ is the direct limit of the top row of the commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & \cdots \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle F(\beta)}\downarrow & & {\scriptstyle F^2(\beta)}\downarrow & & \\ F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & F^3(M) & \xrightarrow{F^3(\beta)} & \cdots. \end{array}$$

Note that the direct limit of the bottom row is $F(\mathscr{M})$, and that the vertical maps induce the isomorphism $\theta\colon \mathscr{M} \longrightarrow F(\mathscr{M})$. If $\beta\colon M \longrightarrow F(M)$ is a generating morphism for $\mathscr{M}$, then the image of M in $\mathscr{M}$ generates $\mathscr{M}$ as a $\mathscr{D}(R)$-module by [1, Corollary 4.4]; this is a key ingredient in the proof of our main result.

2.3 Koszul and local cohomology

Given $f \in R$, there is a map of complexes

$$\begin{array}{ccccccccc}
K^\bullet(f;R) & = & 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & 0 \\
\downarrow & & & & \| & & \downarrow{\scriptstyle f^{p-1}} & & \\
K^\bullet\big(f^p;R\big) & = & 0 & \longrightarrow & R & \xrightarrow{f^p} & R & \longrightarrow & 0 \\
\downarrow & & & & \| & & \downarrow{\scriptstyle \frac{1}{f^p}} & & \\
C^\bullet(f;R) & = & 0 & \longrightarrow & R & \longrightarrow & R_f & \longrightarrow & 0,
\end{array}$$

where $K^\bullet$ denotes the Koszul complex, and $C^\bullet$ the Čech complex. Let $\boldsymbol{f} = f_1, \ldots, f_t$ be a sequence of elements of R. Regarding $K^\bullet(\boldsymbol{f};R)$ and $C^\bullet(\boldsymbol{f};R)$ as the tensor products

$$K^\bullet(f_1;R) \otimes \cdots \otimes K^\bullet(f_t;R) \quad \text{and} \quad C^\bullet(f_1;R) \otimes \cdots \otimes C^\bullet(f_t;R)$$

respectively, one obtains a map of complexes

$$K^\bullet(\boldsymbol{f};R) \longrightarrow K^\bullet\big(\boldsymbol{f}^p;R\big) \longrightarrow C^\bullet(\boldsymbol{f};R),$$

and induced maps on cohomology modules

$$H^k(\boldsymbol{f};R) \xrightarrow{\beta} H^k\big(\boldsymbol{f}^p;R\big) \longrightarrow H^k_{\mathfrak{a}}(R),$$

where $\mathfrak{a}$ is the ideal generated by $\boldsymbol{f}$. By [9, Proposition 1.11 (b)], the map β is a generating homomorphism for the local cohomology module $H^k_{\mathfrak{a}}(R)$; hence the image of $H^k(\boldsymbol{f};R)$ in $H^k_{\mathfrak{a}}(R)$ generates $H^k_{\mathfrak{a}}(R)$ as a $\mathscr{D}(R)$-module, as mentioned at the end of Sect. 2.2.

3 The main theorem

We prove the following result that subsumes Theorem 1.2.

Theorem 3.1 *Let R be a smooth $\mathbb{Z}$-algebra, and $\mathfrak{a}$ an ideal of R generated by elements $\boldsymbol{f} = f_1, \ldots, f_t$. Let k be a nonnegative integer.*

(1) *If a prime integer is a nonzerodivisor on the Koszul cohomology module* $H^k(\boldsymbol{f}; R)$, *then it is a nonzerodivisor on the local cohomology module* $H^k_{\mathfrak{a}}(R)$.
(2) *All but finitely many prime integers are nonzerodivisors on* $H^k_{\mathfrak{a}}(R)$.
(3) *The set of associated primes of the* R*-module* $H^k_{\mathfrak{a}}(R)$ *is finite.*

Proof Let p be a prime integer. The exact sequence

$$0 \longrightarrow R \xrightarrow{\ p\ } R \longrightarrow R/pR \longrightarrow 0$$

induces an exact sequence of Koszul cohomology modules and an exact sequence of local cohomology modules; these fit into a commutative diagram:

$$\begin{array}{ccccccc}
H^{k-1}(\boldsymbol{f}; R) & \xrightarrow{\ \pi\ } & H^{k-1}(\boldsymbol{f}; R/pR) & \longrightarrow & H^k(\boldsymbol{f}; R) & \xrightarrow{\ p\ } & H^k(\boldsymbol{f}; R) \\
{\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow \\
H^{k-1}_{\mathfrak{a}}(R) & \xrightarrow{\ \varphi\ } & H^{k-1}_{\mathfrak{a}}(R/pR) & \xrightarrow{\ d\ } & H^k_{\mathfrak{a}}(R) & \xrightarrow{\ p\ } & H^k_{\mathfrak{a}}(R)
\end{array}$$

The bottom row is a complex of $\mathscr{D}(R)$-modules; in particular, $\varphi(H^{k-1}_{\mathfrak{a}}(R))$ is a $\mathscr{D}(R)$-submodule of $H^{k-1}_{\mathfrak{a}}(R/pR)$. As $\varphi(H^{k-1}_{\mathfrak{a}}(R))$ is annihilated by p, it has a natural structure as a module over the ring $\mathscr{D}(R)/p\mathscr{D}(R)$, which equals $\mathscr{D}(R/pR)$ by Lemma 2.1. Similarly,

$$H^{k-1}_{\mathfrak{a}}(R/pR) \xrightarrow{\ d\ } \text{image}(d) \tag{3.1}$$

is a map of $\mathscr{D}(R/pR)$-modules.

(1) Suppose p is a nonzerodivisor on $H^k(\boldsymbol{f}; R)$. Then the map π is surjective; we need to prove that p is a nonzerodivisor on $H^k_{\mathfrak{a}}(R)$, equivalently, that φ is surjective.

By Sect. 2.3, the image M of α generates $H^{k-1}_{\mathfrak{a}}(R/pR)$ as a $\mathscr{D}(R/pR)$-module. As π is surjective, M is also the image of $\alpha \circ \pi = \varphi \circ \alpha'$. It follows that

$$M \subseteq \varphi\big(H^{k-1}_{\mathfrak{a}}(R)\big).$$

But $\varphi(H^{k-1}_{\mathfrak{a}}(R))$ is a $\mathscr{D}(R/pR)$-submodule of $H^{k-1}_{\mathfrak{a}}(R/pR)$ that contains M. Hence

$$\varphi\big(H^{k-1}_{\mathfrak{a}}(R)\big) = H^{k-1}_{\mathfrak{a}}(R/pR),$$

i.e., φ is surjective, as desired.

(2) Since $H^k(\boldsymbol{f}; R)$ is a finitely generated R-module, it has finitely many associated prime ideals. These finitely many prime ideals contain at most finitely many prime integers; all other prime integers are nonzerodivisors on $H^k(\boldsymbol{f}; R)$, and hence on $H^k_{\mathfrak{a}}(R)$ by (1).

(3) We have proved that the set $\operatorname{Ass}_{\mathbb{Z}} H^k_{\mathfrak{a}}(R)$ is finite; let $\mathfrak{p}$ be an element of this set. It suffices to show that there are at most finitely many elements of $\operatorname{Ass}_R H^k_{\mathfrak{a}}(R)$ that lie over $\mathfrak{p}$.

If $\mathfrak{p}$ is the zero ideal, then each associated prime of $H^k_{\mathfrak{a}}(R)$ lying over $\mathfrak{p}$ is the contraction of an associated prime of

$$H^k_{\mathfrak{a}}(R) \otimes_{\mathbb{Z}} \mathbb{Q} = H^k_{\mathfrak{a}}(R \otimes_{\mathbb{Z}} \mathbb{Q})$$

as an $R \otimes_{\mathbb{Z}} \mathbb{Q}$-module. Since $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is a regular finitely generated $\mathbb{Q}$-algebra, these associated primes are finite in number by [8, Remark 3.7 (i)].

If $\mathfrak{p}$ is generated by a prime integer p, the exactness of

$$H^{k-1}_{\mathfrak{a}}(R/pR) \xrightarrow{\ d\ } H^k_{\mathfrak{a}}(R) \xrightarrow{\ p\ } H^k_{\mathfrak{a}}(R)$$

shows that an associated prime of $H^k_{\mathfrak{a}}(R)$ that contains p is an associated prime of

$$\ker(p) = \operatorname{image}(d).$$

It thus suffices to show that $\operatorname{image}(d)$ has finitely many associated primes as an R-module, or, equivalently, as an R/pR-module.

Recall that (3.1) is a surjection of $\mathscr{D}(R/pR)$-modules. By [9, Corollary 5.10], the module $H^{k-1}_{\mathfrak{a}}(R/pR)$ has finite length as a $\mathscr{D}(R/pR)$-module, and hence so does $\operatorname{image}(d)$. The associated primes of $\operatorname{image}(d)$ are among the minimal primes of its simple $\mathscr{D}(R/pR)$-module subquotients; it thus suffices to show that each simple $\mathscr{D}(R/pR)$-module has a unique associated prime. Indeed, let M be a simple $\mathscr{D}(R/pR)$-module, and $\mathfrak{p}$ a maximal element of $\operatorname{Ass}_{R/pR} M$. Then $H^0_{\mathfrak{p}}(M)$ is a $\mathscr{D}(R/pR)$-submodule of M, and hence it must equal M. But $\mathfrak{p}$ is maximal in $\operatorname{Ass}_{R/pR} M$, so it is the unique associated prime of M. □

We conclude the section with two examples:

Example 3.2 Given a finite set of prime integers S, there exists a polynomial ring R over $\mathbb{Z}$, a monomial ideal $\mathfrak{a}$ in R, and an integer k, such that $H^k_{\mathfrak{a}}(R)$ has p-torsion if and only if $p \in S$; see [16, Example 5.11].

Example 3.3 Let E be an elliptic curve in $\mathbb{P}^2_{\mathbb{Q}}$. Consider the Segre embedding of $E \times \mathbb{P}^1_{\mathbb{Q}}$ in $\mathbb{P}^5_{\mathbb{Q}}$, and let $\mathfrak{a}$ be a lift of the defining ideal to $R = \mathbb{Z}[x_0, \dots, x_5]$, i.e.,

$$\operatorname{Proj}(R/\mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Q}) = E \times \mathbb{P}^1_{\mathbb{Q}}.$$

By [4, page 75] or [11, page 219], the module $H^4_{\mathfrak{a}}(R/pR)$ is zero for infinitely many prime integers p (corresponding to E mod p being supersingular) and nonzero for infinitely many p (corresponding to E mod p being ordinary); see also [15, Corollary 2.2]. Thus,

$$H^4_I(R) \xrightarrow{\ p\ } H^4_I(R)$$

is surjective for infinitely many primes p, and also not surjective for infinitely many p. Theorem 1.2 implies that the map is injective for all but finitely many primes p.

4 Smooth algebras over a Dedekind domain

We indicate how Theorem 1.2 extends to algebras that are smooth over the ring of integers of a number field; first, the local version:

Theorem 4.1 *Let (V, uV) be a discrete valuation ring of mixed characteristic. Let R be a V-algebra that is either smooth over V, or a formal power series ring over V.*

Let $\mathfrak{a}$ be an ideal of R generated by elements $\boldsymbol{f}$.

(1) *If u is a nonzerodivisor on $H^k(\boldsymbol{f}; R)$, then it is a nonzerodivisor on $H^k_{\mathfrak{a}}(R)$.*
(2) *The R-module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals.*

Proof We first reduce to the case where V has a perfect residue field: There exists a discrete valuation ring (V', uV') such that V'/uV' is a perfect field, and $V \longrightarrow V'$ is faithfully flat, see, for example, [2, Chapter IX, Appendice 2]. Take R' to be either $R \otimes_V V'$ or a formal power series ring over V', in the respective cases; note that if R is smooth over V, then R' is smooth over V'. In either case, R' is faithfully flat over R, and it suffices to prove the assertions of the theorem for the ring R'.

We may thus assume that V/uV is a perfect field; it follows that R/uR is an F-finite regular ring. As before, the exact sequence

$$0 \longrightarrow R \xrightarrow{\ u\ } R \longrightarrow R/uR \longrightarrow 0$$

induces the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^{k-1}(\boldsymbol{f}; R) & \xrightarrow{\ \pi\ } & H^{k-1}(\boldsymbol{f}; R/uR) & \longrightarrow & H^k(\boldsymbol{f}; R) & \xrightarrow{\ u\ } & H^k(\boldsymbol{f}; R) \\
{\scriptstyle \alpha'}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow & & \big\downarrow \\
H^{k-1}_{\mathfrak{a}}(R) & \xrightarrow{\ \varphi\ } & H^{k-1}_{\mathfrak{a}}(R/uR) & \xrightarrow{\ d\ } & H^k_{\mathfrak{a}}(R) & \xrightarrow{\ u\ } & H^k_{\mathfrak{a}}(R)
\end{array}$$

The bottom row is a complex of $\mathscr{D}(R, V)$-modules; specifically, image(φ) and image(d) are $\mathscr{D}(R, V)$-modules. Since they are annihilated by u, they are also modules over the ring $\mathscr{D}(R,V)/u\mathscr{D}(R,V)$. If R is smooth over V, then Lemma 2.1 gives

$$\mathscr{D}(R,V)/u\mathscr{D}(R,V)=\mathscr{D}(R/uR,V/uV);$$

the same holds when R is a ring of formal power series over V by (2.1). Moreover, since V/uV is a perfect field, one has

$$\mathscr{D}(R/uR,V/uV)=\mathscr{D}(R/uR).$$

The remainder of the proof now proceeds analogous to that of Theorem 3.1.[1] □

As a consequence, we recover the following result of Lyubeznik, [10, Theorem 1]:

Corollary 4.2 *Let R be an unramified regular local ring of mixed characteristic, or, more generally, assume that the completion of R is a formal power series ring over a discrete valuation ring of mixed characteristic.*

Then each local cohomology module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals.

Proof One reduces to the case where R is a formal power series ring over a discrete valuation ring of mixed characteristic; the result then follows from Theorem 4.1. □

Theorem 4.3 *Let A be the ring of integers of a number field, or, more generally, a Dedekind domain such that for each height one prime ideal $\mathfrak{p}$ of A, the local ring $A_{\mathfrak{p}}$ has mixed characteristic. Let R be a smooth A-algebra. Then each local cohomology module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals.*

Proof Fix a generating set $\boldsymbol{f}$ for $\mathfrak{a}$. The R-module $H^k(\boldsymbol{f};R)$ has finitely many associated prime ideals; let $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ be the contractions of these to the ring A. Let $\mathfrak{p}$ be a height one prime of A that differs from the $\mathfrak{p}_i$. We claim that $\mathfrak{p}$ is not an associated prime of $H^k_{\mathfrak{a}}(R)$, viewed as an A-module.

Indeed, if it is, then $\mathfrak{p}A_{\mathfrak{p}}$ is an associated prime of $H^k_{\mathfrak{a}}(R_{\mathfrak{p}})$ as an $A_{\mathfrak{p}}$-module; but then, by Theorem 4.1 (1), $\mathfrak{p}A_{\mathfrak{p}}$ is an associated prime

[1] *Added in proof*: In the formal power series case one cannot use [8, Theorem 2.4] for a proof of the finiteness of the prime ideals not containing u because the ring is not finitely generated over a field. A proof of this remains the same as in [10, pp. 5880 (from line −5)–5882]; but our proof in the formal power series case of the finiteness of the primes containing u is much simpler than in [10].

of $H^k(\boldsymbol{f}; R_{\mathfrak{p}})$ as an $A_{\mathfrak{p}}$-module, implying that $\mathfrak{p}$ is an associated prime of $H^k(\boldsymbol{f}; R)$ as an A-module, which is false. This proves the claim.

Hence $H^k_{\mathfrak{a}}(R)$ has finitely many associated primes as an A-module. By Theorem 4.1 (2), there are finitely many elements of $\operatorname{Ass}_R H^k_{\mathfrak{a}}(R)$ lying over each element of $\operatorname{Ass}_A H^k_{\mathfrak{a}}(R)$. □

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