Journal of Algebra 571 (2021) 339–349

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Homogeneous prime elements in normal two-dimensional graded rings ☆

Anurag K. Singh [a,*], Ryo Takahashi [b], Kei-ichi Watanabe [c]

[a] *Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 84112, USA*
[b] *Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, 464-8602, Japan*
[c] *Department of Mathematics, College of Humanities and Sciences, Nihon University, Setagaya-ku, Tokyo, 156-8550, Japan*

A R T I C L E   I N F O

Article history:
Received 17 January 2018
Available online 18 July 2018
Communicated by L. Avramov, M. Hochster, B. Ulrich, K. Smith and I. Swanson

Dedicated to Professor Craig Huneke on the occasion of his sixty-fifth birthday

Keywords:
Normal rings
$\mathbb{Q}$-divisors
Prime elements

A B S T R A C T

We prove necessary and sufficient conditions for the existence of homogeneous prime elements in normal $\mathbb{N}$-graded rings of dimension two, in terms of rational coefficient Weil divisors on projective curves.

☆ A.K.S. was supported by NSF grants DMS 1500613 and DMS 1801285, R.T. by JSPS Grant-in-Aid for Scientific Research 16H03923 and 16K05098, and K.W. by Grant-in-Aid for Scientific Research 26400053. This paper started from conversations between R.T. and K.W. at the program *Commutative algebra and related topics* at Okinawa, Japan, supported by MSRI, OIST, and RIMS, Kyoto University. The authors are grateful to these institutes, and to the organizers of the program for providing a very comfortable research atmosphere.

* Corresponding author.
E-mail addresses: singh@math.utah.edu (A.K. Singh), takahashi@math.nagoya-u.ac.jp (R. Takahashi), watanabe@math.chs.nihon-u.ac.jp (K.-i. Watanabe).

https://doi.org/10.1016/j.jalgebra.2018.07.012
0021-8693/

1. Introduction

We investigate the existence of homogeneous prime elements, equivalently, of homogeneous principal prime ideals, in normal $\mathbb{N}$-graded rings R of dimension two. It turns out that there are elegant necessary as well as sufficient conditions for the existence of such prime ideals in terms of rational coefficient Weil divisors, i.e., $\mathbb{Q}$-divisors, on $\operatorname{Proj} R$.

When speaking of an $\mathbb{N}$-*graded ring* R, we assume throughout this paper that R is a finitely generated algebra over its subring R_0, and that R_0 is an algebraically closed field. We say that an $\mathbb{N}$-grading on R is *irredundant* if

$$\gcd\{n \in \mathbb{N} \mid R_n \neq 0\} \;=\; 1.$$

Relevant material on $\mathbb{Q}$-divisors is summarized in §2. Our main result is:

Theorem 1.1. *Let R be a normal ring of dimension 2, with an irredundant $\mathbb{N}$-grading, where R_0 is an algebraically closed field. Set $X := \operatorname{Proj} R$, and let D be a $\mathbb{Q}$-divisor on X such that $R = \oplus_{n\geqslant 0} H^0(X, \mathscr{O}_X(nD))T^n$. Let d be a positive integer.*

(1) *Suppose $x \in R_d$ is a prime element. Set*

$$s := \gcd\{n \in \mathbb{N} \mid [R/xR]_n \neq 0\}.$$

Then the integers d and s are relatively prime, and the divisor sdD is linearly equivalent to a point of X. In particular, $\deg D = 1/sd$.

(2) *Conversely, suppose dD is linearly equivalent to a point P with $P \notin \operatorname{supp}(\operatorname{frac}(D))$. Let g be a rational function on X with*

$$\operatorname{div}(g) \;=\; P - dD.$$

Then $x := gT^d$ is a prime element, and the induced grading on R/xR is irredundant.

The proof of the theorem and further results regarding the number of homogeneous principal prime ideals are included in §3. We next record various examples.

Example 1.2. If a standard $\mathbb{N}$-graded ring R, as in the theorem, has a homogeneous prime element, we claim that R must be a polynomial ring over R_0.

If $x \in R_d$ is a prime element, the theorem implies that $d = 1$. Independent of the theorem, note that R/xR is an $\mathbb{N}$-graded domain of dimension 1, with $[R/xR]_0$ algebraically closed, so R/xR is a numerical semigroup ring by [4, Proposition 2.2.11]. Since it is standard graded, R/xR must be a polynomial ring. But then R is a polynomial ring as well.

Example 1.3. The hypothesis that the underlying field R_0 is algebraically closed is crucial in Theorem 1.1 and Example 1.2: the standard graded ring $\mathbb{Q}[x,y,z]/(x^2+y^2+z^2)$ has a homogeneous prime element x.

Example 1.4. In view of Example 1.2, the ring $R := \mathbb{C}[x,y,z]/(x^2-yz)$, with the standard $\mathbb{N}$-grading, has no homogeneous prime element. However, for nonstandard gradings, there can be homogeneous prime elements:

Fix such a grading with $\deg x = a$, $\deg y = b$, and $\deg z = 2a-b$, where $\gcd(a,b) = 1$ and b is even. Then one has a homogeneous prime element

$$y^{a-b/2} - z^{b/2},$$

which generates the kernel of the $\mathbb{C}$-algebra homomorphism $R \longrightarrow \mathbb{C}[t]$ with

$$x \longmapsto t^a, \quad y \longmapsto t^b, \quad z \longmapsto t^{2a-b}.$$

Example 1.5. The ring $\mathbb{C}[x,y,z]/(x^4+y^2z+xz^2)$, with $\deg x = 4$, $\deg y = 5$, and $\deg z = 6$, has no homogeneous prime elements in view of Theorem 1.1(1), since the corresponding $\mathbb{Q}$-divisor has degree $2/15$ by Proposition 2.2.

Example 1.6. Consider $\mathbb{C}[x,y,z]/(x^2+y^3+z^6)$, with $\deg x = 3$, $\deg y = 2$, and $\deg z = 1$. Then (z) is the unique homogeneous principal prime ideal: the corresponding $\mathbb{Q}$-divisor has degree 1, again by Proposition 2.2.

Example 1.7. Set $R := \mathbb{C}[x,y,z]/(x^2-y^3+z^7)$, with $\deg x = 21$, $\deg y = 14$, and $\deg z = 6$. Then the corresponding $\mathbb{Q}$-divisor has degree $1/42$ by Proposition 2.2, so the degree of a homogeneous prime element must divide 42. In view of the degrees of the generators of R, the possibilities are 6, 14, 21, and 42, and indeed there are prime elements with each of these degrees, namely z, y, x, and $y^3 - \lambda x^2$ for scalars $\lambda \neq 0, 1$, see also Example 3.4.

2. Rational coefficient Weil divisors

We review the construction of normal graded rings in terms of $\mathbb{Q}$-divisors; this is work of Dolgačev [2], Pinkham [6], and Demazure [1]. Let X be a normal projective variety. A $\mathbb{Q}$-*divisor* on X is a $\mathbb{Q}$-linear combination of irreducible subvarieties of X of codimension one. Let $D = \sum n_i V_i$ be such a divisor, where $n_i \in \mathbb{Q}$, and V_i are distinct. Set

$$\lfloor D \rfloor := \sum \lfloor n_i \rfloor V_i,$$

where $\lfloor n \rfloor$ is the greatest integer less than or equal to n. We define

$$\mathscr{O}_X(D) := \mathscr{O}_X(\lfloor D \rfloor).$$

The divisor D is *effective*, denoted $D \geqslant 0$, if each n_i is nonnegative. The *support* of the fractional part of D is the set

$$\operatorname{supp}(\operatorname{frac}(D)) := \{V_i \mid n_i \notin \mathbb{Z}\}.$$

Let $K(X)$ denote the field of rational functions on X. Each $g \in K(X)$ defines a Weil divisor $\operatorname{div}(g)$ by considering the zeros and poles of g with appropriate multiplicity. As these multiplicities are integers, it follows that for a $\mathbb{Q}$-divisor D one has

$$\begin{aligned} H^0(X, \mathscr{O}_X(\lfloor D \rfloor)) &= \{g \in K(X) \mid \operatorname{div}(g) + \lfloor D \rfloor \geqslant 0\} \\ &= \{g \in K(X) \mid \operatorname{div}(g) + D \geqslant 0\} = H^0(X, \mathscr{O}_X(D)). \end{aligned}$$

A $\mathbb{Q}$-divisor D is *ample* if ND is an ample Cartier divisor for some $N \in \mathbb{N}$. In this case, the *generalized section ring* $R(X, D)$ is the $\mathbb{N}$-graded ring

$$R(X, D) := \oplus_{n \geqslant 0} H^0(X, \mathscr{O}_X(nD))T^n,$$

where T is an element of degree 1, transcendental over $K(X)$.

Theorem 2.1 *([1, 3.5]). Let R be an $\mathbb{N}$-graded normal domain that is finitely generated over a field R_0. Let T be a homogeneous element of degree 1 in the fraction field of R. Then there exists a unique ample $\mathbb{Q}$-divisor D on $X := \operatorname{Proj} R$ such that*

$$R_n \ = \ H^0(X, \mathscr{O}_X(nD))T^n \quad \text{for each } n \geqslant 0.$$

The following result is due to Tomari:

Proposition 2.2 *([8, Proposition 2.1]). For R and D as in the theorem above, one has*

$$\lim_{t \to 1} (1-t)^{\dim R} P(R, t) \ = \ (\deg D)^{\dim R - 1},$$

where $P(R, t)$ is the Hilbert series of R.

3. Homogeneous prime elements

Before proceeding with the proof of the main theorem, we record a lemma:

Lemma 3.1. *Let R be a domain with an irredundant $\mathbb{N}$-grading. Let x be a nonzero element of degree $d > 0$, and set*

$$s := \gcd\{n \in \mathbb{N} \mid [R/xR]_n \neq 0\}.$$

Then $\gcd(d,s)=1$. *Moreover, x is a prime element of R if and only if x^s is a prime element of the Veronese subring* $R^{(s)} := \oplus_{n\geqslant 0} R_{ns}$.

Proof. Note that $P(R/xR,t)$ is a rational function of t^s, and that

$$P(R,t) \;=\; \frac{1}{1-t^d}P(R/xR,t).$$

Since the grading on R is irredundant, it follows that $\gcd(d,s)=1$.

We claim that $(xR)^{(s)} = x^s R^{(s)}$. Choose a homogeneous element of $(xR)^{(s)}$, and express it as rx^m with m largest possible. Suppose m is not a multiple of s. By considering its degree, we see that the image of r must be 0 in R/xR, contradicting the maximality of m. It follows that $(xR)^{(s)} \subseteq x^s R^{(s)}$, the reverse containment being trivial. Hence

$$R/xR \;=\; (R/xR)^{(s)} \;=\; R^{(s)}/x^s R^{(s)},$$

which gives the desired equivalence. □

Proof of Theorem 1.1. For a prime element $x\in R_d$, the ring R/xR is an $\mathbb{N}$-graded domain of dimension 1, over an algebraically closed field, so [4, Proposition 2.2.11] implies that it is isomorphic to a numerical semigroup ring. Take s as in Lemma 3.1. Since the Veronese subring $R^{(s)}$ corresponds to the $\mathbb{Q}$-divisor sD, the proof of (1) reduces using the lemma to the case where $s=1$. In this case,

$$P(R/xR,t) \;=\; \frac{1}{1-t} - p(t)$$

for $p(t)$ a polynomial, so

$$P(R,t) \;=\; \frac{1}{1-t^d}\left(\frac{1}{1-t} - p(t)\right),$$

and Proposition 2.2 shows that $\deg D = 1/d$. To complete the proof of (1), it remains to verify that dD is linearly equivalent to a point of X.

The exact sequence

$$0 \longrightarrow R(-d) \xrightarrow{\;x\;} R \longrightarrow R/xR \longrightarrow 0$$

shows that for $n \gg 0$ one has

$$\operatorname{rank} R_{n+d} \;=\; 1 + \operatorname{rank} R_n.$$

Choose $m \gg 0$ such that mdD is integral, and the above holds with $n = md$, i.e.,

$$\operatorname{rank} H^0(X, \mathscr{O}_X(mdD + dD)) \;=\; 1 + \operatorname{rank} H^0(X, \mathscr{O}_X(mdD)).$$

Let g be a rational function on X such that $x = gT^d$. Then $\operatorname{div}(g) + dD \geqslant 0$, and

$$\operatorname{rank} H^0(X, \mathscr{O}_X(mdD + \operatorname{div}(g) + dD)) \;=\; 1 + \operatorname{rank} H^0(X, \mathscr{O}_X(mdD)).$$

Since mdD is an integral divisor, it follows that

$$\lfloor \operatorname{div}(g) + dD \rfloor \;\neq\; 0.$$

Bearing in mind that $\operatorname{div}(g) + dD$ is an effective divisor of degree 1, it follows that

$$\operatorname{div}(g) + dD \;=\; P,$$

for P a point of X.

For (2), we claim that

$$xR \;=\; \oplus_{n \geqslant 0} H^0(X, \mathscr{O}_X(nD - P))T^n,$$

and that this is a prime ideal of R. Note that homogeneous elements of xR have the form

$$gT^d hT^m$$

for $hT^m \in R$, i.e., with h satisfying

$$\operatorname{div}(h) + mD \;\geqslant\; 0.$$

Since $\operatorname{div}(g) = P - dD$, the above condition is equivalent to

$$\operatorname{div}(gh) + (m + d)D - P \;\geqslant\; 0,$$

i.e., to the condition that $gh \in H^0(X, \mathscr{O}_X((m + d)D - P))$. This proves the claim.

To verify that the ideal xR is prime, consider $h_i T^{m_i}$ in $R \smallsetminus xR$, for $i = 1, 2$. Then

$$\operatorname{div}(h_i) + m_i D \;\geqslant\; 0 \quad \text{whereas} \quad \operatorname{div}(h_i) + m_i D - P \;\ngeqslant\; 0.$$

Since P is not in the support of the fractional part of D, it follows that

$$\operatorname{div}(h_1 h_2) + (m_1 + m_2)D - P \;\ngeqslant\; 0,$$

and hence that $h_1 h_2 T^{m_1+m_2} \notin xR$. Thus, xR is indeed prime. It remains to prove that the grading on R/xR is irredundant. Set

$$s := \gcd\{n \in \mathbb{N} \mid [R/xR]_n \neq 0\},$$

in which case

$$P(R/xR,t) \;=\; \frac{1}{1-t^s} - p(t^s),$$

where $p(t^s)$ is a polynomial in t^s, and

$$P(R,t) \;=\; \frac{1}{1-t^d}\left(\frac{1}{1-t^s} - p(t^s)\right).$$

Proposition 2.2 gives the second equality below,

$$\frac{1}{ds} \;=\; \lim_{t\to 1}(1-t)^2 P(R,t) \;=\; \deg D \;=\; \frac{1}{d},$$

implying that $s = 1$. $\square$

Example 3.2. Take $\mathbb{P}^1 := \operatorname{Proj}\mathbb{C}[u,v]$, with points parametrized by u/v, and set

$$D := \frac{1}{2}(0) + \frac{1}{2}(\infty) - \frac{1}{2}(1).$$

Then $R := R(\mathbb{P}^1, D)$ is the $\mathbb{C}$-algebra generated by

$$x := \frac{u-v}{v}T^2, \quad y := \frac{u-v}{u}T^2, \quad z := \frac{(u-v)^2}{uv}T^3,$$

i.e., R is the hypersurface $\mathbb{C}[x,y,z]/(z^2 - xy(x-y))$, with $\deg x = 2 = \deg y$, and $\deg z = 3$. Note that $\deg D = 1/2$, and that $2D$ is an integral divisor. Theorem 1.1(2) shows that

$$\oplus_{n\geqslant 0} H^0(X, \mathscr{O}_X(nD - P))T^n$$

is a prime ideal for $P \in \mathbb{P}^1 \smallsetminus \{0,\infty,1\}$. Indeed, for $P = [\lambda : 1]$ with $\lambda \neq 0, 1$, the displayed ideal is the prime $(x - \lambda y)R$. These are precisely the homogeneous principal prime ideals of R, with the points 0, ∞, and 1 that belong to $\operatorname{supp}(\operatorname{frac}(D))$ corresponding respectively to the ideals xR, yR and $(x-y)R$ that are not prime.

Remark 3.3. Let D be a $\mathbb{Q}$-divisor on $\mathbb{P}^1$ such that $\deg D = 1/d$ where d is a positive integer, and dD is integral. Then the ring $R := R(\mathbb{P}^1, D)$ has infinitely many distinct homogeneous principal prime ideals: all points of $\mathbb{P}^1$ are linearly equivalent, so for each point P there exists a rational function g with

$$\operatorname{div}(g) \;=\; P - dD,$$

and Theorem 1.1(2) implies that gT^dR is a prime ideal for each point P with

$$P \in \mathbb{P}^1 \setminus \operatorname{supp}(\operatorname{frac}(D)).$$

This explains the infinitely many prime ideals in Example 3.2, and also in Example 3.4 below; the latter, moreover, has homogeneous prime elements of different degrees.

Example 3.4. On $\mathbb{P}^1 := \operatorname{Proj} \mathbb{C}[u, v]$, consider the $\mathbb{Q}$-divisor

$$D := \frac{1}{2}(\infty) - \frac{1}{3}(0) - \frac{1}{7}(1).$$

Then $R := R(\mathbb{P}^1, D)$ is the ring $\mathbb{C}[x, y, z]/(x^2 - y^3 + z^7)$, where

$$z := \frac{u^2(u-v)}{v^3}T^6, \quad y := \frac{u^5(u-v)^2}{v^7}T^{14}, \quad x := \frac{u^7(u-v)^3}{v^{10}}T^{21}.$$

For each point $P = [\lambda : 1]$ in $\mathbb{P}^1 \smallsetminus \{0, \infty, 1\}$, i.e., with $\lambda \neq 0, 1$, one has a prime ideal

$$\oplus_{n \geqslant 0} H^0(X, \mathscr{O}_X(nD - P))T^n \;=\; (y^3 - \lambda x^2)R.$$

These, along with xR, yR, and zR, are precisely the homogeneous principal prime ideals.

Example 3.5. Set X to be the elliptic curve $\operatorname{Proj} \mathbb{C}[u, v, w]/(v^2w - u^3 + w^3)$. Then

$$\operatorname{div}(v/w) \;=\; P_1 + P_2 + P_3 - 3O,$$

where $O = [0 : 1 : 0]$ is the point at infinity, and

$$P_1 = [1 : 0 : 1], \quad P_2 = [\theta : 0 : 1], \quad P_3 = [\theta^2 : 0 : 1],$$

for θ a primitive cube root of unity. Take

$$D := \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 - O.$$

The ring $R := R(X, D)$ has generators

$$x := \frac{w}{v}T^2, \quad y := \frac{w}{v}T^3, \quad z := \frac{uw}{v^2}T^4,$$

so $R = \mathbb{C}[x, y, z]/(x^6 + y^4 - z^3)$. Since $\deg D = 1/2$, the only possible homogeneous prime elements are in degree 2. Indeed,

$$2D \;=\; P_1 + P_2 + P_3 - 2O \;=\; \operatorname{div}(v/w) + O,$$

and $O \notin \operatorname{supp}(\operatorname{frac}(D))$, so $(w/v)T^2 = x$ is a prime element; note that xR is the *unique* homogeneous principal prime ideal of R, in contrast with Examples 3.2 and 3.4.

Example 3.6. With X and O as in the previous example, note that

$$\operatorname{div}(u/w) \;=\; Q_1 + Q_2 - 2O,$$

where

$$Q_1 = [0 : i : 1], \quad Q_2 = [0 : -i : 1].$$

Consider the $\mathbb{Q}$-divisor

$$D \;=\; \frac{1}{2}Q_1 + \frac{1}{2}Q_2 - \frac{1}{2}O.$$

Then the ring $R := R(X, D)$ has generators

$$x := \frac{w}{u}T^2, \quad y := \frac{w}{u}T^3, \quad z := \frac{w}{u}T^4, \quad t := \frac{vw^2}{u^3}T^6,$$

and presentation

$$R \;=\; \mathbb{C}[x, y, z, t]/(y^2 - xz,\ x^6 - z^3 + t^2).$$

Once again, since $\deg D = 1/2$, the only possibility for homogeneous prime elements is in degree 2. We see that

$$2D \;=\; Q_1 + Q_2 - O \;=\; \operatorname{div}(u/w) + O.$$

However, since $O \in \operatorname{supp}(\operatorname{frac}(D))$, Theorem 1.1(2) does not apply. Indeed, $(w/u)T^2 = x$ is not a prime element. The key point is that X is not rational, and there does not exist a point P, linearly equivalent to $2D$, with $P \notin \operatorname{supp}(\operatorname{frac}(D))$.

4. Rational singularities

Let H be a numerical semigroup. For $\mathbb{F}$ a field and t an indeterminate, set

$$\mathbb{F}[H] := \mathbb{F}[t^n \mid n \in H].$$

Question 4.1. Does $\mathbb{F}[H]$ deform to a normal $\mathbb{N}$-graded ring, i.e., does there exist a normal $\mathbb{N}$-graded ring R, with $x \in R$ homogeneous, such that $R/xR \cong \mathbb{F}[H]$?

Question 4.2. For which numerical semigroups H does there exist R, as above, such that R has rational singularities?

The following is a partial answer:

Proposition 4.3. *Let R be a normal ring of dimension 2, with an irredundant $\mathbb{N}$-grading, where $R_0 = \mathbb{F}$ is an algebraically closed field. Suppose x_0 is a homogeneous prime element that is part of a minimal reduction of R_+, and that the induced grading on R/x_0R is irredundant. Then the following are equivalent:*

(1) *The ring R has rational singularities.*
(2) *There exist minimal $\mathbb{F}$-algebra generators $x_0, \dots, x_r$ for R, with x_i homogeneous, and*

$$r + \deg x_0 > \deg x_1 > \cdots > \deg x_r = r. \tag{4.3.1}$$

Proof. Note that R/x_0R is a numerical semigroup ring; let H denote the semigroup.

(1) $\Longrightarrow$ (2): The element x_0 extends to a minimal generating set $x_0, \dots, x_r$ for R. Since $R/x_0R = \mathbb{F}[H]$ is a numerical semigroup ring, the degrees of $x_1, \dots, x_r$ are distinct; after reindexing, we may assume that

$$\deg x_1 > \cdots > \deg x_r.$$

Since R is a 2-dimensional ring with rational singularities, it has minimal multiplicity by [5, Theorem 3.1], namely

$$e(R) \;=\; \operatorname{edim}(R) - 1.$$

As x_0 is part of a minimal reduction of R_+, the ring R/x_0R has minimal multiplicity as well, i.e., $e(R/x_0R) = r$. It follows that $\deg x_r = r$. By [7, Corollary 3.2], the Frobenius number of H is $\deg x_1 - \deg x_r = \deg x_1 - r$, which is the a-invariant of $\mathbb{F}[H]$. But then

$$a(R) + \deg x_0 \;=\; a(\mathbb{F}[H]) \;=\; \deg x_1 - r.$$

Since R is a ring of positive dimension with rational singularities, $a(R)$ must be negative by [3,9], implying that $r + \deg x_0 > \deg x_1$ as desired.

(2) $\Longrightarrow$ (1): Since R is normal by assumption, one has only to verify that $a(R) < 0$ in view of the above references. This is immediate since the a-invariant of $\mathbb{F}[H]$, equivalently the Frobenius number of H, is $\deg x_1 - r$. $\square$

Example 4.4. Consider the $\mathbb{Q}$-divisor

$$D := \frac{5}{7}(0) - \frac{4}{7}(\infty)$$

on $\mathbb{P}^1 := \operatorname{Proj} \mathbb{C}[u, v]$, with points parametrized by u/v. Then $R := R(\mathbb{P}^1, D)$ has generators

$$w := \frac{v^2}{u^2}T^3, \quad x := \frac{v^3}{u^3}T^5, \quad y := \frac{v^4}{u^4}T^7, \quad z := \frac{v^5}{u^5}T^7.$$

The relations are readily seen to be the size two minors of the matrix

$$\begin{pmatrix} w & x & z \\ x & y & w^3 \end{pmatrix}.$$

Each point $P = [\lambda : 1]$ with $\lambda \neq 0$ gives a prime ideal

$$\oplus_{n \geqslant 0} H^0(X, \mathscr{O}_X(nD - P))T^n \;=\; (y - \lambda z)R,$$

and these are precisely the homogeneous principal prime ideals of R.

For example,

$$R/(y-z)R \;=\; \mathbb{C}[t^3, t^5, t^7].$$

Since $(y-z, w)R$ is a minimal reduction of R_+ and the grading on $R/(y-z)R$ is irredundant, Proposition 4.3 applies. The ring R has rational singularities since $a(R) = -3$, and the inequalities (4.3.1) indeed hold since

$$3 + \deg(y-z) > \deg y > \deg x > \deg w = 3.$$

Example 4.5. Take R as in Example 1.6, i.e., $R := \mathbb{C}[x,y,z]/(x^2+y^3+z^6)$, with $\deg x = 3$, $\deg y = 2$, and $\deg z = 1$. Then z is a prime element such that the induced grading on R/zR is irredundant; z is also part of the minimal reduction $(z,y)R$ of R_+. Since $a(R) = 0$, the ring R does not have rational singularities; likewise, (4.3.1) does not hold since

$$2 + \deg z \not> \deg x.$$

References

[1] M. Demazure, Anneaux gradués normaux, in: Introduction à la théorie des singularités II, in: Travaux en Cours, vol. 37, Hermann, Paris, 1988, pp. 35–68.
[2] I.V. Dolgačev, Automorphic forms, and quasihomogeneous singularities, Funktsional. Anal. i Prilozhen. 9 (1975) 67–68.
[3] H. Flenner, Rationale quasihomogene Singularitäten, Arch. Math. 36 (1981) 35–44.
[4] S. Goto, K.-i. Watanabe, On graded rings I, J. Math. Soc. Japan 30 (1978) 179–213.
[5] C. Huneke, K.-i. Watanabe, Upper bound of multiplicity of F-pure rings, Proc. Amer. Math. Soc. 143 (2015) 5021–5026.
[6] H. Pinkham, Normal surface singularities with C^* action, Math. Ann. 227 (1977) 183–193.
[7] J.C. Rosales, P.A. García-Sánchez, Numerical Semigroups, Dev. Math., vol. 20, Springer, New York, 2009.
[8] M. Tomari, Multiplicity of filtered rings and simple $K3$ singularities of multiplicity two, Publ. Res. Inst. Math. Sci. 38 (2002) 693–724.
[9] K.-i. Watanabe, Rational singularities with k^*-action, in: Commutative Algebra, Trento, 1981, in: Lect. Notes Pure Appl. Math., vol. 84, Dekker, New York, 1983, pp. 339–351.