

F-REGULARITY DOES NOT DEFORM

By ANURAG K. SINGH

Abstract. We show that the property of F-regularity does not deform, and thereby settle a long-standing open question in the theory of tight closure. Specifically, we construct a three dimensional \mathbb{N} -graded domain R which is not F-regular (or even F-pure), but has a quotient R/tR which is F-regular. Examples are constructed over fields of characteristic $p > 0$, as well as over fields of characteristic zero.

1. Introduction. Throughout this paper, all rings are commutative, Noetherian, and have an identity element. The theory of *tight closure* was developed by Melvin Hochster and Craig Huneke in [HH2] and draws attention to rings which have the property that all their ideals are tightly closed, called *weakly F-regular* rings. The term *F-regular* is reserved for rings all of whose localizations are weakly F-regular. A natural question that arose with the development of the theory was whether the property of F-regularity deforms, i.e., if (R, m, K) is a local ring such that R/tR is F-regular for some nonzerodivisor $t \in m$, must R be F-regular? (See the Epilogue of [Ho].) Hochster and Huneke showed that this is indeed true if the ring R is Gorenstein, [HH3], and their work has been followed by various attempts at extending this result, see [AKM], [Si], [Sm3]. Our primary goal here is to settle this question by constructing a family of examples to show that F-regularity does not deform. We shall throughout be considering \mathbb{N} -graded rings, but local examples can be obtained, in all cases, by localizing at the homogeneous maximal ideals. Our main result is:

THEOREM 1.1. *There exists an \mathbb{N} -graded ring R of dimension three (finitely generated over a field $R_0 = K$ of characteristic $p > 2$) which is not F-pure, but has an F-regular quotient R/tR where $t \in m$ is a homogeneous nonzerodivisor.*

Specifically, for positive integers m and n satisfying $m - m/n > 2$, consider the ring $R = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix

$$\begin{pmatrix} A^2 + T^m & B & D \\ C & A^2 & B^n - D \end{pmatrix}.$$

Then the ring R/tR is F-regular, whereas R is not F-regular. If p and m are relatively prime integers, then the ring R is not F-pure.

A notion closely related to (and frequently the same as) F-regularity is that of *strong F-regularity*. A recent result of G. Lyubeznik and K. E. Smith states that the properties of weak F-regularity, F-regularity, and strong F-regularity agree for \mathbb{N} -graded F-finite rings, see [LS]. In light of this result, we frequently make no distinction between these notions.

The first result on the deformation of F-regularity is a theorem of Hochster and Huneke which states that for a Gorenstein local ring (R, m, K) , if R/tR is F-regular for some nonzerodivisor $t \in m$, then R is F-regular. They show that for Gorenstein rings the properties of F-regularity and F-rationality coincide, and that F-rationality deforms, see [HH3, Theorem 4.2]. This result is generalized in [Si] where the author uses the idea of passing to an anti-canonical cover $S = \bigoplus_{i \geq 0} I^{(i)}$, where I represents the inverse of the canonical module in $\text{Cl}(R)$. Strong F-regularity is shown to deform in the case that the symbolic powers $I^{(i)}$ satisfy the Serre condition S_3 for all $i \geq 0$, and the ring S is Noetherian.

This question has also been settled for \mathbb{Q} -Gorenstein rings, i.e., rings in which the canonical module is a torsion element of the divisor class group. For \mathbb{Q} -Gorenstein rings essentially of finite type over a field of characteristic zero, K. E. Smith showed that the property of F-regular type (a characteristic zero analogue of F-regularity) does deform, see [Sm3]. The point is that in this setting F-regular type is equivalent to log-terminal singularities, and log-terminal singularities deform by J. Kollár's result on "inversion of adjunction," see [Ko]. For \mathbb{Q} -Gorenstein rings of characteristic p , a purely algebraic proof that F-regularity deforms was provided by I. Aberbach, M. Katzman, and B. MacCrimmon in [AKM].

Before proceeding with formal definitions in the next section, we would like to point out that although tight closure is primarily a characteristic p notion, it has strong connections with the study of singularities of algebraic varieties over fields of characteristic zero. Specifically, let R be a ring which is essentially of finite type over a field of characteristic zero; then R has rational singularities if and only if it is of F-rational type, see [Ha], [Sm1]. In the \mathbb{Q} -Gorenstein case, we have some even more remarkable connections: F-regular type is equivalent to log-terminal singularities and F-pure type implies (and is conjectured to be equivalent to) log-canonical singularities, see [Sm2], [Wa3].

2. Frobenius closure and tight closure. By an \mathbb{N} -graded ring R , we will always mean a ring $R = \bigoplus_{n \geq 0} R_n$, finitely generated over a field $R_0 = K$.

Let R be a Noetherian ring of characteristic $p > 0$. The letter e denotes a variable nonnegative integer, and q its e th power, i.e., $q = p^e$. For an ideal $I = (x_1, \dots, x_n) \subseteq R$, let $I^{[q]} = (x_1^q, \dots, x_n^q)$.

For a reduced ring R of characteristic $p > 0$, $R^{1/q}$ shall denote the ring obtained by adjoining all q th roots of elements of R . A ring R is said to be *F-finite* if $R^{1/p}$ is module-finite over R . A finitely generated algebra R over a field K is F-finite if and only if $K^{1/p}$ is a finite field extension of K . We use R° to denote the complement of the union of the minimal primes of R .

Definition 2.1. Let R be a ring of characteristic p , and I an ideal of R . For an element x of R , we say that $x \in I^F$, the *Frobenius closure* of I , if there exists $q = p^e$ such that $x^q \in I^{[q]}$.

An element x of R is said to be in I^* , the *tight closure* of I , if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. If $I = I^*$ we say that the ideal I is *tightly closed*. It is easily seen that $I \subseteq I^F \subseteq I^*$.

A ring R is said to be *F-pure* if the Frobenius homomorphism is pure, i.e., $F: M \rightarrow F(M)$ is injective for all R -modules M . Note that this implies $I^F = I$ for all ideals I of R .

A ring R is *weakly F-regular* if every ideal of R is tightly closed, and is *F-regular* if every localization is weakly F-regular. An F-finite ring R is *strongly F-regular* if for every element $c \in R^\circ$, there exists an integer $q = p^e$ such that the R -linear inclusion $R \rightarrow R^{1/q}$ sending 1 to $c^{1/q}$ splits as a map of R -modules. R is *F-rational* if, in every local ring of R , all ideals generated by systems of parameters are tightly closed.

It follows from the definitions that a weakly F-regular ring is F-rational as well as F-pure. We summarize some basic results regarding these notions from [HH1, Theorem 3.1], [HH3, Theorem 4.2] and [LS, Corollaries 4.3, 4.4].

THEOREM 2.2.

- (1) *Regular rings are F-regular; if they are F-finite, they are also strongly F-regular. Strongly F-regular rings are F-regular.*
- (2) *Direct summands of F-regular rings are F-regular.*
- (3) *F-rational rings are normal. An F-rational ring which is a homomorphic image of a Cohen-Macaulay ring is itself Cohen-Macaulay.*
- (4) *An F-rational Gorenstein ring is F-regular. If it is F-finite, then it is also strongly F-regular.*
- (5) *The notions of weak F-regularity and F-regularity agree for \mathbb{N} -graded rings. For F-finite \mathbb{N} -graded rings, these are also equivalent to strong F-regularity.*

3. A review of rational coefficient Weil divisors. The examples constructed in the following section are best understood in the setting of \mathbb{Q} -divisors. Also, an interpretation of the graded pieces of certain local cohomology modules using \mathbb{Q} -divisors provided the original heuristic ideas which led to these examples. We recall some notation and results from [De], [Wa1], [Wa2].

Definition 3.1. By a *rational coefficient Weil divisor* (or a \mathbb{Q} -divisor) on a normal projective variety X , we mean a linear combination of codimension one irreducible subvarieties of X , with coefficients in \mathbb{Q} . For $E = \sum n_i V_i$ with $n_i \in \mathbb{Q}$, we set $[E] = \sum [n_i] V_i$, where $[n]$ denotes the greatest integer less than or equal to n , and define $\mathcal{O}_X(E) = \mathcal{O}_X([E])$.

Let $E = \sum (p_i/q_i)V_i$ where the integers p_i and q_i are relatively prime and $q_i > 0$. We define $E' = \sum ((q_i - 1)/q_i)V_i$ to be the *fractional part* of E . Note that with this definition we have $-[-nE] = [nE + E']$ for any integer n .

For an ample \mathbb{Q} -divisor E (i.e., NE is an ample Cartier divisor for some $N \in \mathbb{N}$), we construct the *generalized section ring*:

$$S = S(X, E) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nE)).$$

In this notation, Demazure's result ([De, 3.5]) states that every \mathbb{N} -graded normal ring arises as a generalized section ring $S = S(X, E)$ where E is an ample \mathbb{Q} -divisor on $X = \text{Proj } S$.

Let X be a smooth projective variety of dimension d with canonical divisor K_X , and let E be an ample \mathbb{Q} -divisor on X . If ω denotes the graded canonical module of the ring $S = S(X, E)$, K.-i. Watanabe showed in [Wa1] and [Wa2] that

$$[\omega^{(i)}]_n = H^0(X, \mathcal{O}_X(i(K_X + E') + nE))$$

and

$$[H_m^{d+1}(\omega^{(i)})]_n = H^d(X, \mathcal{O}_X(i(K_X + E') + nE)).$$

Let $E_S(K) = H_m^{d+1}(\omega)$ denote the injective hull of K as a graded S -module. The Frobenius action on the n th graded piece of $E_S(K)$ can then be identified with

$$H^d(X, \mathcal{O}_X(K_X + E' + nE)) \xrightarrow{F} H^d(X, \mathcal{O}_X(p(K_X + E' + nE))),$$

and in particular the Frobenius action on $[H_m^{d+1}(\omega)]_0$, the socle of $E_S(K)$, can be identified with

$$H^d(X, \mathcal{O}_X(K_X + E')) \xrightarrow{F} H^d(X, \mathcal{O}_X(p(K_X + E'))).$$

If the ring S is F-pure this Frobenius action must be injective, and consequently $H^d(X, \mathcal{O}_X(p(K_X + E')))$ must be nonzero. Heuristically, the dimension of the vector space $H^d(X, \mathcal{O}_X(p(K_X + E')))$ may be regarded as a measure of the F-purity of S . With this in mind, we choose E such that $h^d(X, \mathcal{O}_X(p(K_X + E')))$ while nonzero, is "small." This motivates our choice of the \mathbb{Q} -divisor E on \mathbb{P}^1 , see the proof of Proposition 4.3.

4. The main construction. When working with quotients of polynomial rings, we shall use lower-case letters to denote the images of the corresponding variables.

Remark 4.1. Let K be a field of characteristic p . For positive integers m and n , consider the ring $R = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix

$$\mathfrak{M}_{m,n} = \begin{pmatrix} A^2 + T^m & B & D \\ C & A^2 & B^n - D \end{pmatrix}.$$

The ring R is graded by setting the weights of $a, b, c, d,$ and t to be $m, 2m, 2m, 2mn,$ and 2 respectively. This ring is the specialization of a Cohen-Macaulay ring, and so is itself Cohen-Macaulay. The elements t, c and d form a homogeneous system of parameters for R , and so the element $t \in m$ is indeed a nonzerodivisor.

We record the following crucial lemma.

LEMMA 4.2. *Let m and n be positive integers satisfying $m - m/n > 2$. Consider the ring $R = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$ (see §4.1). If k is a positive integer such that $k(m - m/n - 2) \geq 1$, then*

$$(b^n t^{m-1})^{2mk+1} \in (a^{2mk+1}, d^{2mk+1}).$$

Proof. Let $\tau = A^2 + T^m$ and $\alpha = A^2$. It suffices to work in the polynomial ring $K[\tau, \alpha, B, C, D]$ and establish that

$$B^{n(2mk+1)}(\tau - \alpha)^{2k(m-1)} \in (\alpha^{mk+1}, D^{2mk+1}) + \mathfrak{a}$$

where \mathfrak{a} is the ideal generated by the size two minors of the matrix

$$\begin{pmatrix} \tau & B & D \\ C & \alpha & B^n - D \end{pmatrix}.$$

Taking the binomial expansion of $(\tau - \alpha)^{2k(m-1)}$, it suffices to show that

$$B^{n(2mk+1)}(\tau, \alpha)^{2k(m-1)} \in (\alpha^{mk+1}, D^{2mk+1}) + \mathfrak{a}.$$

This would follow if we show that for all integers i where $1 \leq i \leq mk + 1$, we have

$$B^{n(2mk+1)}\alpha^{mk+1-i}\tau^{mk-2k+i-1} \in (\alpha^{mk+1}, D^{2mk+1}) + \mathfrak{a},$$

and so it is certainly enough to show that

$$B^{n(2mk+1)}\tau^{mk-2k+i-1} \in (\alpha^i, D^{2mk+1}) + \mathfrak{a}.$$

Since $\alpha D - B(B^n - D) \in \mathfrak{a}$, it suffices to establish that

$$B^{n(2mk+1)} \tau^{mk-2k+i-1} \in (B^i(B^n - D)^i, D^{2mk+1}, B^n \tau - D(C + \tau)).$$

Working modulo the element $B^i(B^n - D)^i$, we may reduce $B^{n(2mk+1)}$ to a polynomial in B and D such that the highest power of B that occurs is less than $i(n + 1)$. Consequently it suffices to show that

$$B^{n(2mk+1-j)} \tau^{mk-2k+i-1} D^j \in (D^{2mk+1}, B^n \tau - D(C + \tau))$$

where $n(2mk + 1 - j) < i(n + 1)$, i.e., $j \geq 2mk + (1 - i)(1 + 1/n)$. With this simplification, it is enough to check that

$$B^{n(2mk+1-j)} \tau^{mk-2k+i-1} \in (D^{2mk+1-j}, B^n \tau - D(C + \tau)).$$

It only needs to be verified that $mk - 2k + i - 1 \geq 2mk + 1 - j$ since, working modulo $B^n \tau - D(C + \tau)$, we can then express $B^{n(2mk+1-j)} \tau^{mk-2k+i-1}$ as a multiple of $D^{2mk+1-j}$. Finally, note that

$$(mk - 2k + i - 1) - (2mk + 1 - j) = j - mk - 2k + i - 2 \geq k \left(m - \frac{m}{n} - 2 \right) - 1 \geq 0$$

since $i \leq mk + 1$, $j \geq 2mk + (1 - i)(1 + 1/n)$ and $k(m - m/n - 2) \geq 1$. □

PROPOSITION 4.3. *Let $S = K[A, B, C, D]/J$ where the characteristic of the field K is a prime $p > 2$, and J is the ideal generated by the size two minors of the matrix*

$$\begin{pmatrix} A^2 & B & D \\ C & A^2 & B^n - D \end{pmatrix}.$$

Then S is an F -regular ring.

Proof. There are various ways to establish this. We can identify S with the generalized section ring $\bigoplus_{i \geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(iE))X^i$, where $\mathbb{P}^1 = \text{Proj } K[X, Y]$, and E is the rational coefficient Weil divisor

$$E = \frac{1}{2}V(X) + \frac{1}{2}V(Y) + \frac{1}{2n}V(X + Y).$$

Under this identification,

$$A = X, \quad B = \frac{X^3}{Y}, \quad C = XY \quad \text{and} \quad D = \frac{X^{3n+1}}{Y^n(X + Y)}.$$

One may now appeal to Watanabe’s classification in [Wa2] to conclude that S is F -regular.

For an alternate proof, it is easily verified that S is the Veronese subring

$$H^{(2n+1)} = \bigoplus_{i \in \mathbb{N}} [H]_{i(2n+1)}$$

where H is the hypersurface

$$K[A, X, Y]/(A^2 - XY(X^n - Y))$$

and the variables A, X and Y have weights $2n + 1, 2$ and $2n$ respectively. Here $B = XY^2, C = X(X^n - Y)^2$ and $D = Y^{2n+1}$. Since the characteristic of K is greater than 2, a routine computation shows that the hypersurface H is F-regular, and consequently its direct summand S is also F-regular. \square

Remark 4.4. Although we do not use this fact, we mention that the hypersurface H in the proof above is obtained as the cyclic cover

$$S \oplus \omega \oplus \omega^{(2)} \oplus \dots \oplus \omega^{(2n)}$$

where ω is the canonical module of the ring S .

PROPOSITION 4.5. *Let K be a field of characteristic $p > 2$ and consider the ring $R = R_{m,n} = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$ (see §4.1). If $m - m/n > 2$, then R is not F-regular. If in addition p and m are relatively prime, then R is not F-pure.*

Proof. First note that $b^n t^{m-1} \notin (a, d)$. To establish that R is not F-regular we shall show that $b^n t^{m-1} \in (a, d)^*$.

For a suitably large arbitrary positive integer e , let $q = p^e = 2mk + \delta$ where k and δ are integers such that $k(m - m/n - 2) \geq 1$ and $-2m + 2 \leq \delta \leq 1$. To see that $b^n t^{m-1} \in (a, d)^*$, it suffices to show that

$$(b^n t^{m-1})^{q+2m-1} \in (a^q, d^q).$$

Since $q + 2m - 1 = 2mk + \delta + 2m - 1 \geq 2mk + 1$ and $q \leq 2mk + 1$, it suffices to check that

$$(b^n t^{m-1})^{2mk+1} \in (a^{2mk+1}, d^{2mk+1}),$$

but this is precisely the assertion of Lemma 4.2.

For the second assertion, note that since $p > 2$, the integers p and $2m$ are relatively prime and we may choose a positive integer e such that $q = p^e = 2mk + 1$ for some $k > 0$. Taking a higher power of p , if necessary, we may also assume that $k(m - m/n - 2) \geq 1$. But now $(b^n t^{m-1})^q \in (a^q, d^q)$ by Lemma 4.2, and so $b^n t^{m-1} \in (a, d)^F$. Hence the ring R is not F-pure. \square

Proof of Theorem 1.1. We have already noted in §4.1 that the element t is a nonzerodivisor in R , and Proposition 4.3 establishes that the ring R/tR is F-regular. Since $m - m/n > 2$, Proposition 4.5 shows that R fails to be F-regular, and is not even F-pure if p and m are relatively prime integers. \square

5. The characteristic zero case. Hochster and Huneke have developed a notion of tight closure for rings essentially of finite type over fields of characteristic zero, see [HH2], [HH4]. However we can also define notions corresponding to F-regularity, F-purity, and F-rationality in characteristic zero, without explicitly considering a closure operation for rings of characteristic zero. We include a brief summary, and discuss how the examples constructed above also show that the property *F-regular type* does not deform.

Suppose $R = K[X_1, \dots, X_n]/I$ is a ring finitely generated over a field K of characteristic zero, choose a finitely generated \mathbb{Z} -algebra A such that

$$R_A = A[X_1, \dots, X_n]/I_A$$

is a free A -algebra with $R \cong R_A \otimes_A K$. Note that the fibers of the homomorphism $A \rightarrow R_A$ over maximal ideals of A are finitely generated algebras over fields of positive characteristic.

Definition 5.1. Let R be a ring which is finitely generated over a field of characteristic zero. Then R is said to be of *F-regular type* if there exists a finitely generated \mathbb{Z} -algebra $A \subseteq K$ and a finitely generated A -algebra R_A as above such that $R \cong R_A \otimes_A K$ and, for all maximal ideals μ in a Zariski dense subset of $\text{Spec } A$, the fiber rings $R_A \otimes_A A/\mu$ are F-regular.

Similarly, R is said to be of *F-pure type* if for all maximal ideals μ in a Zariski dense subset of $\text{Spec } A$, the fiber rings $R_A \otimes_A A/\mu$ are F-pure.

Remark 5.2. Some authors use the term F-pure type (F-regular type) to mean that $R_A \otimes_A A/\mu$ is F-pure (F-regular) for all maximal ideals μ in a Zariski dense open subset of $\text{Spec } A$.

THEOREM 5.3. *For positive integers m and n satisfying $m - m/n > 2$, consider the ring $R = \mathbb{Q}[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$ of §4.1. Then the ring R is not of F-pure type, whereas R/tR is of F-regular type.*

Proof. If p is a prime integer which does not divide $2m$, the fiber of $\mathbb{Z} \rightarrow R_{\mathbb{Z}}$ over $p\mathbb{Z}$ is not F-pure by Proposition 4.5, and consequently the ring R is not of F-pure type. On the other hand, Proposition 4.3 shows that R/tR is of F-regular type since the fiber of $\mathbb{Z} \rightarrow (R/tR)_{\mathbb{Z}}$ over $p\mathbb{Z}$ is F-regular for all primes $p > 2$. \square

Remark 5.4. R. Fedder first constructed examples to show that F-purity does not deform, see [Fe]. However Fedder pointed out that his examples were less

than satisfactory in two ways: firstly the rings were not integral domains, and secondly his arguments did not work in the characteristic zero setting, i.e., did not comment on the deformation of the property *F-pure type*. In [Si] the author constructed various examples which overcame both these shortcomings, but left at least one issue unresolved—although the rings R were integral domains (which were not F-pure), the F-pure quotient rings R/tR were not integral domains. The examples we have constructed here also settle this remaining issue.

6. Conditions on fibers. The examples constructed in the previous section are also relevant from the point of view of the behavior of F-regularity under base change. We first recall a theorem of Hochster and Huneke, [HH3, Theorem 7.24].

THEOREM 6.1. *Let $(A, m, K) \rightarrow (R, n, L)$ be a flat local homomorphism of local rings of characteristic p such that A is weakly F-regular, R is excellent, and the generic and closed fibers are regular. Then the ring R is weakly F-regular.*

It is a natural question to ask what properties are inherited by an excellent ring R if, as above, $(A, m, K) \rightarrow (R, n, L)$ is a flat local homomorphism, the ring A is F-regular and the generic and closed fibers are F-regular. Our examples can be used to show that even if (A, m, K) is a discrete valuation ring and the generic and closed fibers of $(A, m, K) \rightarrow (R, n, L)$ are F-regular, then the ring R need not be F-regular.

Once again, we construct \mathbb{N} -graded examples, and examples with local rings can be obtained by the obvious localizations at the homogeneous maximal ideals. Let $A = K[T]$ be a polynomial ring in one variable, and $R = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$, see §4.1. As before, K is a field of characteristic $p > 2$, and m and n are positive integers such that $m - m/n > 2$.

The generic fiber of the inclusion $A \rightarrow R$ is a localization of R_t , whereas the fiber over the homogeneous maximal ideal of A is R/tR . We have earlier established that R/tR is F-regular, and only need to show that the ring R_t is F-regular. In the following proposition we show that the R is, in fact, locally F-regular on the punctured spectrum.

PROPOSITION 6.2. *Let K be a field of characteristic $p > 2$. For positive integers m and n consider the ring $R = R_{m,n} = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$ (see §4.1). Then the ring R_P is F-regular for all prime ideals P in $\text{Spec } R - \{m\}$.*

Proof. A routine verification shows that the singular locus of R is $V(J)$ where the defining ideal is $J = (a, b, c(c + t^m), d)$. Consequently we need to show that the two local rings R_P and R_Q are F-regular where $P = (a, b, c, d)$ and $Q = (a, b, c + t^m, d)$.

After localizing at the prime P , we may write $d = b^n(a^2 + t^m)/(c + a^2 + t^m)$ and so R_P is a localization of the ring

$$K[T, A, B, C]/(A^2(A^2 + T^m) - BC)$$

at the prime ideal (a, b, c) . Since $a^2 + t^m$ is a unit, the hypersurface R_P is easily seen to be F-regular.

Localizing at the prime Q , we have $b = a^2(a^2 + t^m)/c$ and so R_Q is a localization of the ring

$$K[T, A, C, D]/(C^n D(C + A^2 + T^m) - A^{2n}(A^2 + T^m)^{n+1})$$

at the prime ideal $(a, c + t^m, d)$. Again we have a hypersurface which, it can be easily verified, is F-regular. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, EAST HALL, 525 EAST UNIVERSITY AVENUE, ANN ARBOR, MI 48109-1109

Current address: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN STREET, URBANA, IL 61801

Electronic mail: SINGH6@MATH.UIUC.EDU

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