TIGHT CLOSURE: APPLICATIONS AND QUESTIONS

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These notes are based on five lectures at the *The 4th Japan-Vietnam Joint Seminar on Commutative Algebra*, that took place at Meiji University in February 2009. For the most part, each of the sections below is independent of the others.

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1. Magic Squares

A *magic square* is a matrix with nonnegative integer entries such that each row and each column has the same sum, called the *line sum*. Let $H_n(r)$ be the number of $n \times n$ magic squares with line sum r. Then

$$H_n(0) = 1\,, \qquad H_n(1) = n!\,, \qquad \text{and} \qquad \sum_{n \geqslant 0} \frac{H_n(2)x^n}{(n!)^2} = \frac{e^{x/2}}{\sqrt{1-x}}\,;$$

the first two formulae are elementary, and the third was proved by Anand, Dumir, and Gupta [ADG] . On the other hand, viewing $H_n(r)$ as a function of $r \geqslant 0$, one has

$$H_1(r) = 1\,, \qquad H_2(r) = r+1\,, \qquad H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}\,.$$

Once again, the first two are trivial, keeping in mind that $H_2(r)$ counts the matrices

$$\begin{pmatrix} i & r-i \\ r-i & i \end{pmatrix} \qquad \text{for } 0 \leqslant i \leqslant r\,.$$

The formula for $H_3(r)$ may be found in MacMahon [MaP, § 407]; we will compute it here in Example 1.2. It was conjectured in [ADG] that the function $H_n(r)$ agrees with a degree $(n-1)^2$ polynomial in r for all integers $r \geqslant 0$. This—and more—was proved by Stanley [St1]; see also [St2, St3]. We give a tight closure proof of the following:

Theorem 1.1 (Stanley)**.** *Let $H_n(r)$ be the number of $n \times n$ magic squares with line sum r. Then $H_n(r)$ agrees with a degree $(n-1)^2$ polynomial in r for all integers $r \geqslant 0$.*

Let K be a field. Let (x_{ij}) be an $n \times n$ matrix of indeterminates, for n a fixed positive integer, and set R to be the polynomial ring

$$R = K[x_{ij} \mid 1 \leqslant i, j \leqslant n]\,.$$

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Set S to be the K-subalgebra of R generated by the monomials

$$\prod_{i,j} x_{ij}^{a_{ij}} \qquad \text{such that } \big(a_{ij}\big) \text{ is an } n \times n \text{ magic square}\,.$$

For example, in the case $n = 2$, the magic squares are

$$\begin{pmatrix} i & r-i \\ r-i & i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (r-i) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{for } 0 \leqslant i \leqslant r\,,$$

and it follows that

$$S = K[x_{11}x_{22},\ x_{12}x_{21}]\,.$$

Quite generally, the Birkhoff-von Neumann theorem states that each magic square is a sum of permutation matrices. Thus, S is generated over K by the $n!$ monomials

$$\prod_{i=1}^{n} x_{i\,\sigma(i)} \qquad \text{for } \sigma \text{ a permutation of } \{1, \dots, n\}\,.$$

Consider the $\mathbb{Q}$-grading on R with $R_0 = K$ and $\deg x_{ij} = 1/n$ for each i, j. Then

$$\deg\Big(\prod_{i=1}^{n} x_{i\,\sigma(i)}\Big) = 1\,,$$

so S is a *standard* $\mathbb{N}$*-graded* K*-algebra*, by which we mean $S_0 = K$ and $S = K[S_1]$. Let

$$P(S,t) = \sum_{r \geqslant 0} \big(\operatorname{rank}_K S_r\big)\, t^r\,,$$

which is the Hilbert-Poincaré series of S. Then $H_n(r)$ is the coefficient of t^r in $P(S,t)$.

Example 1.2. In the case $n = 3$, the permutation matrices satisfy the linear relation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so S has a presentation

$$S = K\,[y_1, y_2, y_3, y_4, y_5, y_6]\,/\,(y_1y_2y_3 - y_4y_5y_6)$$

where

$$\begin{array}{lll} y_1 \longmapsto x_{11}x_{22}x_{33}, & y_2 \longmapsto x_{12}x_{23}x_{31}, & y_3 \longmapsto x_{13}x_{21}x_{32}, \\ y_4 \longmapsto x_{13}x_{22}x_{31}, & y_5 \longmapsto x_{11}x_{23}x_{32}, & y_6 \longmapsto x_{12}x_{21}x_{33}. \end{array}$$

Since S is a hypersurface of degree 3, its Hilbert-Poincaré series is

$$P(S,t) \;=\; \frac{1-t^3}{(1-t)^6} \;=\; \frac{1+t+t^2}{(1-t)^5}\,;$$

the coefficient of t^r in this series is readily seen to be

$$H_3(r) \;=\; \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

Returning to the general case, since S is standard $\mathbb{N}$-graded, one has

$$P(S,t) = \frac{f(t)}{(1-t)^d}$$

where $f(1) \neq 0$ and $d = \dim S$. Since $H_n(r)$ is the coefficient of t^r in $P(S,t)$, it follows that $H_n(r)$ agrees with a degree $d-1$ polynomial in r for *large* positive integers r. It remains to compute the dimension d, and to show that $H_n(r)$ agrees with a polynomial in r for *each* integer $r \geqslant 0$.

The dimension of S may be computed as the transcendence degree of the fraction field of S over K; this equals the number of monomials in S that are algebraically independent over K. Since monomials are algebraically independent precisely when their exponent vectors are linearly independent, the dimension d of S is the rank of the $\mathbb{Q}$-vector space spanned by the $n \times n$ magic squares. The rank of this vector space may be computed by counting the choices "$-$" that may be made when forming an $n \times n$ matrix over $\mathbb{Q}$ with constant line sum; the remaining entries "$*$" below are forced:

$$\begin{pmatrix} - & - & \cdots & - & - \\ - & - & \cdots & - & * \\ - & - & \cdots & - & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ - & - & \cdots & - & * \\ * & * & * & * & * \end{pmatrix}.$$

Hence $d = (n-1)^2 + 1$. It follows that $H_n(r)$ agrees with a degree $(n-1)^2$ polynomial in r, at least for large positive integers r.

To see that $H_n(r)$ agrees with a polynomial in r for each $r \geqslant 0$, it suffices to show that $\deg f(t) \leqslant d-1$; indeed, if this is the case, we may write

$$f(t) = \sum_{i=0}^{d-1} a_i (1-t)^i \,,$$

and so

$$P(S,t) \;=\; \sum_{i=0}^{d-1} \frac{a_i}{(1-t)^{d-i}} \;=\; \sum_{i=1}^{d} \frac{a_{d-i}}{(1-t)^i} \,,$$

which is a power series where the coefficient of t^r agrees with a polynomial in r for each integer $r \geqslant 0$. The rest of this section is devoted to proving that $\deg f(t) \leqslant d-1$, and to developing the requisite tight closure theory along the way.

Consider the K-linear map $\rho \colon R \longrightarrow S$ that fixes a monomial $\prod x_{ij}^{a_{ij}}$ when (a_{ij}) is a magic square, and maps it to 0 otherwise. Since the sum of two magic squares is a magic square, and the sum of a magic square and a non-magic square is a non-magic square, ρ is a homomorphism of S-modules. As ρ fixes S, the inclusion $S \subseteq R$ is S-split, which implies that S is a direct summand of R as an S-module.

Thus far, the field K was arbitrary; for the rest of this section, assume K is an algebraically closed field of prime characteristic p. We will also assume, for simplicity, that all rings, ideals, and elements in question are homogeneous.

Let A be a domain of prime characteristic p, and let q denote a varying positive integer power of p. For an ideal $\mathfrak{a}$ of A, define

$$\mathfrak{a}^{[q]} = (a^q \mid a \in \mathfrak{a})\,.$$

The *tight closure* of $\mathfrak{a}$, denoted $\mathfrak{a}^*$, is the ideal

$$\left\{z \in A \mid \text{ there exists a nonzero } c \in R \text{ with } cz^q \in \mathfrak{a}^{[q]} \text{ for each } q = p^e\right\}\,.$$

While the results hold in greater generality, the following will suffice for our needs:

Lemma 1.3. *Let R and S be as defined earlier. Then:*

1. *For each homogeneous ideal $\mathfrak{a}$ of R, one has $\mathfrak{a}^* = \mathfrak{a}$.*
2. *For each homogeneous ideal $\mathfrak{a}$ of S, one has $\mathfrak{a}^* = \mathfrak{a}$.*
3. *The ring S is Cohen-Macaulay, i.e., each homogeneous system of parameters for S is a regular sequence.*
4. *If $\boldsymbol{y} = y_1, \dots, y_d$ is a homogeneous system of parameters for S consisting of elements of degree 1, then $S_{\geqslant d} \subseteq \boldsymbol{y}S$.*

Since S is standard $\mathbb{N}$-graded with S_0 an algebraically closed field, S indeed has a homogeneous system of parameters $\boldsymbol{y}$ consisting of degree 1 elements. Using (3),

$$P(S,t) = \frac{P(S/\boldsymbol{y}S, t)}{(1-t)^d}\,.$$

But then the polynomial $f(t) = P(S/\boldsymbol{y}S, t)$ has degree at most $d-1$ by (4). Thus, the lemma above completes the proof of Theorem 1.1.

Proof of Lemma 1.3. (1) Let z be an element of $\mathfrak{a}^*$. Without loss of generality, assume z is homogeneous. Then there exists a nonzero homogeneous element c of positive degree such that $cz^q \in \mathfrak{a}^{[q]}$ for each $q = p^e$. Taking q-th roots, one has $c^{1/q}z \in \mathfrak{a}R^{1/q}$, i.e.,

$$c^{1/q} \ \in\ (\mathfrak{a}R^{1/q} :_{R^{1/q}} z) \ =\ (\mathfrak{a} :_R z)R^{1/q}\,,$$

where the equality above holds because $R^{1/q} = K[\boldsymbol{x}^{1/q}]$ is a free R-module. But then $(\mathfrak{a} :_R z)R^{1/q}$ contains elements of arbitrarily small positive degree, so $(\mathfrak{a} :_R z) = R$.

(2) If $z \in \mathfrak{a}^*$ for a homogeneous ideal $\mathfrak{a}$ of S, then $z \in \mathfrak{a}R^*$. But $\mathfrak{a}R^* = \mathfrak{a}R$ by (1), so z belongs to $\mathfrak{a}R \cap S$. This ideal equals $\mathfrak{a}$ since S is a direct summand of R.

(3) Let $\boldsymbol{y}$ be a homogeneous system of parameters for S. Then $A = K[\boldsymbol{y}]$ is a Noether normalization for S, i.e., the elements $\boldsymbol{y}$ are algebraically independent over K, and S is integral over $K[\boldsymbol{y}]$. Let N be the largest integer with $A^N \subseteq S$. Then S/A^N is a finitely generated A-torsion module, and is thus annihilated by a nonzero element c of A.

Suppose $sy_{i+1} \in (y_1, \dots, y_i)S$ for a homogeneous element s of S. Taking Frobenius powers, one has $s^q y_{i+1}^q \in (y_1^q, \dots, y_i^q)S$ for each $q = p^e$. Since $cS \subseteq A^N$, multiplying by the element c yields

$$cs^q y_{i+1}^q \ \in\ (y_1^q, \dots, y_i^q)A^N \qquad \text{for each } q = p^e\,.$$

But $\boldsymbol{y}$ is a regular sequence on the free A-module A^N, so

$$cs^q \ \in\ (y_1^q, \dots, y_i^q)A^N \ \subseteq\ (y_1^q, \dots, y_i^q)S \qquad \text{for each } q = p^e\,.$$

It follows that $s \in (y_1, \dots, y_i)S^* = (y_1, \dots, y_i)S$.

(4) Let z be a homogeneous element of S having degree at least d. Since z is integral over A, there exists a homogeneous equation

$$z^k + a_1 z^{k-1} + \cdots + a_k \;=\; 0 \qquad \text{with } a_i \in A\,.$$

But then

$$z^N \;\in\; A + Az + \cdots + Az^{k-1} \qquad \text{for all } N \geqslant 0\,,$$

in particular, for $q = p^e$, one has

$$z^{q+k-1} \;=\; b_0 + b_1 z + \cdots + b_{k-1} z^{k-1} \qquad \text{where } b_i \in A\,.$$

Note that

$$\deg b_i \;\geqslant\; \deg b_{k-1} \;=\; \deg z^q \;\geqslant\; qd\,,$$

i.e., $b_i \in A_{\geqslant qd}$. This implies that

$$b_i \;\in\; (\boldsymbol{y}A)^{qd} \;\subseteq\; (y_1^q, \ldots, y_d^q)A\,,$$

so $z^{q+k-1} \in (y_1^q, \ldots, y_d^q)S$ for each q. Hence $z \in \boldsymbol{y}S^*$, but $\boldsymbol{y}S^* = \boldsymbol{y}S$ by (2). □

2. Splinters

We saw a glimpse of tight closure theory in the previous section. The theory was developed by Hochster and Huneke [HH1], and has had enormous impact. It is a closure operation on ideals, first defined for rings of prime characteristic using the Frobenius map, and then extended to rings of characteristic zero by reduction mod p. The theory leads to powerful results on unrelated topics such as rings of invariants—this is the appropriate framework for much of the previous section—integral closure of ideals and Briançon-Skoda theorems, and symbolic powers of ideals.

Rings of characteristic $p > 0$ in which all ideals are tightly closed are *weakly F-regular.* A local ring R of characteristic $p > 0$ is *F-rational* if each ideal generated by a system of parameters is tightly closed. If R is not necessarily local, we say R is *F-rational* if $R_{\mathfrak{p}}$ is F-rational for each prime ideal $\mathfrak{p}$. Lemma 1.3 extends to the theorem below.

Theorem 2.1. *The following hold for rings of prime characteristic:*

(1) Regular rings are weakly F-regular.
(2) Direct summands of weakly F-regular rings are weakly F-regular.
(3) F-rational rings are normal; an F-rational ring that is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.
(4) F-rational Gorenstein rings are weakly F-regular.
(5) Let R be an $\mathbb{N}$-graded ring that is finitely generated over a field R_0. If R is weakly F-regular, then so is each localization $R_{\mathfrak{p}}$.

Proof. For (1) and (2) see [HH1, Theorem 4.6, Proposition 4.12]; (3) is part of [HH5, Theorem 4.2], and (4) is [HH5, Corollary 4.7]. Lastly, (5) is [LS, Corollary 4.4]. □

The class of weakly F-regular rings includes determinantal rings, homogeneous coordinate rings of Grassmann varieties, normal monomial rings, and, more generally, rings of invariants of linearly reductive groups. While Brenner and Monsky [BM] have constructed striking examples demonstrating that the operation of taking tight closure of an ideal need not commute with localization, the following remains unanswered:

Question 2.2. Does weak F-regularity localize, i.e., if R is a weakly F-regular ring, is each localization $R_{\mathfrak{p}}$ also weakly F-regular?

By Lyubeznik-Smith [LS], the answer is affirmative for $\mathbb{N}$-graded rings R with R_0 a field. We discuss an approach to Question 2.2 via splitting in module-finite extensions:

Definition 2.3. An integral domain R is *splinter* if it is a direct summand, as an R-module, of each module-finite extension ring.

If a ring R is a direct summand of an extension ring S, then $\mathfrak{a}S \cap R = \mathfrak{a}$ for each ideal $\mathfrak{a}$ of R. The converse holds when R is approximately Gorenstein, [Ho2, Proposition 5.5]; in particular, if R is an excellent domain and S a finite extension, then R is a direct summand of S if and only if $\mathfrak{a}S \cap R = \mathfrak{a}$ for each ideal $\mathfrak{a}$ of R.

It is readily verified that splinter rings are normal: Suppose a fraction a/b is integral over a splinter ring R. Since $R \subseteq R[a/b]$ is a finite extension, it must split. But then

$$a \ \in \ bR[a/b] \cap R \ = \ bR\,,$$

by which, $a/b \in R$.

Characteristic zero: Let R be a normal domain containing $\mathbb{Q}$. For each module-finite extension domain S, the field trace map $\mathrm{Tr}\colon \mathrm{frac}(S) \longrightarrow \mathrm{frac}(R)$ provides a splitting

$$\frac{1}{[\mathrm{frac}(S) : \mathrm{frac}(R)]}\,\mathrm{Tr}\colon S \longrightarrow R\,.$$

Thus, an integral domain containing $\mathbb{Q}$ is splinter if and only if it is normal.

Mixed characteristic: In this case, the *monomial conjecture*, Conjecture 3.8, is equivalent to the conjecture that every regular local ring is splinter, which is the *direct summand conjecture*. Heitmann [He] has verified this for rings of dimension up to three.

Positive characteristic: Hochster-Huneke [HH4, Theorem 5.25] proved that weakly F-regular rings of positive characteristic are splinters, Theorem 2.4 below, and that the converse holds for Gorenstein rings, [HH4, Theorem 6.7]. The rest of this section is mostly devoted to the case of positive characteristic, but first some definitions:

Let R be an integral domain. The *absolute integral closure* R^+ of R is the integral closure of R in an algebraic closure of its fraction field. The *plus closure* of an ideal $\mathfrak{a}$ of R is $\mathfrak{a}^+ = \mathfrak{a}R^+ \cap R$. It follows from the earlier discussion that an excellent domain R is splinter if and only if each ideal of R equals its plus closure.

Theorem 2.4. *Let R be an integral domain of positive characteristic. Then*

$$\mathfrak{a}^+ \subseteq \mathfrak{a}^*$$

for each ideal $\mathfrak{a}$ of R. Hence each weakly F-regular excellent domain of positive characteristic is splinter.

Proof. Suppose $z \in \mathfrak{a}^+$. Then there exists a finite extension domain S with $z \in \mathfrak{a}S$. Fix a splitting of the inclusion of fields $\operatorname{frac}(R) \subseteq \operatorname{frac}(S)$, and consider its restriction $S \longrightarrow \operatorname{frac}(R)$. Since S is module-finite over R, one may multiply by a nonzero element c of R to obtain an R-module homomorphism $\varphi\colon S \longrightarrow R$; note that $\varphi(1) = c$.

For each $q = p^e$, one has $z^q \in \mathfrak{a}^{[q]}S$. Applying φ, one obtains

$$\varphi(z^q) \in \mathfrak{a}^{[q]} \qquad \text{for each } q = p^e .$$

But $\varphi(z^q) = cz^q$, so $z \in \mathfrak{a}^*$. □

As we saw, $\mathfrak{a}^+ \subseteq \mathfrak{a}^*$ in domains of positive characteristic. Smith [Sm1] proved that $\mathfrak{a}^+ = \mathfrak{a}^*$ when $\mathfrak{a}$ is a parameter ideal in an excellent domain. Brenner and Monsky [BM] have constructed examples with $\mathfrak{a}^+ \neq \mathfrak{a}^*$

We next sketch the theory of tight closure for modules. The *Frobenius functor* $\mathcal{F}$ is the base change functor $R \otimes_R -$ on the category of R-modules, where R is viewed as an R-module via the Frobenius endomorphism $F\colon R \longrightarrow R$. The e-th iteration $\mathcal{F}^e$ agrees with base change under $F^e\colon R \longrightarrow R$. Note that $\mathcal{F}^e(R) = R$, and that $\mathcal{F}^e(R/\mathfrak{a}) = R/\mathfrak{a}^{[p^e]}$.

For each R-module M, one has a natural map $M \longrightarrow \mathcal{F}^e(M)$ with $m \longmapsto 1 \otimes m$; to keep track of the iteration e, we denote the image by m^{p^e}. Let $N \subseteq M$ be R-modules. The induced map $\mathcal{F}^e(N) \longrightarrow \mathcal{F}^e(M)$ need not be injective; its image is denoted by $N_M^{[p^e]}$, and is the R-span of elements n^{p^e} for $n \in N$.

The tight closure of N in M, denoted N_M^*, is the set of all $m \in M$ for which there exists an element c in R°—the complement of the minimal prime of R—with

$$cm^{p^e} \in N_M^{[p^e]} \qquad \text{for all integers } e \gg 0 .$$

With this definition, R is *strongly F-regular* if $N_M^* = N$ for each pair of R-modules $N \subseteq M$; we do not require M or N to be finitely generated.

Strong F-regularity can be tested on indecomposable injective modules:

Proposition 2.5. *Let R be a Noetherian ring of prime characteristic. The following statements are equivalent:*

(1) *R is strongly F-regular, i.e., $N_M^* = N$ for all R-modules $N \subseteq M$;*
(2) *for each maximal ideal $\mathfrak{m}$ of R, one has $0_E^* = 0$, where E is the injective hull of $R/\mathfrak{m}$ as an R-module;*
(3) *for each maximal ideal $\mathfrak{m}$ of R, one has $u \notin 0_E^*$, where E is the injective hull of $R/\mathfrak{m}$, and u is an element generating the socle of E.*

Corollary 2.6. *Let R be a Noetherian ring of prime characteristic. If R is strongly F-regular, then so is $W^{-1}R$ for each multiplicative subset W of R.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R disjoint from W. Let E be the injective hull of $R/\mathfrak{p}$ as an R-module. Then E is also the injective hull of $R/\mathfrak{p}$ as a $W^{-1}R$-module. By the above proposition, it suffices to verify that 0 is tightly closed in E, the tight closure being computed over $W^{-1}R$. But $\mathcal{F}^e_{W^{-1}R}(E) = \mathcal{F}^e_R(E)$, and each element of $(W^{-1}R)^\circ$ has the form c/w for $c \in R^\circ$ and $w \in W$. □

Divisorial ideals. Let R be a normal domain. An ideal $\mathfrak{a}$ of R is *divisorial* if each of its associated primes has height one. In this case, the primary decomposition of $\mathfrak{a}$ has the form $\mathfrak{a} = \bigcap_i \mathfrak{p}_i^{(n_i)}$, and $\mathfrak{a}$ determines an element

$$[\mathfrak{a}] = \sum_i n_i[\mathfrak{p}_i] \qquad \text{in } \operatorname{Cl}(R)\,,$$

the divisor class group of R. For divisorial ideals $\mathfrak{a}$ and $\mathfrak{b}$, one has $[\mathfrak{a}] = [\mathfrak{b}]$ in $\operatorname{Cl}(R)$ if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are isomorphic as R-modules. Each divisorial ideal is a finitely generated, torsion-free, reflexive R-module of rank one, and each such module is isomorphic to a divisorial ideal.

Let R be a normal domain and $\mathfrak{a}$ a divisorial ideal. Let t be an indeterminate. The *symbolic Rees algebra* $\mathcal{R}(\mathfrak{a})$ is the ring

$$R \;\oplus\; \mathfrak{a}t \;\oplus\; \mathfrak{a}^{(2)}t^2 \;\oplus\; \mathfrak{a}^{(3)}t^3 \;\oplus\; \cdots\,,$$

viewed as a subring of $R[t]$. In general, for $\mathfrak{a}$ a divisorial ideal, $\mathcal{R}(\mathfrak{a})$ need not be Noetherian, e.g., if R is the homogeneous coordinate ring of an elliptic curve, and $\mathfrak{a}$ is a prime ideal such that $[\mathfrak{a}]$ has infinite order in $\operatorname{Cl}(R)$. On the other hand, if R is a two-dimensional ring with rational singularities, Lipman [Li] proved that $\operatorname{Cl}(R)$ is a torsion group; it follows that in this case $\mathcal{R}(\mathfrak{a})$ is Noetherian for each divisorial ideal $\mathfrak{a}$. For rings of dimension three, the hypothesis that R has rational singularities is no longer sufficient; see Cutkosky [Cu]. However, if R is a Gorenstein $\mathbb{C}$-algebra of dimension three, with rational singularities, then $\mathcal{R}(\mathfrak{a})$ is Noetherian for each divisorial ideal $\mathfrak{a}$; this is due to Kawamata, [Ka]. We do not know the answer to the following:

Question 2.7. If R is a splinter domain of positive characteristic, and $\mathfrak{a}$ a divisorial ideal, is the symbolic Rees algebra $\mathcal{R}(\mathfrak{a})$ Noetherian?

Suppose $(R, \mathfrak{m})$ is a normal local ring with canonical module ω. Let $\mathfrak{a}$ be a divisorial ideal that is an inverse for ω in $\operatorname{Cl}(R)$ i.e., such that

$$[\mathfrak{a}] + [\omega] = 0 \qquad \text{in } \operatorname{Cl}(R)\,.$$

Following [Wa], we say that the symbolic Rees algebra $\mathcal{R}(\mathfrak{a})$ is the *anti-canonical cover* of R. When the anti-canonical cover is Noetherian, we are able to prove that splinter rings are precisely those that are strongly F-regular:

Theorem 2.8. *Let R be an excellent local ring of positive characteristic that is a homomorphic image of a Gorenstein ring. If R is splinter and the anti-canonical cover of R is Noetherian, then R is strongly F-regular.*

Key ingredients of the proof are a criterion for when $\mathcal{R}(\mathfrak{a})$ is Noetherian from [GHNV], and a local cohomology computation from [Wa]: The ring $\mathcal{R} = \mathcal{R}(\mathfrak{a})$ has an $\mathbb{N}$-grading with $\mathcal{R}_n = \mathfrak{a}^{(n)}t^n$. Let $\mathfrak{M}$ be the unique homogeneous maximal ideal of $\mathcal{R}$, namely

$$\mathfrak{M} = \mathcal{R}_{\geqslant 1} + \mathfrak{m}\mathcal{R}\,.$$

Let $d = \dim R$. By [Wa, Theorem 2.2], if $\mathcal{R}$ is Noetherian, then one has

$$\begin{aligned} H^{d+1}_{\mathfrak{M}}(\mathcal{R}) &\cong \bigoplus_{n<0} H^d_{\mathfrak{m}}(\mathfrak{a}^{(n)})t^n \\ &\cong \bigoplus_{n>0} H^d_{\mathfrak{m}}(\omega^{(n)})t^{-n} . \end{aligned}$$

Theorem 2.8 yields the following corollary on localizations of weakly F-regular rings; this extends earlier results of Williams [Wi] and MacCrimmon [MaB]:

Corollary 2.9. *Let R be an excellent normal ring of positive characteristic that is a homomorphic image of a Gorenstein ring. Suppose the anti-canonical cover of $R_{\mathfrak{p}}$ is Noetherian for each $\mathfrak{p} \in \operatorname{Spec} R \smallsetminus \operatorname{MaxSpec} R$.*

If R is weakly F-regular, so is $W^{-1}R$, for each multiplicative subset W of R.

Proof. By [HH1, Corollary 4.15], a ring S is weak F-regular if and only if $S_{\mathfrak{m}}$ is weakly F-regular for each maximal ideal $\mathfrak{m}$. Thus, to prove that $W^{-1}R$ is weakly F-regular, it suffices to consider the case $W = R \smallsetminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal in $\operatorname{Spec} R \smallsetminus \operatorname{MaxSpec} R$.

As R is weakly F-regular, it is splinter by Theorem 2.4. Since a localization of a splinter is splinter, the ring $R_{\mathfrak{p}}$ is splinter. But then Theorem 2.8 implies that $R_{\mathfrak{p}}$ is strongly F-regular, hence also weakly F-regular. □

A splinter ring of positive characteristic is pseudorational by Smith [Sm1, Sm2], and a two-dimensional pseudorational ring is $\mathbb{Q}$-Gorenstein by Lipman [Li]. Thus, we have:

Corollary 2.10. *Let R be a two-dimensional ring of positive characteristic. Then R is splinter if and only if it is weakly F-regular.*

Using results of [Ha, HW, Sm2, MS], Theorem 2.8 also provides a characterization of rings of characteristic zero with log terminal singularities:

Corollary 2.11. *Let R be a $\mathbb{Q}$-Gorenstein ring that is finitely generated over a field of characteristic zero. Then R has log terminal singularities if and only if it is of splinter-type, i.e., for almost all primes p, the characteristic p models of R are splinter.*

3. Annihilators of local cohomology

This is based on joint work with Paul Roberts and V. Srinivas, [RSS]. Let R be an integral domain of characteristic $p > 0$. An element z of R belongs to the tight closure of an ideal $\mathfrak{a}$ if, by definition, there exists a nonzero element c of R with

$$cz^q \in \mathfrak{a}^{[q]} \qquad \text{for each } q = p^e .$$

If this is the case, taking q-th roots in the above display, it follows that

$$c^{1/q}z \in \mathfrak{a}R^{1/q} \qquad \text{for each } q = p^e ,$$

and hence that

$$c^{1/q}z \in \mathfrak{a}R^+ \qquad \text{for each } q = p^e .$$

Fix a valuation $v\colon R \smallsetminus \{0\} \longrightarrow \mathbb{Z}_{\geqslant 0}$, and extend it to $v\colon R^+ \smallsetminus \{0\} \longrightarrow \mathbb{Q}_{\geqslant 0}$. The elements $c^{1/q} \in R^+$ have arbitrarily small positive order as q varies, and multiply z

into $\mathfrak{a}R^+$. The surprising thing is that this essentially characterizes tight closure; first a definition from [HH2]:

Definition 3.1. Let $(R, \mathfrak{m})$ be a complete local domain of arbitrary characteristic. Fix a valuation v that is positive on $\mathfrak{m} \smallsetminus \{0\}$, and extend it to $v\colon R^+ \smallsetminus \{0\} \longrightarrow \mathbb{Q}_{\geqslant 0}$. The *dagger closure* $\mathfrak{a}^\dagger$ of an ideal $\mathfrak{a}$ is the ideal consisting of all elements $z \in R$ for which there exist elements $u \in R^+$, having arbitrarily small positive order, with $uz \in \mathfrak{a}R^+$.

Theorem 3.2. [HH2, Theorem 3.1] *Let $(R, \mathfrak{m})$ be a complete local domain of positive characteristic. Fix a valuation as above. Then, for each ideal $\mathfrak{a}$ of R, one has $\mathfrak{a}^\dagger = \mathfrak{a}^*$.*

While tight closure is defined in characteristic zero by reduction to prime characteristic, the definition of dagger closure is characteristic-free. However, dagger closure is quite mysterious in characteristic zero and in mixed characteristic. We focus next on an example; for graded domains, we use the grading in lieu of a valuation. Whenever R is an $\mathbb{N}$-graded domain that is finitely generated over a field R_0, there are *some* elements of R^+ that can be assigned a $\mathbb{Q}$-degree such that they satisfy a homogeneous equation of integral dependence over R, and we work with such elements.

Example 3.3. Let $R = \mathbb{F}_p[x, y, z]/(x^3 + y^3 + z^3)$ where $p \neq 3$. Then $z^2 \in (x, y)^*$. One way to see this is to use the definition of tight closure with $c = z$ as the multiplier.

Another way is to consider the local cohomology module $H^2_{\mathfrak{m}}(R)$ as computed via the Čech complex on x, y; it is easily verified that the assertion $z^2 \in (x, y)^*$ is equivalent to the assertion that the element

$$\eta \ = \ \left[\frac{z^2}{xy}\right] \qquad \text{of } H^2_{\mathfrak{m}}(R)$$

belongs to the submodule $0^*_{H^2_{\mathfrak{m}}(R)}$. To verify that $\eta \in 0^*_{H^2_{\mathfrak{m}}(R)}$, first note that the standard grading on R induces a grading on $H^2_{\mathfrak{m}}(R)$ under which $\deg \eta = 0$. Using F for the Frobenius action on $H^2_{\mathfrak{m}}(R)$, the element

$$F^e(\eta) \ = \ \left[\frac{z^{2p^e}}{x^{p^e} y^{p^e}}\right]$$

has degree 0 as well. Since $H^2_{\mathfrak{m}}(R)$ has no elements of positive degree, each $c \in R_{>0}$ must annihilate $F^e(\eta)$ for each $e \geqslant 0$. Hence $\eta \in 0^*_{H^2_{\mathfrak{m}}(R)}$, equivalently, $z^2 \in (x, y)^*$.

Yet another way is to identify $F^e\colon R \longrightarrow R$ with the inclusion $R \hookrightarrow R^{1/p^e}$ as below:

(3.3.1)
$$\begin{array}{ccc} R & \xrightarrow{F^e} & R \\ \| & & \downarrow \cong \\ R & \longrightarrow & R^{1/p^e}\,. \end{array}$$

Note that while the upper horizontal map is not degree-preserving, the lower one is, where one endows R^{1/p^e} with the natural $\frac{1}{p^e}\mathbb{N}$-grading. Since $H^2_{\mathfrak{m}}(R^{1/p^e})$ has no elements of positive degree, each element of R^{1/p^e} having positive degree must annihilate the image of η in $H^2_{\mathfrak{m}}(R^{1/p^e})$, and hence also the image of η in $H^2_{\mathfrak{m}}(R^+)$. In particular, for each e, there exist elements of R^+ having degree $1/p^e$ that annihilate the image of η in

$H^2_{\mathfrak{m}}(R^+)$. This point of view is useful when computing the corresponding dagger closure in characteristic zero:

Example 3.4. Let $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$. We show that $z^2 \in (x, y)^\dagger$. For this, it suffices to show that the image of the element

$$\eta = \left[\frac{z^2}{xy}\right]$$

under the natural map $H^2_{\mathfrak{m}}(R) \longrightarrow H^2_{\mathfrak{m}}(R^+)$ is annihilated by elements of R^+ having arbitrarily small positive degree.

Let φ be the $\mathbb{C}$-algebra automorphism of R with

$$\varphi(x) = x^3 - \omega y^3\,, \qquad \varphi(y) = y^3 - \omega x^3\,, \qquad \varphi(z) = (1-\omega)xyz\,,$$

where ω is a primitive third root of unity. As in (3.3.1), identify $\varphi^e \colon R \longrightarrow R$ with a graded embedding $R \hookrightarrow R^{\varphi^e}$, i.e.,

$$\begin{array}{ccc} R & \xrightarrow{\varphi^e} & R \\ \Big\| & & \Big\downarrow \cong \\ R & \longrightarrow & R^{\varphi^e} \end{array}$$

where R^{φ^e} is $\frac{1}{3^e}\mathbb{N}$-graded. Note that R^{φ^e} may be viewed as a subalgebra of R^+. Since $H^2_{\mathfrak{m}}(R^{\varphi^e})$ has no elements of positive degree, each element of R^{φ^e} having positive degree annihilates the image of η in $H^2_{\mathfrak{m}}(R^{\varphi^e})$, and hence the image of η in $H^2_{\mathfrak{m}}(R^+)$.

In Example 3.4, the ring R is the homogeneous coordinate ring of an elliptic curve, and hence has several degree-increasing endomorphisms: if E is an elliptic curve and N a positive integer, consider the endomorphism of E that takes a point P to $N \cdot P$ under the group law. Then there exists a homogeneous coordinate ring R of E such that the map $P \longmapsto N \cdot P$ corresponds to a ring endomorphism $\varphi \colon R \longrightarrow R$ with $\varphi(R_1) \subseteq R_{N^2}$. Arguably, Example 3.4 is atypical in that the endomorphism exhibited satisfies $\varphi(R_1) \subseteq R_3$. Perhaps the following is more convincing:

Example 3.5. Let $R = \mathbb{C}[x, y, z]/(x^3 - xz^2 - y^2z)$. Consider the group law on $\operatorname{Proj} R$ with $[0 : 1 : 0]$ as the identity. It is a routine verification that the group inverse is

$$-[a : b : 1] = [a \ : \ -b \ : \ 1]$$

and that the formula for doubling a point is

$$2\,[a : b : 1] = [2ab^3 + 6a^2b + 2b \ : \ b^4 - 3ab^2 - 9a^2 + 1 \ : \ 8b^3]\,.$$

The endomorphism $P \longrightarrow 2 \cdot P$ corresponds to the ring endomorphism

$$\begin{aligned} \varphi(x) &= 2xy^3 + 6x^2yz + 2yz^3\,, \\ \varphi(y) &= y^4 - 3xy^2z - 9x^2z^2 + z^4\,, \\ \varphi(z) &= 8y^3z\,. \end{aligned}$$

This time, one indeed has $\varphi(R_1) \subseteq R_{2^2}$.

Using the Albanese map from a projective variety to its Albanese variety—which is an abelian variety—and endomorphisms of the abelian variety coming from the group law, we were able to prove the following, [RSS, Theorem 3.4, Corollary 3.5]:

Theorem 3.6. *Let R be an $\mathbb{N}$-graded domain that is finitely generated over a field R_0 of characteristic zero. Given a positive real number ϵ, there exists a $\mathbb{Q}$-graded finite extension domain S, such that the image of the induced map*

$$H^2_{\mathfrak{m}}(R)_0 \longrightarrow H^2_{\mathfrak{m}}(S)$$

is annihilated by an element of S having degree less than ϵ.

Moreover, if $\dim R = 2$, there exists an extension S as above such that the image of $H^2_{\mathfrak{m}}(R)_{\geqslant 0}$ in $H^2_{\mathfrak{m}}(S)$ is annihilated by an element of S having degree less than ϵ.

The homological conjectures. The motivation for studying dagger closure arises from the *homological conjectures*; these are a collection of conjectures in local algebra, due to Auslander, Bass, Hochster, Serre, and others, which have proved to be a source of wonderful mathematics. Peskine and Szpiro [PS] made huge progress on these; subsequently, Hochster's theorem [Ho1] that every local ring containing a field has a big Cohen-Macaulay module settled most of the conjectures in the equal characteristic case. The mixed characteristic case has proved more formidable: some of the conjectures including Auslander's zerodivisor conjecture and Bass' conjecture were proved by Roberts [Ro] for rings of mixed characteristic, while others such as Hochster's *monomial conjecture*, Conjecture 3.8 below, and its equivalent formulations the *direct summand conjecture*, *the canonical element conjecture*, and the *improved new intersection conjecture* remain unresolved. Heitmann [He] proved these equivalent conjectures for rings of dimension up to three; the key ingredient is:

Theorem 3.7 (Heitmann)**.** *Let $(R, \mathfrak{m})$ be a local domain of dimension 3 and mixed characteristic p. For each $n \in \mathbb{N}$, there exists a finite extension domain S, such that the image of the induced map*

$$H^2_{\mathfrak{m}}(R) \longrightarrow H^2_{\mathfrak{m}}(S)$$

is annihilated by $p^{1/n}$.

Note that once a valuation v on $R^+ \smallsetminus \{0\}$ is fixed, $v(p^{1/n}) = v(p)/n$ takes arbitrarily small positive values as n gets large.

Conjecture 3.8 (Hochster's Monomial Conjecture)**.** Let $x_1, \dots, x_d$ be a system of parameters for a local ring $(R, \mathfrak{m})$. Then

$$x_1^t \cdots x_d^t \notin \left(x_1^{t+1}, \dots, x_d^{t+1}\right) R \qquad \text{for all } t \in \mathbb{N}\,.$$

If $x_1, \dots, x_d$ form a regular sequence on an R-module, it is readily seen that they satisfy the assertion of the monomial conjecture; such a module is called a *big Cohen-Macaulay module*; "big" emphasizes that the module need not be finitely generated. Hochster and Huneke [HH3] extended the result of [Ho1] by proving that every local ring containing a field has a big Cohen-Macaulay *algebra*; moreover, for R a local domain of positive characteristic, they showed that R^+ is a big Cohen-Macaulay algebra. It turns out that $R^{+\mathrm{sep}}$, the subalgebra of *separable* elements of R^+, is also a big Cohen-Macaulay algebra, [Si1]. In another direction, Huneke and Lyubeznik [HL] obtained the following refinement of [HH3], with a simple and elegant proof:

Theorem 3.9 (Huneke-Lyubeznik)**.** *Let $(R, \mathfrak{m})$ be a local domain of positive characteristic that is a homomorphic image of a Gorenstein ring. Then there exists a finite extension domain S such that the image of the induced map*

$$H^k_{\mathfrak{m}}(R) \longrightarrow H^k_{\mathfrak{m}}(S)$$

is zero for each $k < \dim R$.

The hypothesis of "positive characteristic" in the above theorem cannot be replaced by "characteristic zero." For example, let R be a normal domain of characteristic zero that is not Cohen-Macaulay. If S is a finite extension of R, then field trace provides an R-linear splitting of $R \hookrightarrow S$, so $H^k_{\mathfrak{m}}(R) \longrightarrow H^k_{\mathfrak{m}}(S)$ is a split inclusion as well. Perhaps the best that one can hope for is an affirmative answer to the following:

Question 3.10. Let $(R, \mathfrak{m})$ be a domain with a valuation v that is positive on $\mathfrak{m} \smallsetminus \{0\}$. Extend v to $R^+ \smallsetminus \{0\} \longrightarrow \mathbb{Q}_{\geqslant 0}$. Given a real number $\epsilon > 0$, and integer $k < \dim R$, does there exists a subalgebra S of R^+ such that the image of the induced map

$$H^k_{\mathfrak{m}}(R) \longrightarrow H^k_{\mathfrak{m}}(S)$$

is annihilated by an element of S having order less than ϵ?

A related question in the characteristic zero graded setting is:

Question 3.11. Let R be an $\mathbb{N}$-graded domain that is finitely generated over a field R_0 of characteristic zero. Given a real number $\epsilon > 0$ and integer $k \geqslant 0$, does there exist a $\mathbb{Q}$-graded finite extension domain S, such that the image of the induced map

$$H^k_{\mathfrak{m}}(R)_{\geqslant 0} \longrightarrow H^k_{\mathfrak{m}}(S)$$

is annihilated by an element of S of degree less than ϵ?

This is straightforward for $k = 0, 1$; the first nontrivial case is $H^2_{\mathfrak{m}}(R)_0$, which is settled by Theorem 3.6. However, the question remains unresolved for $H^2_{\mathfrak{m}}(R)_1$. Some test cases include the diagonal subalgebras constructed in [KSSW] with $H^2_{\mathfrak{m}}(R)_0 = 0$ and $H^2_{\mathfrak{m}}(R)_1 \neq 0$. Another concrete, unresolved case of Question 3.11 is:

Question 3.12. Set $R = \mathbb{Q}[x_0, \dots, x_d]/(x_0^n + \cdots + x_d^n)$, where $n > d$. Is the image of $H^d_{\mathfrak{m}}(R)_{\geqslant 0}$ in $H^d_{\mathfrak{m}}(R^+)$ killed by elements of R^+ having arbitrarily small positive degree?

By Theorem 3.6, the answer is affirmative for $d = 2$. In terms of dagger closure, Question 3.12 would have an affirmative answer if

$$x_0^d \in (x_1, \dots, x_d)^{\dagger}.$$

Affirmative answers to these would give reasons to be optimistic about the following:

Question 3.13. Does dagger closure have the "colon capturing" property in characteristic zero, i.e., if $x_1, \dots, x_d$ is a system of parameters for R, is it true that

$$(x_1, \dots, x_{k-1}) :_R x_k \subseteq (x_1, \dots, x_{k-1})^{\dagger} \qquad \text{for each } k\,?$$

According to Hochster and Huneke [HH2, page 244] "it is important to raise (and answer) this question."

4. Bockstein homomorphisms in local cohomology

This is based on joint work with Uli Walther. Let R be a polynomial ring in finitely many variables over the ring of integers. Let $\mathfrak{a}$ be an ideal of R, and let p be a prime integer. Taking local cohomology $H^{\bullet}_{\mathfrak{a}}(-)$, the exact sequence

$$0 \longrightarrow R \xrightarrow{\ p\ } R \longrightarrow R/pR \longrightarrow 0$$

induces an exact sequence

$$H^k_{\mathfrak{a}}(R/pR) \xrightarrow{\ \delta\ } H^{k+1}_{\mathfrak{a}}(R) \xrightarrow{\ p\ } H^{k+1}_{\mathfrak{a}}(R) \xrightarrow{\ \pi\ } H^{k+1}_{\mathfrak{a}}(R/pR)\,.$$

The *Bockstein homomorphism* β^k_p is the composition

$$\pi \circ \delta \colon H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR)\,.$$

Fix $\mathfrak{a} \subseteq R$; we prove that for all but finitely many prime integers p, the Bockstein homomorphisms β^k_p are zero. More precisely:

Theorem 4.1. *Let R be a polynomial ring in finitely many variables over the ring of integers. Let $\mathfrak{a} = (f_1, \dots, f_t)$ be an ideal of R, and let p be a prime integer.*

If p is a nonzerodivisor on the Koszul cohomology module $H^{k+1}(\boldsymbol{f}; R)$, then the Bockstein homomorphism $\beta^k_p \colon H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR)$ is zero.

This is motivated by Lyubeznik's conjecture [Ly1, Remark 3.7] which states that for regular rings R, each local cohomology module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals. This conjecture has been verified for regular rings of positive characteristic by Huneke and Sharp [HS], and for regular local rings of characteristic zero as well as unramified regular local rings of mixed characteristic by Lyubeznik [Ly1, Ly2]. It remains unresolved for polynomial rings over $\mathbb{Z}$, where it implies that for fixed $\mathfrak{a} \subseteq R$, the Bockstein homomorphisms β^k_p are zero for almost all prime integers p; the above theorem thus provides supporting evidence for Lyubeznik's conjecture.

The situation is quite different for hypersurfaces, as compared with regular rings:

Example 4.2. Consider the hypersurface

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$$

and ideal $\mathfrak{a} = (x, y, z)R$. A variation of the argument given in [Si2] shows that

$$\beta^2_p \colon H^2_{\mathfrak{a}}(R/pR) \longrightarrow H^3_{\mathfrak{a}}(R/pR)$$

is nonzero for each prime integer p.

Huneke [Hu, Problem 4] asked whether local cohomology modules of Noetherian rings have finitely many associated prime ideals. The answer to this is negative since $H^3_{\mathfrak{a}}(R)$ in the hypersurface example has p-torsion elements for each prime integer p, and hence has infinitely many associated primes; see [Si2]. Indeed, the issue of p-torsion appears to be central in studying Lyubeznik's conjecture for finitely generated $\mathbb{Z}$-algebras.

We outline the proof of Theorem 4.1. One first verifies that the Bockstein homomorphism $H^k_{(\boldsymbol{f})}(R/pR) \longrightarrow H^{k+1}_{(\boldsymbol{f})}(R/pR)$ depends only on $\boldsymbol{f}$ mod pR, more precisely:

Lemma 4.3. *Let M be an R-module, and let p be an element of R that is M-regular. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R with $\operatorname{rad}(\mathfrak{a}+pR) = \operatorname{rad}(\mathfrak{b}+pR)$. Then there exists a commutative diagram*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^k_{\mathfrak{a}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{a}}(M/pM) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^k_{\mathfrak{b}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{b}}(M/pM) & \longrightarrow & \cdots
\end{array}$$

where the horizontal maps are the respective Bockstein homomorphisms, and the vertical maps are natural isomorphisms.

Proof. It suffices to consider the case $\mathfrak{a} = \mathfrak{b} + yR$, where $y \in \operatorname{rad}(\mathfrak{b}+pR)$. For each R-module N, one has an exact sequence

$$\longrightarrow H^{k-1}_{\mathfrak{b}}(N)_y \longrightarrow H^k_{\mathfrak{a}}(N) \longrightarrow H^k_{\mathfrak{b}}(N) \longrightarrow H^k_{\mathfrak{b}}(N)_y \longrightarrow$$

which is functorial in N; see for example [ILLM, Exercise 14.4]. Using this for

$$0 \longrightarrow M \xrightarrow{\ p\ } M \longrightarrow M/pM \longrightarrow 0\,,$$

one obtains the commutative diagram below, with exact rows and columns.

$$\begin{array}{ccccccc}
H^{k-1}_{\mathfrak{b}}(M/pM)_y & \longrightarrow & H^k_{\mathfrak{b}}(M)_y & \xrightarrow{\ p\ } & H^k_{\mathfrak{b}}(M)_y & \longrightarrow & H^k_{\mathfrak{b}}(M/pM)_y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^k_{\mathfrak{a}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{a}}(M) & \xrightarrow{\ p\ } & H^{k+1}_{\mathfrak{a}}(M) & \longrightarrow & H^{k+1}_{\mathfrak{a}}(M/pM) \\
{\scriptstyle \theta^k}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \theta^{k+1}} \\
H^k_{\mathfrak{b}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{b}}(M) & \xrightarrow{\ p\ } & H^{k+1}_{\mathfrak{b}}(M) & \longrightarrow & H^{k+1}_{\mathfrak{b}}(M/pM) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^k_{\mathfrak{b}}(M/pM)_y & \longrightarrow & H^{k+1}_{\mathfrak{b}}(M)_y & \xrightarrow{\ p\ } & H^{k+1}_{\mathfrak{b}}(M)_y & \longrightarrow & H^{k+1}_{\mathfrak{b}}(M/pM)_y
\end{array}$$

Since $H^\bullet_{\mathfrak{b}}(M/pM)$ is y-torsion, it follows that $H^\bullet_{\mathfrak{b}}(M/pM)_y = 0$. Hence the maps $\theta^\bullet$ are isomorphisms, and the desired result follows. ☐

Another ingredient in the proof of Theorem 4.1 is the existence of endomorphisms of the polynomial ring $R = \mathbb{Z}[x_1, \dots, x_n]$. For p a nonzerodivisor on $H^{k+1}(\boldsymbol{f}; R)$, consider the endomorphism φ of R with $\varphi(x_i) = x_i^p$ for each i. Since

$$H^{k+1}(\boldsymbol{f}; R) \xrightarrow{\ p\ } H^{k+1}(\boldsymbol{f}; R)$$

is injective and φ is flat, it follows that

$$H^{k+1}(\varphi^e(\boldsymbol{f}); R) \xrightarrow{\ p\ } H^{k+1}(\varphi^e(\boldsymbol{f}); R)$$

is injective for each $e \geqslant 0$. Thus, the Bockstein map on Koszul cohomology

$$H^k(\varphi^e(\boldsymbol{f}); R/pR) \longrightarrow H^{k+1}(\varphi^e(\boldsymbol{f}); R/pR)$$

must be the zero map. Suppose $\eta \in H^k_{\mathfrak{a}}(R/pR)$. Then η has a lift in $H^k(\varphi^e(\boldsymbol{f}); R/pR)$ for large e. But then the commutativity of the diagram

$$\begin{array}{ccc}
H^k(\varphi^e(\boldsymbol{f}); R/pR) & \longrightarrow & H^{k+1}(\varphi^e(\boldsymbol{f}); R/pR) \\
\downarrow & & \downarrow \\
H^k_{(\varphi^e(\boldsymbol{f}))}(R/pR) & \longrightarrow & H^{k+1}_{(\varphi^e(\boldsymbol{f}))}(R/pR) \\
\downarrow & & \downarrow \\
H^k_{(\boldsymbol{f})}(R/pR) & \longrightarrow & H^{k+1}_{(\boldsymbol{f})}(R/pR)\,,
\end{array}$$

where each horizontal map is a Bockstein homomorphism, implies that η maps to zero in $H^{k+1}_{\mathfrak{a}}(R/pR)$.

Stanley-Reisner ideals. For $\mathfrak{a}$ the Stanley-Reisner ideal of a simplicial complex, the following theorem connects Bockstein homomorphisms on reduced simplicial cohomology groups with those on local cohomology modules. First, some notation:

Let Δ be a simplicial complex, and τ a subset of its vertex set. The *link* of τ is

$$\operatorname{link}_\Delta(\tau) = \{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset \ \text{ and } \ \sigma \cup \tau \in \Delta\}\,.$$

Given $\boldsymbol{u} \in \mathbb{Z}^n$, we set $\widetilde{\boldsymbol{u}} = \{i \mid u_i < 0\}$.

Theorem 4.4. *Let Δ be a simplicial complex with vertices $1, \dots, n$. Set R to be the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$, and let $\mathfrak{a} \subseteq R$ be the Stanley-Reisner ideal of Δ.*

For each prime integer p, the following are equivalent:

(1) the Bockstein homomorphism $H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR)$ is zero;
(2) the Bockstein homomorphism

$$\widetilde{H}^{n-k-2-|\widetilde{\boldsymbol{u}}|}(\operatorname{link}_\Delta(\widetilde{\boldsymbol{u}}); \mathbb{Z}/p\mathbb{Z}) \longrightarrow \widetilde{H}^{n-k-1-|\widetilde{\boldsymbol{u}}|}(\operatorname{link}_\Delta(\widetilde{\boldsymbol{u}}); \mathbb{Z}/p\mathbb{Z})$$

is zero for each $\boldsymbol{u} \in \mathbb{Z}^n$ with $\boldsymbol{u} \leqslant \boldsymbol{0}$.

Example 4.5. Let Λ_m be the *m-fold dunce cap*, i.e., the quotient of the unit disk obtained by identifying each point on the boundary circle with its translates under rotation by $2\pi/m$; the 2-fold dunce cap Λ_2 is the real projective plane.

Suppose m is the product of distinct primes $p_1, \dots, p_r$. It is readily computed that the Bockstein homomorphisms

$$\widetilde{H}^1(\Lambda_m; \mathbb{Z}/p_i) \longrightarrow \widetilde{H}^2(\Lambda_m; \mathbb{Z}/p_i)$$

are nonzero. Let Δ be the simplicial complex corresponding to a triangulation of Λ_m, and let $\mathfrak{a}$ in $R = \mathbb{Z}[x_1, \dots, x_n]$ be the corresponding Stanley-Reisner ideal. The theorem then implies that the Bockstein homomorphisms

$$H^{n-3}_{\mathfrak{a}}(R/p_iR) \longrightarrow H^{n-2}_{\mathfrak{a}}(R/p_iR)$$

are nonzero for each p_i. It follows that the local cohomology module $H_{\mathfrak{a}}^{n-2}(R)$ has a p_i-torsion element for each $i = 1, \dots, r$.

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