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p-TORSION ELEMENTS IN LOCAL COHOMOLOGY MODULES

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Abstract. For every prime integer p, M. Hochster conjectured the existence of certain p-torsion elements in a local cohomology module over a regular ring of mixed characteristic. We show that Hochster's conjecture is false. We next construct an example where a local cohomology module over a hypersurface has p-torsion elements for every prime integer p, and consequently has infinitely many associated prime ideals.

For a commutative Noetherian ring R and an ideal $\mathfrak{a} \subset R$, the finiteness properties of the local cohomology modules $H^i_{\mathfrak{a}}(R)$ have been studied by various authors. In this paper we focus on the following question raised by C. Huneke [Hu, Problem 4]: if M is a finitely generated R-module, is the number of associated primes ideals of $H^i_{\mathfrak{a}}(M)$ always finite?

In the case that the ring R is regular and contains a field of prime characteristic $p > 0$, Huneke and Sharp showed in [HS] that the set of associated prime ideals of $H^i_{\mathfrak{a}}(R)$ is finite. If R is a regular *local* ring containing a field of characteristic zero, G. Lyubeznik showed that $H^i_{\mathfrak{a}}(R)$ has only finitely many associated prime ideals, see [Ly1] and also [Ly2, Ly3]. Recently Lyubeznik has also proved this result for unramified regular local rings of mixed characteristic, [Ly4]. Our computations here support Lyubeznik's conjecture [Ly1, Remark 3.7 (iii)] that local cohomology modules over all regular rings have only finitely many associated prime ideals.

In Section 4 we construct an example of a hypersurface R such that the local cohomology module $H^3_{\mathfrak{a}}(R)$ has p-torsion elements for every prime integer p, and consequently has infinitely many associated prime ideals.

For some of the other work related to this question, we refer the reader to the papers [BL, BRS, He].

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1. Hochster's conjecture

Consider the polynomial ring over the integers $R = \mathbb{Z}[u, v, w, x, y, z]$ where $\mathfrak{a}$ is the ideal generated by the size two minors of the matrix

$$M = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix},$$

i.e., $\mathfrak{a} = (\Delta_1, \Delta_2, \Delta_3)R$ where $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, and $\Delta_3 = uy - vx$. M. Hochster conjectured that for every prime integer p, there exist certain p-torsion elements in the local cohomology module $H^3_{\mathfrak{a}}(R)$, and consequently that $H^3_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals. We first describe the construction of these elements. For an arbitrary prime integer p, consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{\ p\ } R \longrightarrow R/pR \longrightarrow 0,$$

which induces the following exact sequence of local cohomology modules:

$$(*) \qquad 0 \to H^2_{\mathfrak{a}}(R) \to H^2_{\mathfrak{a}}(R) \to H^2_{\mathfrak{a}}(R/pR) \to H^3_{\mathfrak{a}}(R) \to H^3_{\mathfrak{a}}(R) \to 0.$$

It can be shown that the module $H^3_{\mathfrak{a}}(R/pR)$ is zero using a result of Peskine and Szpiro, see Proposition 1.2 below. In the ring R we have the equation

$$u\Delta_1 + v\Delta_2 + w\Delta_3 = 0$$

arising from the determinant of the matrix

$$\begin{pmatrix} u & v & w \\ u & v & w \\ x & y & z \end{pmatrix}.$$

Let $q = p^e$ where e is a positive integer. Taking the qth power of the above equation, we see that

$$(**) \qquad (u\Delta_1)^q + (v\Delta_2)^q + (w\Delta_3)^q \equiv 0 \mod p,$$

and this yields a relation on the elements $\bar{\Delta}_1^q,\ \bar{\Delta}_2^q,\ \bar{\Delta}_3^q \in R/pR$, where $\bar{\Delta}_i$ denotes the image of Δ_i in R/pR. This relation may be viewed as an element $\mu_q \in H^2_{\mathfrak{a}}(R/pR)$. Hochster conjectured that for every prime integer there exists a choice of $q = p^e$ such that

$$\mu_q \notin \operatorname{Image}\Big(H^2_{\mathfrak{a}}(R) \to H^2_{\mathfrak{a}}(R/pR)\Big),$$

and consequently that the image of μ_q under the connecting homomorphism in the exact sequence $(*)$ is a nonzero p-torsion element of $H^3_{\mathfrak{a}}(R)$.

We note an equivalent form of Hochster's conjecture which is convenient to work with. Recall that the module $H^3_{\mathfrak{a}}(R)$ may be viewed as the direct limit

$$\varinjlim \frac{R}{(\Delta_1^k, \Delta_2^k, \Delta_3^k)R},$$

where the maps in the direct limit system are induced by multiplication by the element $\Delta_1\Delta_2\Delta_3$. The equation $(**)$ shows that

$$\lambda_q = \frac{(u\Delta_1)^q + (v\Delta_2)^q + (w\Delta_3)^q}{p}$$

has integer coefficients, i.e., that λ_q is an element of the ring R. Let

$$\eta_q = [\lambda_q + (\Delta_1^q,\ \Delta_2^q,\ \Delta_3^q)R] \ \in\ H^3_{\mathfrak{a}}(R).$$

Lemma 1.1. *With the notation as above, the following statements are equivalent:*

(1) $\mu_q \notin \operatorname{Image}\Big(H^2_{\mathfrak{a}}(R) \to H^2_{\mathfrak{a}}(R/pR)\Big)$,

(2) η_q *is a nonzero p-torsion element of* $H^3_{\mathfrak{a}}(R)$.

Proof. (2) $\Rightarrow$ (1) If $\mu_q \in \operatorname{Image}\big(H^2_{\mathfrak{a}}(R) \to H^2_{\mathfrak{a}}(R/pR)\big)$, the relation $(\bar{u}^q, \bar{v}^q, \bar{w}^q)$ on the elements $\bar{\Delta}_1^q,\ \bar{\Delta}_2^q,\ \bar{\Delta}_3^q \in R/pR$ lifts to a relation in $H^2_{\mathfrak{a}}(R)$, i.e., there exists an integer k and elements $\alpha_i \in R$ where $\alpha_1\Delta_1^{q+k} + \alpha_2\Delta_2^{q+k} + \alpha_3\Delta_3^{q+k} = 0$, and

$$(u^q\Delta_2^k\Delta_3^k,\ v^q\Delta_3^k\Delta_1^k,\ w^q\Delta_1^k\Delta_2^k) \equiv (\alpha_1,\ \alpha_2,\ \alpha_3) \mod p.$$

Hence we have

$$\begin{gathered}(u^q\Delta_2^k\Delta_3^k - \alpha_1)\Delta_1^{q+k} + (v^q\Delta_3^k\Delta_1^k - \alpha_2)\Delta_2^{q+k} + (w^q\Delta_1^k\Delta_2^k - \alpha_3)\Delta_3^{q+k} = \\ \Big((u\Delta_1)^q + (v\Delta_2)^q + (w\Delta_3)^q\Big)(\Delta_1\Delta_2\Delta_3)^k \in (p\Delta_1^{q+k},\ p\Delta_2^{q+k},\ p\Delta_3^{q+k})R,\end{gathered}$$

and so $\lambda_q(\Delta_1\Delta_2\Delta_3)^k \in (\Delta_1^{q+k},\ \Delta_2^{q+k},\ \Delta_3^{q+k})R$ and $\eta_q = 0$.

The argument that (1) $\implies$ (2) is similar. $\square$

By the above lemma, an equivalent form of Hochster's conjecture is that for every prime integer p, there is an integer $q = p^e$ such that η_q is a nonzero p-torsion element of $H^3_{\mathfrak{a}}(R)$. Since $p\lambda_q \in (\Delta_1^q,\ \Delta_2^q,\ \Delta_3^q)R$, it is immediate that

$$p \cdot \eta_q = [p\lambda_q + (\Delta_1^q,\ \Delta_2^q,\ \Delta_3^q)R] = 0 \ \in\ H^3_{\mathfrak{a}}(R).$$

We shall show that Hochster's conjecture is false by showing that for all $q = p^e$, $\eta_q = 0$ in $H^3_{\mathfrak{a}}(R)$. More specifically we show that if $k = q - 1$ then

$$\lambda_q(\Delta_1\Delta_2\Delta_3)^k \in (\Delta_1^{q+k},\ \Delta_2^{q+k},\ \Delta_3^{q+k})R.$$

We record the following result, [PS, Proposition 4.1], from which it follows that $H^3_{\mathfrak{a}}(R/pR) = 0$. We use this result with $A = R/P$ and $I = (\bar{\Delta}_1, \bar{\Delta}_2, \bar{\Delta}_3)$. Then $\operatorname{height} I = 2$, and it is well known that the determinantal ring A/I is Cohen-Macaulay.

Proposition 1.2 (Peskine-Szpiro)**.** *Let A be a regular domain of prime characteristic $p > 0$, and let I be an ideal of A such that A/I is Cohen-Macaulay. If* $\operatorname{height} I = h$, *then $H^i_I(A) = 0$ for $i > h$.*

Remark 1.3. For a field K, let $R_K = K \otimes_{\mathbb{Z}} R$ where $R = \mathbb{Z}[u,v,w,x,y,z]$ as above. We record an argument due to Hochster which shows that $H^3_{\mathfrak{a}}(R_{\mathbb{Q}})$ is nonzero. Consider the action of $SL_2(\mathbb{Q})$ on $R_{\mathbb{Q}}$ where where $\alpha \in SL_2(\mathbb{Q})$ sends the entries of the matrix M to the entries of αM. The invariant subring for this action is $A = \mathbb{Q}[\Delta_1, \Delta_2, \Delta_3]$ and, since $SL_2(\mathbb{Q})$ is linearly reductive, A is a direct summand of $R_{\mathbb{Q}}$ as an A–module. Consequently $H^3_{\mathfrak{a}}(A)$ is a nonzero submodule of $H^3_{\mathfrak{a}}(R_{\mathbb{Q}})$. Hence while $H^3_{\mathfrak{a}}(R_{\mathbb{Z}/p\mathbb{Z}}) = 0$, we have $H^3_{\mathfrak{a}}(R_{\mathbb{Q}}) \neq 0$; it is then only natural to expect that the study of $H^3_{\mathfrak{a}}(R)$ would be interesting!

2. The multi-grading

We now work with a fixed prime integer p, an arbitrary prime power $q = p^e$, and set $\lambda = \lambda_q$. We first use a multi-grading to reduce the question whether

$$(\#) \qquad \lambda(\Delta_1\Delta_2\Delta_3)^k \in (\Delta_1^{q+k},\ \Delta_2^{q+k},\ \Delta_3^{q+k})R$$

to a question in a polynomial ring in three variables. We assign weights as follows:

$$\begin{aligned} u &: (1,0,0,0) & \qquad x &: (1,0,0,1) \\ v &: (0,1,0,0) & y &: (0,1,0,1) \\ w &: (0,0,1,0) & z &: (0,0,1,1) \end{aligned}$$

With this grading,

$$\deg(\Delta_1) = (0,1,1,1), \qquad \deg(\Delta_2) = (1,0,1,1), \qquad \deg(\Delta_3) = (1,1,0,1),$$

and λ is a homogeneous element of degree (q,q,q,q). Hence in a homogeneous equation of the form

$$\lambda(\Delta_1\Delta_2\Delta_3)^k = c_1\Delta_1^{q+k} + c_2\Delta_2^{q+k} + c_3\Delta_3^{q+k},$$

we must have

$$\deg(c_1) = (q+2k,k,k,2k), \quad \deg(c_2) = (k,q+2k,k,2k), \\ \deg(c_3) = (k,k,q+2k,2k).$$

We use this to examine the monomials which can occur in c_i. If c_1 involves a monomial of the form $u^a v^b w^c x^d y^i z^j$, then

$$(a+d,\ b+i,\ c+j,\ d+i+j) = (q+2k,\ k,\ k,\ 2k),$$

and so $b = k-i,\ c = k-j,\ d = 2k-i-j$ and $a = q+i+j$. Hence c_1 is a $\mathbb{Z}$-linear combination of monomials of the form

$$u^{q+i+j}v^{k-i}w^{k-j}x^{2k-i-j}y^i z^j = u^q(uy)^i(vx)^{k-i}(uz)^j(wx)^{k-j}.$$

Let $[s_1,s_2]^k$ denote the set of all monomials $s_1^i s_2^{k-i}$ for $0 \le i \le k$, and let $[s_1,s_2]\cdot[t_1,t_2]$ denote the set of products of all pairs of monomials from $[s_1,s_2]$

and $[t_1, t_2]$. With this notation, c_1 is a $\mathbb{Z}$-linear combination of monomials in $u^q \cdot [uy, vx]^k \cdot [uz, wx]^k$. After similar computations for c_2 and c_3, we may conclude that (#) holds if and only if $\lambda(\Delta_1\Delta_2\Delta_3)^k$ is a $\mathbb{Z}$-linear combination of elements from

$$\begin{aligned}
&\Delta_1^{q+k} u^q \cdot [uy, vx]^k \cdot [uz, wx]^k,\\
&\Delta_2^{q+k} v^q \cdot [vz, wy]^k \cdot [vx, uy]^k,\\
&\Delta_3^{q+k} w^q \cdot [wx, uz]^k \cdot [wy, vz]^k.
\end{aligned}$$

We may divide throughout by the element $(uvw)^{q+2k}$, and study this issue in the polynomial ring $\mathbb{Z}[\frac{x}{u}, \frac{y}{v}, \frac{z}{w}]$. Let

$$A = \frac{z}{w} - \frac{x}{u}, \qquad B = \frac{x}{u} - \frac{y}{v}, \qquad T = -\frac{x}{u}.$$

The condition (#) is then equivalent to the statement that the element

$$\frac{\lambda(\Delta_1\Delta_2\Delta_3)^k}{(uvw)^{q+2k}} = \frac{1}{p}\Big((A+B)^q + (-A)^q + (-B)^q\Big)\Big((A+B)AB\Big)^k$$

is a $\mathbb{Z}$-linear combination, in the polynomial ring $\mathbb{Z}[A, B, T]$, of elements of

$$\begin{aligned}
&(A+B)^{q+k} \cdot [T, A]^k \cdot [T, B]^k,\\
&A^{q+k} \cdot [T, B]^k \cdot [T+B, A+B]^k,\\
&B^{q+k} \cdot [T, A]^k \cdot [T-A, A+B]^k.
\end{aligned}$$

We show that the above statement is indeed true for $k = q-1$ using the following equational identity, which will be proved in the next section:

$$\begin{aligned}
&(A+B)^{2k+1} \sum_{n=0}^{k} \binom{k}{n} T^n \sum_{i=0}^{n} (-1)^{k+i} A^{k-i} B^{k-n+i} \binom{k+i}{k}\binom{k+n-i}{k}\\
&- A^{2k+1} \sum_{n=0}^{k} \binom{k}{n} (-1)^n (T+B)^n \sum_{i=0}^{n} (A+B)^{k-i} B^{k-n+i} \binom{k+i}{k}\binom{k+n-i}{k}\\
&- B^{2k+1} \sum_{n=0}^{k} \binom{k}{n} (T-A)^n \sum_{i=0}^{n} (A+B)^{k-i} A^{k-n+i} \binom{k+i}{k}\binom{k+n-i}{k} = 0.
\end{aligned}$$

Separating the terms with $n = 0$ and dividing by p, the above identity gives us

$$\begin{aligned}
&\frac{1}{p}\Big((A+B)^{k+1}(-1)^k - A^{k+1} - B^{k+1}\Big)\Big(AB(A+B)\Big)^k \\
&= -\,(A+B)^{2k+1}\sum_{n=1}^{k}\binom{k}{n}T^n\sum_{i=0}^{n}(-1)^{k+i}A^{k-i}B^{k-n+i}\frac{1}{p}\binom{k+i}{k}\binom{k+n-i}{k} \\
&\quad + A^{2k+1}\sum_{n=1}^{k}\binom{k}{n}(-1)^n(T+B)^n\sum_{i=0}^{n}(A+B)^{k-i}B^{k-n+i}\frac{1}{p}\binom{k+i}{k}\binom{k+n-i}{k} \\
&\quad + B^{2k+1}\sum_{n=1}^{k}\binom{k}{n}(T-A)^n\sum_{i=0}^{n}(A+B)^{k-i}A^{k-n+i}\frac{1}{p}\binom{k+i}{k}\binom{k+n-i}{k}.
\end{aligned}$$

To ensure that the expression on the right hand side of the equality is indeed a $\mathbb{Z}$-linear combination of elements of the required form, it suffices to establish that the prime integer p divides

$$\binom{k+i}{k}\binom{k+n-i}{k},$$

when $1 \le n \le k$ and $0 \le i \le n$, where $k = q-1 = p^e - 1$. Note that this condition ensures that $i \ge 1$ or $n - i \ge 1$, and so it suffices to show that p divides

$$\binom{k+r}{k} = \binom{p^e-1+r}{p^e-1},$$

for $1 \le r \le p^e - 1$. Since

$$\begin{aligned}
\binom{p^e-1+r}{p^e-1} &= \left(\frac{p^e}{r}\right)\left(\frac{p^e+1}{1}\right)\left(\frac{p^e+2}{2}\right)\cdots\left(\frac{p^e+r-1}{r-1}\right) \\
&= \left(\frac{p^e}{r}\right)\left(\frac{p^e}{1}+1\right)\left(\frac{p^e}{2}+1\right)\cdots\left(\frac{p^e}{r-1}+1\right),
\end{aligned}$$

this is easily seen to be true.

3. The equational identity

We first record some identities with binomial coefficients that we shall be using. These identities can be easily proved using Zeilberger's algorithm (see [PWZ]) and the Maple package `EKHAD`, but we include these proofs for the sake of completeness.

When the range of a summation is not specified, it is assumed to extend over all integers. We set $\binom{k}{i} = 0$ if $i < 0$ or if $k < i$.

Lemma 3.1.

$$(1)\qquad \sum_{i=0}^{m}(-1)^i\binom{2k+1}{s+i}\binom{k+i}{k}\binom{k+m-i}{k} = \begin{cases}\binom{m+s-k-1}{m}\binom{2k+m+1}{m+s} & \text{if } s>k,\\ 0 & \text{if } k\ge s\ge k+1-m,\end{cases}$$

$$(2)\qquad \sum_{j=0}^{s}(-1)^j\binom{s}{j}\binom{a+j}{k} = (-1)^s\binom{a}{k-s},$$

$$(3)\qquad \sum_{i=0}^{s}\binom{k-i}{k-s}\binom{k+i}{k}\binom{k+m-i}{m} = \binom{k+m-s}{m}\binom{2k+m+1}{s}.$$

Proof. (1) Let

$$F(m,i) = (-1)^i\binom{2k+1}{s+i}\binom{k+i}{k}\binom{k+m-i}{k},$$
$$G(m,i) = \frac{i(s+i)(k+m+1-i)}{m+1-i}F(m,i), \quad \text{and} \quad H(m) = \sum_i F(m,i).$$

It is easily seen that

$$G(m,i+1) = -(k+i+1)(2k+1-s-i)F(m,i) \quad \text{and}$$
$$F(m+1,i) = \frac{k+m+1-i}{m+1-i}F(m,i),$$

and these can then be used to verify the relation

$$G(m,i+1)-G(m,i) = (2k+m+2)(m+s-k)F(m,i)-(m+1)(m+s+1)F(m+1,i).$$

Summing with respect to i gives us

$$0 = (2k+m+2)(m+s-k)H(m)-(m+1)(m+s+1)H(m+1).$$

Using this recurrence,

$$H(m) = \frac{(2k+m+1)\cdots(2k+2)(m+s-k-1)\cdots(s-k)}{(m)\cdots(1)(m+s)\cdots(s+1)}H(0),$$

and the required result follows.

(2) Let

$$F(s,j) = (-1)^j\binom{s}{j}\binom{a+j}{k}, \qquad G(s,j) = \frac{-j(a+j-k)}{s+1-j}F(s,j),$$

and $H(s) = \sum_j F(s,j)$. It is easily seen that

$$G(s,j+1) = (a+j+1)F(s,j) \quad \text{and} \quad F(s+1,j) = \frac{s+1}{s+1-j}F(s,j),$$

and these can then be used to verify the relation

$$G(s,j+1)-G(s,j)=(k-s)F(s,j)+(a-k+s+1)F(s+1,j).$$

Summing with respect to j gives us

$$0=(k-s)H(s)+(a-k+s+1)H(s+1).$$

Using this recurrence,

$$H(s)=\frac{(-1)^s(k-s+1)\cdots(k)}{(a-k+s)\cdots(a-k+1)}H(0)=(-1)^s\binom{a}{k-s}.$$

(3) Let

$$\begin{aligned}F(s,i)&=\binom{k-i}{k-s}\binom{k+i}{k}\binom{k+m-i}{m},\\ G(s,i)&=\frac{i(k-s)(k+m+1-i)}{s+1-i}F(s,i),\end{aligned}$$

and $H(s)=\sum_i F(s,i)$. It is easily seen that

$$G(s,i+1)=(k-s)(k+i+1)F(s,i)\quad\text{ and }\quad F(s+1,i)=\frac{k-s}{s+1-i}F(s,i),$$

and these can then be used to verify the relation

$$G(s,i+1)-G(s,i)=(k-s)(2k+m-s+1)F(s,i)-(s+1)(k+m-s)F(s+1,i).$$

Summing with respect to j gives us

$$0=(k-s)(2k+m-s+1)H(s)-(s+1)(k+m-s)H(s+1).$$

Using this recurrence,

$$\begin{aligned}H(s)&=\frac{(k-s+1)\cdots(k)(2k+m-s+2)\cdots(2k+m+1)}{(s)\cdots(1)(k+m-s+1)\cdots(k+m)}H(0)\\ &=\binom{k+m-s}{m}\binom{2k+m+1}{s}.\end{aligned}$$

□

We are now ready to prove the equational identity

$$\begin{aligned}&(A+B)^{2k+1}\sum_{n=0}^{k}\binom{k}{n}T^n\sum_{i=0}^{n}(-1)^{k+i}A^{k-i}B^{k-n+i}\binom{k+i}{k}\binom{k+n-i}{k}\\ &-A^{2k+1}\sum_{n=0}^{k}\binom{k}{n}(-1)^n(T+B)^n\sum_{i=0}^{n}(A+B)^{k-i}B^{k-n+i}\binom{k+i}{k}\binom{k+n-i}{k}\\ &-B^{2k+1}\sum_{n=0}^{k}\binom{k}{n}(T-A)^n\sum_{i=0}^{n}(A+B)^{k-i}A^{k-n+i}\binom{k+i}{k}\binom{k+n-i}{k}=0.\end{aligned}$$

Examining the coefficient of T^m for all $0 \le m \le k$, we need to show

$$\begin{aligned}
&(A+B)^{2k+1}\binom{k}{m}\sum_{i=0}^{m}(-1)^{k+i}A^{k-i}B^{k-m+i}\binom{k+i}{k}\binom{k+m-i}{k} \\
&- A^{2k+1}\sum_{n=m}^{k}\binom{k}{n}\binom{n}{m}(-1)^n\sum_{i=0}^{n}(A+B)^{k-i}B^{k-m+i}\binom{k+i}{k}\binom{k+n-i}{k} \\
&- B^{2k+1}\sum_{n=m}^{k}\binom{k}{n}\binom{n}{m}(-1)^{n-m}\sum_{i=0}^{n}(A+B)^{k-i}A^{k-m+i}\binom{k+i}{k}\binom{k+n-i}{k} \\
&= 0.
\end{aligned}$$

Dividing by $\binom{k}{m}(AB)^{k-m}$ and using the fact that

$$\binom{k}{m}\binom{k-m}{n-m} = \binom{k}{n}\binom{n}{m},$$

we need to establish that

$$\begin{aligned}
&(A+B)^{2k+1}\sum_{i=0}^{m}(-1)^{k+i}A^{m-i}B^{i}\binom{k+i}{k}\binom{k+m-i}{k} \\
&- A^{k+m+1}\sum_{n=m}^{k}\binom{k-m}{n-m}(-1)^n\sum_{i=0}^{n}(A+B)^{k-i}B^{i}\binom{k+i}{k}\binom{k+n-i}{k} \\
&- B^{k+m+1}\sum_{n=m}^{k}\binom{k-m}{n-m}(-1)^{n-m}\sum_{i=0}^{n}(A+B)^{k-i}A^{i}\binom{k+i}{k}\binom{k+n-i}{k} = 0.
\end{aligned}$$

For $0 \le r \le m-1$, the coefficient of $A^{k+m-r}B^{k+1+r}$ is

$$\sum_{i=0}^{m}(-1)^{k+i}\binom{2k+1}{k-r+i}\binom{k+i}{k}\binom{k+m-i}{k},$$

which is zero by lemma 3.1 (1) since $k \ge k-r \ge k+1-m$.

For $0 \le r \le k$, the coefficient of $A^{k+m+1+r}B^{k-r}$ as well as the coefficient of $(-1)^m A^{k-r}B^{k+m+1+r}$ is

$$
\begin{aligned}
&\sum_{i=0}^{m}(-1)^{k+i}\binom{2k+1}{k+1+r+i}\binom{k+i}{k}\binom{k+m-i}{k} \\
&\qquad\qquad\qquad - \sum_{n=m}^{k}(-1)^n\binom{k-m}{n-m}\sum_i\binom{k-i}{r}\binom{k+i}{k}\binom{k+n-i}{k} \\
&= (-1)^k\binom{m+r}{m}\binom{2k+m+1}{k-r} \\
&\qquad\qquad\qquad - \sum_{i=0}^{k-r}\binom{k-i}{r}\binom{k+i}{k}\sum_{n=m}^{k}(-1)^n\binom{k-m}{n-m}\binom{k+n-i}{k} \\
&= (-1)^k\binom{m+r}{m}\binom{2k+m+1}{k-r} - \sum_{i=0}^{k-r}(-1)^k\binom{k-i}{r}\binom{k+i}{k}\binom{k+m-i}{m} \\
&= 0,
\end{aligned}
$$

using identities established in lemma 3.1.

4. A local cohomology module with infinitely many associated prime ideals

Consider the hypersurface of mixed characteristic

$$R = \mathbb{Z}[U, V, W, X, Y, Z]/(UX + VY + WZ),$$

and the ideal $\mathfrak{a} = (x, y, z)R$. We show that the local cohomology module $H^3_{\mathfrak{a}}(R)$ has p-torsion elements for infinitely many prime integers p, and consequently that it has infinitely many associated prime ideals. We have

$$H^3_{\mathfrak{a}}(R) = \varinjlim \frac{R}{(x^k, y^k, z^k)R},$$

where the maps in the direct limit system are induced by multiplication by the element xyz. Let p be a prime integer. It is easily seen that

$$\lambda = \frac{(ux)^p + (vy)^p + (wz)^p}{p}$$

has integer coefficients, and we claim that the element

$$\eta = [\lambda + (x^p,\ y^p,\ z^p)R] \ \in\ H^3_{\mathfrak{a}}(R)$$

is nonzero and p-torsion. It is immediate that $p \cdot \eta = 0$, and what really needs to be established is that η is a nonzero element of $H^3_{\mathfrak{a}}(R)$, i.e., that

$$\lambda(xyz)^k \notin (x^{p+k},\ y^{p+k},\ z^{p+k})R \ \text{ for all } \ k \in \mathbb{N}.$$

We shall accomplish this using an $\mathbb{N}^4$-grading. We assign weights as follows:

$$\begin{aligned} u &: (0,1,1,1) & x &: (1,0,0,0) \\ v &: (1,0,1,1) & y &: (0,1,0,0) \\ w &: (1,1,0,1) & z &: (0,0,1,0) \end{aligned}$$

With this grading, λ is a homogeneous element of degree (p,p,p,p). Hence in a homogeneous equation of the form

$$\lambda(xyz)^k = c_1 x^{p+k} + c_2 y^{p+k} + c_3 z^{p+k},$$

we must have

$$\deg(c_1) = (0, p+k, p+k, p), \qquad \deg(c_2) = (p+k, 0, p+k, p),$$
$$\deg(c_3) = (p+k, p+k, 0, p).$$

We may use this to examine the monomials which can occur in c_i, and it is easily seen that the only monomial that can occur in c_1 with a nonzero coefficient is $u^p y^k z^k$, and similarly that c_2 is an integer multiple of $v^p z^k x^k$ and c_3 is an integer multiple of $w^p x^k y^k$. Hence $\lambda(xyz)^k \in (x^{p+k}, y^{p+k}, z^{p+k})R$ if and only if

$$\begin{aligned} \lambda(xyz)^k \in (u^p y^k z^k x^{p+k},\ v^p z^k x^k y^{p+k},\ w^p x^k y^k z^{p+k})R& \\ = (xyz)^k \Big((ux)^p,\ (vy)^p,\ (wz)^p\Big)R&, \end{aligned}$$

i.e., if and only if $\lambda \in ((ux)^p, (vy)^p, (wz)^p)R$. To complete our argument it suffices to show that $\lambda \notin (p, (ux)^p, (vy)^p, (wz)^p)R$, i.e., that

$$\frac{(ux)^p + (vy)^p + (-1)^p(ux+vy)^p}{p} \notin \Big(p,\ (ux)^p,\ (vy)^p\Big)R.$$

After making the specializations $u \mapsto 1,\ v \mapsto 1,\ w \mapsto 1$, it is enough to verify that

$$\frac{x^p + y^p + (-1)^p(x+y)^p}{p} \notin (p,\ x^p,\ y^p)\mathbb{Z}[x,y]$$

and this holds since the coefficient of $x^{p-1}y$ in $(x^p + y^p + (-1)^p(x+y)^p)/p$ is $(-1)^p$.

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