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A CONNECTEDNESS RESULT IN POSITIVE CHARACTERISTIC

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Dedicated to Professor Paul Roberts on the occasion of his sixtieth birthday

Abstract. Let $(R,\mathfrak{m})$ be a complete local ring of dimension at least two, which contains a separably closed coefficient field of positive characteristic. Using a vanishing theorem of Peskine-Szpiro, Lyubeznik proved that the local cohomology module $H^1_{\mathfrak{m}}(R)$ is Frobenius-torsion if and only if the punctured spectrum of R is connected in the Zariski topology. We give a simple proof of this theorem and, more generally, a formula for the number of connected components in terms of the Frobenius action on $H^1_{\mathfrak{m}}(R)$.

1. Introduction

All rings considered in this note are commutative and Noetherian. We give a simple proof of the following result due to Lyubeznik:

Theorem 1.1 ([Ly2, Corollary 4.6])**.** *Let $(R,\mathfrak{m})$ be a complete local ring of dimension at least two, with a separably closed coefficient field of positive characteristic. Then the e-th iteration of the Frobenius map*

$$F\colon H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(R)$$

is zero for $e \gg 0$ if and only if $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$ *is connected in the Zariski topology.*

We also obtain, by similar methods, the following theorem:

Theorem 1.2. *Let $(R,\mathfrak{m})$ be a complete local ring of positive dimension, with an algebraically closed coefficient field of positive characteristic. Then the number of connected components of* $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$ *is*

$$1 + \dim_K \bigcap_{e\in\mathbb{N}} F^e(H^1_{\mathfrak{m}}(R)).$$

In Section 5 we describe how this provides an algorithm to determine the number of geometrically connected components of projective algebraic sets defined over a finite field: computer algebra algorithms for primary decomposition can be used to determine the number of connected components over finite extensions of the fields $\mathbb{F}_p$ or $\mathbb{Q}$, but not over the algebraic closures of these fields. In the case of characteristic zero, de Rham cohomology allows for the computation of the number

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of geometrically connected components via D-module methods, [Wal], and we show that the Frobenius provides analogous methods in the case of positive characteristic.

Theorem 1.1 is obtained in [Ly2] as a corollary of the following two theorems of Lyubeznik and Peskine-Szpiro:

Theorem 1.3 ([Ly2, Theorem 1.1])**.** *Let $(A, \mathfrak{M})$ be a regular local ring containing a field of positive characteristic, and let $\mathfrak{A}$ be an ideal of A. Then $H^i_{\mathfrak{A}}(A) = 0$ if and only if there exists an integer $e \geq 1$ such that the e-th Frobenius iteration*

$$F^e \colon H^{\dim A - i}_{\mathfrak{M}}(A/\mathfrak{A}) \to H^{\dim A - i}_{\mathfrak{M}}(A/\mathfrak{A})$$

is the zero map.

Theorem 1.4 ([PS, Chapter III, Theorem 5.5])**.** *Let $(A, \mathfrak{M})$ be a complete regular local ring with a separably closed coefficient field of positive characteristic, and let $\mathfrak{A}$ be an ideal of A. Then $H^i_{\mathfrak{A}}(A) = 0$ for $i \geq \dim A - 1$ if and only if $\dim(A/\mathfrak{A}) \geq 2$ and $\operatorname{Spec}(A/\mathfrak{A}) \smallsetminus \{\mathfrak{M}\}$ is connected.*

Our proof of Theorem 1.1 is "simple" in the sense that it does not rely on vanishing theorems such as those of [PS]—indeed, the only ingredient, aside from elementary considerations, is the local duality theorem. Results analogous to Theorem 1.4 were proved by Hartshorne in the projective case [HaR, Theorem 7.5], and by Ogus in equicharacteristic zero using de Rham cohomology [Og, Corollary 2.11]. Combining these results, one has:

Theorem 1.5. *Let $(A, \mathfrak{M})$ be a regular local ring containing a field, and let $\mathfrak{A}$ be an ideal of A. Then $H^i_{\mathfrak{A}}(A) = 0$ for $i \geq \dim A - 1$ if and only if*

(1) $\dim(A/\mathfrak{A}) \geq 2$, *and*
(2) $\operatorname{Spec}(A/\mathfrak{A}) \smallsetminus \{\mathfrak{M}\}$ *is formally geometrically connected (see Definition* 2.1*).*

Huneke and Lyubeznik [HL, Theorem 2.9] gave a characteristic free proof of this using a generalization of a result of Faltings, [Fa, Satz 1]. Some other applications of local cohomology theory which yield strong results on the connectedness properties of algebraic varieties may be found in the papers [BR] and [HH], where the authors obtain generalizations of Faltings' connectedness theorem.

For the convenience of the reader, we include an Appendix with some facts about Frobenius actions; see Section 6.

2. Preliminary remarks

Notation. When R is the homomorphic image of a ring A, we use upper-case letters $\mathfrak{P}, \mathfrak{Q}, \mathfrak{M}, \mathfrak{A}, \mathfrak{B}$ for ideals of A, and corresponding lower-case letters $\mathfrak{p}, \mathfrak{q}, \mathfrak{m}, \mathfrak{a}, \mathfrak{b}$ for their images in R.

Definition 2.1. Let $(R, \mathfrak{m})$ be a local ring. A field $K \subseteq R$ is a *coefficient field* for R if the composition $K \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism. Every complete local ring containing a field has a coefficient field.

We recall some notions from [Ra, Chapitre VIII]. Let $(R, \mathfrak{m}, K)$ be a local ring and let $\overline{f(T)} \in K[T]$ denote the image of a polynomial $f(T) \in R[T]$. Then R is *Henselian* if for every monic polynomial $f(T) \in R[T]$, every factorization of $\overline{f(T)}$ as a product of relatively prime monic polynomials in $K[T]$ lifts to a factorization of $f(T)$ as a product of monic polynomials in $R[T]$. Hensel's Lemma is precisely the statement that every complete local ring is Henselian. The *Henselization* of a local ring R is a local ring R^{h}, with the property that every local homomorphism from R to a Henselian local ring factors uniquely through R^{h}. The ring R^{h} is obtained by taking the direct limit of all local étale extensions S of R for which $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ induces an isomorphism of residue fields $R/\mathfrak{m} \xrightarrow{\cong} S/\mathfrak{n}$.

A local ring $(R, \mathfrak{m}, K)$ is said to be *strictly Henselian* if it is Henselian and its residue field K is separably closed. It is easily seen that R is strictly Henselian if and only if every monic polynomial $f(T) \in R[T]$ for which $\overline{f(T)} \in K[T]$ is separable splits into linear factors in $R[T]$. Every local ring has a *strict Henselization* R^{sh}, such that every local homomorphism from R to a strictly Henselian ring factors through R^{sh}. The strict Henselization of a field K is its separable closure K^{sep}. In general, the strict Henselization of a local ring $(R, \mathfrak{m}, K)$ is obtained by fixing an embedding $\iota\colon K \to K^{\mathrm{sep}}$, and taking the direct limit of local étale extensions $(S, \mathfrak{n}, L)$ of $(R, \mathfrak{m}, K)$ with $L \hookrightarrow K^{\mathrm{sep}}$, for which the induced map $K \to L \to K^{\mathrm{sep}}$ agrees with $\iota\colon K \to K^{\mathrm{sep}}$.

The *punctured spectrum* of a local ring $(R, \mathfrak{m})$ is the set $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$, with the topology induced by the Zariski topology on $\operatorname{Spec} R$. We say that *the punctured spectrum of R is formally geometrically connected* if the punctured spectrum of $\hat{R}^{\mathrm{sh}}$, the completion of the strict Henselization of the completion of R, is connected. If R is an $\mathbb{N}$-graded ring which is finitely generated over a field $R_0 = K$, then $\operatorname{Proj} R$ is said to be *geometrically connected* if $\operatorname{Proj}(R \otimes_K K^{\mathrm{sep}})$ is connected.

Definition 2.2. Let $\mathfrak{a}$ be an ideal of a ring R. A ring homomorphism $\varphi\colon R \to S$ induces a map of local cohomology modules $H^i_{\mathfrak{a}}(R) \xrightarrow{\varphi} H^i_{\mathfrak{a}S}(S)$. In particular, if R contains a field of characteristic $p > 0$, then the Frobenius homomorphism $F\colon R \to R$ induces an additive map

$$H^i_{\mathfrak{a}}(R) \xrightarrow{F} H^i_{\mathfrak{a}^{[p]}}(R) = H^i_{\mathfrak{a}}(R),$$

called the *Frobenius action* on $H^i_{\mathfrak{a}}(R)$. An element $\eta \in H^i_{\mathfrak{a}}(R)$ is *F-torsion* if $F^e(\eta) = 0$ for some $e \in \mathbb{N}$. The module $H^i_{\mathfrak{a}}(R)$ is *F-torsion* if each element is F-torsion. The image of F^e need not be an R-module, but it is a K-vector space when K is perfect. In this case the *F-stable* part of $H^i_{\mathfrak{a}}(R)$ is the vector space

$$H^i_{\mathfrak{a}}(R)_{\mathrm{st}} = \bigcap_{e \in \mathbb{N}} F^e(H^i_{\mathfrak{a}}(R)).$$

Some results about F-torsion modules and F-stable subspaces are summarized in Section 6. For a very general theory of F-modules, we refer the reader to [Ly1].

Remark 2.3. Consider a local ring $(R, \mathfrak{m})$ of positive dimension. The punctured spectrum of R is disconnected if and only if the minimal primes of R can be partitioned into two sets $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ such that $\operatorname{rad}(\mathfrak{p}_i + \mathfrak{q}_j) = \mathfrak{m}$ for all pairs $\mathfrak{p}_i, \mathfrak{q}_j$. Consider the graph Γ whose vertices are the minimal primes of R, and there is an edge between minimal primes $\mathfrak{p}$ and $\mathfrak{p}'$ if and only if $\operatorname{rad}(\mathfrak{p} + \mathfrak{p}') \neq \mathfrak{m}$. It follows that the punctured spectrum of R is connected if and only if the graph Γ is connected. If the graph Γ is connected, take a spanning tree, i.e., a connected acyclic subgraph, containing all the vertices of Γ. This spanning tree must contain a vertex $\mathfrak{p}_i$ with only one edge, so $\Gamma \smallsetminus \{\mathfrak{p}_i\}$ is connected as well.

Let $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ be incomparable prime ideals of a local domain A. Then their images $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are precisely the minimal primes of the ring $R = A/(\mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_n)$. From the above discussion, we conclude that if the punctured spectrum of R is connected, then there exists i such that the punctured spectrum of the ring

$$A/(\mathfrak{P}_1 \cap \cdots \cap \hat{\mathfrak{P}}_i \cap \cdots \cap \mathfrak{P}_n)$$

is connected as well.

Theorems 1.1 and 1.2 assert that connectedness issues for $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$ are determined by the Frobenius action on $H^1_{\mathfrak{m}}(R)$. We next record an observation about the length of $H^1_{\mathfrak{m}}(R)$.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a local ring which is a homomorphic image of a Gorenstein domain. Then $H^1_{\mathfrak{m}}(R)$ has finite length if and only if $\operatorname{ann}_R \mathfrak{p} = 0$ for every prime ideal $\mathfrak{p}$ of R with $\dim R/\mathfrak{p} = 1$.*

Proof. If $\dim R = 0$, then $H^1_{\mathfrak{m}}(R) = 0$, and R has no primes with $\dim R/\mathfrak{p} = 1$. If $\dim R = 1$, then $H^1_{\mathfrak{m}}(R)$ has infinite length and $\dim R/\mathfrak{p} = 1$ for some minimal prime $\mathfrak{p}$ of R. For the rest of the proof we hence assume that $\dim R \geq 2$.

Let $R = A/\mathfrak{Q}$ where A is a Gorenstein domain. Localizing A at the inverse image of $\mathfrak{m}$, we may assume that $(A, \mathfrak{M})$ is a local ring. Using local duality over A, the module $H^1_{\mathfrak{m}}(R) = H^1_{\mathfrak{M}}(A/\mathfrak{Q})$ has finite length if and only if $\operatorname{Ext}_A^{\dim A-1}(A/\mathfrak{Q}, A)$ has finite length as an A-module. Since $\operatorname{Ext}_A^{\dim A-1}(A/\mathfrak{Q}, A)$ is finitely generated, this is equivalent to the vanishing of

$$\operatorname{Ext}_A^{\dim A-1}(A/\mathfrak{Q}, A)_{\mathfrak{P}} = \operatorname{Ext}_{A_{\mathfrak{P}}}^{\dim A-1}(A_{\mathfrak{P}}/\mathfrak{Q}A_{\mathfrak{P}}, A_{\mathfrak{P}})$$

for all $\mathfrak{P} \in \operatorname{Spec} A \smallsetminus \{\mathfrak{M}\}$. Using local duality over the Gorenstein local ring $(A_{\mathfrak{P}}, \mathfrak{P}A_{\mathfrak{P}})$, this is equivalent to the vanishing of

$$H^{\dim A_{\mathfrak{P}}-\dim A+1}_{\mathfrak{P}A_{\mathfrak{P}}}(A_{\mathfrak{P}}/\mathfrak{Q}A_{\mathfrak{P}}) = H^{\dim A_{\mathfrak{P}}-\dim A+1}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$$

for all $\mathfrak{P} \in \operatorname{Spec} A \smallsetminus \{\mathfrak{M}\}$. This local cohomology module vanishes for $\mathfrak{P} \notin V(\mathfrak{Q})$. Since $\dim A_{\mathfrak{P}} - \dim A + 1 \leq 0$ for $\mathfrak{P} \in \operatorname{Spec} A \smallsetminus \{\mathfrak{M}\}$, we need only consider primes $\mathfrak{P} \in V(\mathfrak{Q})$ with $\dim A_{\mathfrak{P}} = \dim A - 1$. Since A is a catenary local domain, $\dim A_{\mathfrak{P}}$ equals $\dim A - 1$ precisely when $\dim A/\mathfrak{P} = 1$, equivalently $\dim R/\mathfrak{p} = 1$. Hence $H^1_{\mathfrak{m}}(R)$ has finite length if and only if $H^0_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = H^0_{\mathfrak{p}}(R)$ vanishes for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = 1$, i.e., if and only if $\operatorname{ann}_R \mathfrak{p} = 0$ for all $\mathfrak{p}$ with $\dim R/\mathfrak{p} = 1$. □

3. Main results

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a strictly Henselian local domain containing a field of positive characteristic. If R is a homomorphic image of a Gorenstein domain and $\dim R \geq 2$, then $H^1_{\mathfrak{m}}(R)$ is F-torsion.*

Proof. Suppose there exists $\eta \in H^1_{\mathfrak{m}}(R)$ which is not F-torsion. Since R is a domain, Proposition 2.4 implies that $H^1_{\mathfrak{m}}(R)$ has finite length. Hence for all integers $e \gg 0$, the element $F^e(\eta)$ belongs to the R-module spanned by $\eta, F(\eta), F^2(\eta), \dots, F^{e-1}(\eta)$. Amongst all equations of the form

$$F^{e+k}(\eta) + r_1 F^{e+k-1}(\eta) + \cdots + r_e F^k(\eta) = 0 \qquad (3.1.1)$$

with $r_i \in R$ for all i, choose one where the number of nonzero coefficients r_i that occur is minimal. We claim that r_e must be a unit. Note that $H^1_{\mathfrak{m}}(R)$ is killed by $\mathfrak{m}^{q'}$ for some $q' = p^{e'}$. If $r_e \in \mathfrak{m}$, then applying $F^{e'}$ to equation (3.1.1), we get

$$F^{e'+e+k}(\eta) + r_1^{q'} F^{e'+e+k-1}(\eta) + \cdots + r_e^{q'} F^{e'+k}(\eta) = 0.$$

But $r_e^{q'} F^{e'+k}(\eta) \in \mathfrak{m}^{q'} H^1_{\mathfrak{m}}(R) = 0$, so this is an equation with fewer nonzero coefficients, contradicting the minimality assumption. This shows that $r_e \in R$ is a unit. Since η is not F-torsion, neither is $F^k(\eta)$, so after replacing η if necessary, we have an equation of the form

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0 \tag{3.1.2}$$

where r_e is a unit and $\eta \in H^1_{\mathfrak{m}}(R)$ is not F-torsion. Let $\eta = [(y_1/x_1, \dots, y_d/x_d)]$ where $H^1_{\mathfrak{m}}(R)$ is regarded as the cohomology of a Čech complex on a system of parameters $x_1, \dots, x_d$ for R. Then (3.1.2) implies that there exists $r_{e+1} \in R$ such that each $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in R[T].$$

Now $f'(T) = r_e$ is a unit, so $\overline{f(T)} \in R/\mathfrak{m}[T]$ is a separable polynomial. Since R is strictly Henselian, the polynomial $f(T)$ splits in $R[T]$, and hence any root of $f(T)$ in the fraction field of R is an element of R. In particular, $y_1/x_1 = \cdots = y_d/x_d$ is an element of R, and so $\eta = 0$. □

We next prove the connectedness criterion, Theorem 1.1. By Proposition 6.1, the module $H^1_{\mathfrak{m}}(R)$ is F-torsion if and only if there exists e such that $F^e(H^1_{\mathfrak{m}}(R)) = 0$. In view of this, the following theorem is equivalent to Theorem 1.1.

Theorem 3.2. *Let* $(R, \mathfrak{m})$ *be a local ring with* $\dim R > 0$*, which contains a field of positive characteristic. Then* $H^1_{\mathfrak{m}}(R)$ *is* F*-torsion if and only if* $\dim R \geq 2$ *and the punctured spectrum of* R *is formally geometrically connected.*

Proof. Quite generally, for a local ring $(R, \mathfrak{m})$ we have $H^i_{\mathfrak{m}}(\hat{R}) = H^i_{\mathfrak{m}}(R)$. Moreover, $S = \hat{\hat{R}}^{\mathrm{sh}}$ is a faithfully flat extension of R, and $H^i_{\mathfrak{m}}(R) \otimes_R S \cong H^i_{\mathfrak{m}S}(S)$ is F-torsion if and only if $H^i_{\mathfrak{m}}(R)$ is F-torsion. Hence we may assume that R is a complete local ring with a separably closed coefficient field.

Suppose that $H^1_{\mathfrak{m}}(R)$ is F-torsion. The local cohomology module $H^{\dim R}_{\mathfrak{m}}(R)$ is not F-torsion by Proposition 6.2, so $\dim R \geq 2$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{a} + \mathfrak{b}$ is $\mathfrak{m}$-primary and $\mathfrak{a} \cap \mathfrak{b} = 0$. Let

$$x_1 = y_1 + z_1, \quad \dots, \quad x_d = y_d + z_d$$

be a system of parameters for R where $y_i \in \mathfrak{a}$ and $z_i \in \mathfrak{b}$. Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} = 0$, we have $y_i z_j = 0$ for all i, j, and hence

$$y_i(y_j + z_j) = y_j(y_i + z_i)\,.$$

These relations give an element of $H^1_{\mathfrak{m}}(R)$ regarded as the cohomology of a Čech complex on $x_1, \dots, x_d$, namely

$$\eta = \left[\left(\frac{y_1}{x_1}, \dots, \frac{y_d}{x_d}\right)\right] \in H^1_{\mathfrak{m}}(R).$$

The hypotheses imply that $F^e(\eta) = 0$ for some e, so there exists $q = p^e$ and $r \in R$ such that $(y_i/x_i)^q = r$ in R_{x_i} for all $1 \leq i \leq d$. Hence there exists $t \in \mathbb{N}$ such that $x_i^t y_i^q = r x_i^{q+t}$, i.e.,

$$(y_i + z_i)^t y_i^q = r(y_i + z_i)^{q+t}\,.$$

But $y_i z_i = 0$, so these equations simplify to give $(1-r)y_i^{q+t} = r z_i^{q+t}$. Since R is a local ring, either r or $1 - r$ must be a unit. If r is a unit, then $z_i^{q+t} \in \mathfrak{a}$ for all i, and so $\mathfrak{a}$ is $\mathfrak{m}$-primary. Similarly if $1-r$ is a unit, then $\mathfrak{b}$ is $\mathfrak{m}$-primary. This proves that the punctured spectrum of R is connected.

For the converse, assume that $\dim R \geq 2$ and that the punctured spectrum of R is connected. Let $\mathfrak{n}$ denote the nilradical of R. Note that $\operatorname{Spec} R$ is homeomorphic to $\operatorname{Spec} R/\mathfrak{n}$. Moreover, $\mathfrak{n}$ supports a Frobenius action and is F-torsion. The long exact sequence of local cohomology relating $H^1_{\mathfrak{m}}(R)$ and $H^1_{\mathfrak{m}}(R/\mathfrak{n})$ implies that if $H^1_{\mathfrak{m}}(R/\mathfrak{n})$ is F-torsion, then so is $H^1_{\mathfrak{m}}(R)$, and hence there is no loss of generality in assuming that R is reduced. Let $R = A/(\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n)$ where $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ are incomparable prime ideals of a power series ring $A = K[[x_1, \ldots, x_m]]$ over a separably closed field K. We use induction on n to prove that $H^1_{\mathfrak{m}}(R)$ is F-torsion; the case $n = 1$ follows from Theorem 3.1, so we assume $n > 1$ below.

If $\dim R/\mathfrak{p}_i = 1$ for some i, then $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$ is the disjoint union of $V(\mathfrak{p}_i) \smallsetminus \{\mathfrak{m}\}$ and $V(\mathfrak{p}_1 \cap \cdots \cap \hat{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_n) \smallsetminus \{\mathfrak{m}\}$, contradicting the connectedness assumption. Hence $\dim R/\mathfrak{p}_i \geq 2$ for all i. By Remark 2.3, after relabeling the minimal primes if necessary, we may assume that the punctured spectrum of $A/\mathfrak{Q}$ is connected where $\mathfrak{Q} = \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n$. The short exact sequence

$$0 \to A/(\mathfrak{P}_1 \cap \mathfrak{Q}) \to A/\mathfrak{P}_1 \oplus A/\mathfrak{Q} \to A/(\mathfrak{P}_1 + \mathfrak{Q}) \to 0$$

induces a long exact sequence of local cohomology modules containing the piece

$$(3.2.1) \qquad H^0_{\mathfrak{M}}(A/(\mathfrak{P}_1 + \mathfrak{Q})) \to H^1_{\mathfrak{M}}(A/(\mathfrak{P}_1 \cap \mathfrak{Q})) \to H^1_{\mathfrak{M}}(A/\mathfrak{P}_1) \oplus H^1_{\mathfrak{M}}(A/\mathfrak{Q})\,.$$

Since $\operatorname{rad}(\mathfrak{P}_1 + \mathfrak{P}_i) \neq \mathfrak{M}$ for some $i > 1$, it follows that $\dim A/(\mathfrak{P}_1 + \mathfrak{Q}) \geq 1$. Proposition 6.2 now implies that $H^0_{\mathfrak{M}}(A/(\mathfrak{P}_1 + \mathfrak{Q}))$ is F-torsion. By the inductive hypothesis, $H^1_{\mathfrak{M}}(A/\mathfrak{P}_1)$ and $H^1_{\mathfrak{M}}(A/\mathfrak{Q})$ are F-torsion as well. The exact sequence (3.2.1) implies that $H^1_{\mathfrak{M}}(A/(\mathfrak{P}_1 \cap \mathfrak{Q})) = H^1_{\mathfrak{m}}(R)$ is F-torsion. □

The following lemma will be used in the proof of Theorem 1.2.

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a complete local domain with an algebraically closed coefficient field of positive characteristic. Then $H^1_{\mathfrak{m}}(R)_{\mathrm{st}}$, the F-stable part of the module $H^1_{\mathfrak{m}}(R)$, is zero.*

Proof. If $\dim R = 0$, then $H^1_{\mathfrak{m}}(R) = 0$, and if $\dim R \geq 2$, then the assertion follows from Theorem 3.1. The remaining case is $\dim R = 1$. Theorem 6.3 implies that $H^1_{\mathfrak{m}}(R)_{\mathrm{st}}$ has a vector space basis $\eta_1, \ldots, \eta_r$ such that $F(\eta_i) = \eta_i$.

Let $\eta \in H^1_{\mathfrak{m}}(R)_{\mathrm{st}}$ be an element with $F(\eta) = \eta$. Considering $H^1_{\mathfrak{m}}(R)$ as the cohomology of a suitable Čech complex, let η be the class of y/x in $R_x/R = H^1_{\mathfrak{m}}(R)$, where $y \in R$ and $x \in \mathfrak{m}$. Since $F(\eta) = \eta$, there exists $r \in R$ such that

$$\left(\frac{y}{x}\right)^p - \frac{y}{x} - r = 0\,,$$

and so $y/x \in R_x$ is a root of the polynomial $f(T) = T^p - T - r \in R[T]$. The polynomial $\overline{f(T)} \in K[T]$ is separable and R is strictly Henselian, so $f(T)$ splits in $R[T]$. Since y/x is a root of $f(T)$ in the fraction field of R, it must then be an element of R, and hence $\eta = 0$. □

Proof of Theorem 1.2. We may assume R to be reduced by Proposition 6.5. First consider the case where the punctured spectrum of R is connected. If $\dim R \geq 2$, then $H^1_{\mathfrak{m}}(R)$ is F-torsion by Theorem 3.2, so $H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = 0$. If $\dim R = 1$, then R is a domain, and Lemma 3.3 implies that $H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = 0$.

We continue by induction on the number of connected components of the punctured spectrum of R. If the punctured spectrum of R is disconnected, then $R =$

$A/(\mathfrak{A} \cap \mathfrak{B})$ where $(A, \mathfrak{M})$ is a power series ring over the field K, and $\mathfrak{A}$ and $\mathfrak{B}$ are radical ideals of A which are not $\mathfrak{M}$-primary, but $\mathfrak{A} + \mathfrak{B}$ is $\mathfrak{M}$-primary. There is a short exact sequence

$$0 \to A/(\mathfrak{A} \cap \mathfrak{B}) \to A/\mathfrak{A} \oplus A/\mathfrak{B} \to A/(\mathfrak{A} + \mathfrak{B}) \to 0\,.$$

Since $H^0_{\mathfrak{M}}(A/\mathfrak{A}) = H^0_{\mathfrak{M}}(A/\mathfrak{B}) = H^1_{\mathfrak{M}}(A/(\mathfrak{A}+\mathfrak{B})) = 0$, the resulting exact sequence of local cohomology gives us

$$0 \to H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B})) \to H^1_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B})) \to H^1_{\mathfrak{M}}(A/\mathfrak{A}) \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B}) \to 0\,.$$

By Theorem 6.4, we have a K-vector space isomorphism

$$H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = H^1_{\mathfrak{M}}(A/(\mathfrak{A} \cap \mathfrak{B}))_{\mathrm{st}} \cong H^0_{\mathfrak{M}}(A/(\mathfrak{A} + \mathfrak{B}))_{\mathrm{st}} \oplus H^1_{\mathfrak{M}}(A/\mathfrak{A})_{\mathrm{st}} \oplus H^1_{\mathfrak{M}}(A/\mathfrak{B})_{\mathrm{st}}\,.$$

Since $H^0_{\mathfrak{M}}(A/(\mathfrak{A}+\mathfrak{B}))_{\mathrm{st}} = K$ by Proposition 6.2, the inductive hypothesis completes the proof. □

We next record the graded versions of the results proved in this section:

Theorem 3.4. *Let R be an $\mathbb{N}$-graded ring of positive dimension, which is finitely generated over a field $R_0 = K$ of characteristic $p > 0$.*

(1) *If R is a domain with $\dim R \geq 2$, and K is separably closed, then $H^1_{\mathfrak{m}}(R)$ is F-torsion.*
(2) *The module $H^1_{\mathfrak{m}}(R)$ is F-torsion if and only if $\dim R \geq 2$ and $\operatorname{Proj} R$ is geometrically connected.*
(3) *Let K be a perfect field, and let $\overline{K}$ denote its algebraic closure. Then the number of connected components of $\operatorname{Proj}(R \otimes_K \overline{K})$ is*

$$1 + \dim_K H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = 1 + \dim_K([H^1_{\mathfrak{m}}(R)]_0)_{\mathrm{st}}\,.$$

Proof. (1) Note that $H^1_{\mathfrak{m}}(R)$ is a $\mathbb{Z}$-graded R-module, and that

$$F\colon [H^1_{\mathfrak{m}}(R)]_n \to [H^1_{\mathfrak{m}}(R)]_{np} \qquad \text{for all } n \in \mathbb{Z}\,.$$

The module $H^1_{\mathfrak{m}}(R)$ has finite length, so all elements of $H^1_{\mathfrak{m}}(R)$ of positive or negative degree are F-torsion; it remains to show that elements $\eta \in [H^1_{\mathfrak{m}}(R)]_0$ are F-torsion as well. Let η be a element of $[H^1_{\mathfrak{m}}(R)]_0$ which is not F-torsion. As in the proof of Theorem 3.1, after a change of notation we may assume that

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta = 0$$

where all r_i are in $[R]_0 = K$, and r_e is nonzero. Let $\eta = [(y_1/x_1, \ldots, y_d/x_d)]$ where $H^1_{\mathfrak{m}}(R)$ is regarded as the cohomology of a homogeneous Čech complex. Then there exists $r_{e+1} \in K$ such that $y_i/x_i \in R_{x_i}$ is a root of the polynomial

$$f(T) = T^{p^e} + r_1 T^{p^{e-1}} + \cdots + r_e T + r_{e+1} \in K[T]\,.$$

But $f(T)$ is a separable polynomial, so it splits in $K[T]$. The element $y_i/x_i = y_j/x_j$ is a root of $f(T)$ in the fraction field of R, so it must be one of the roots of $f(T)$ in K. It follows that $\eta = 0$, which completes the proof of (1).

The proof of (2) is now similar to that of Theorem 3.2 and is left to the reader. For (3), note that $F^e(H^1_{\mathfrak{m}}(R))$ is a K-vector space since K is perfect, and that

$$\dim_K H^1_{\mathfrak{m}}(R)_{\mathrm{st}} = \dim_{\overline{K}} H^1_{\mathfrak{m}}(R \otimes_K \overline{K})_{\mathrm{st}}\,.$$

Thus we may assume $K = \overline{K}$, and the proof is similar to that of Theorem 1.2. □

Remark 3.5. Theorem 3.4(3) generalizes, in the case of positive characteristic, the well-known fact that the number of connected components of $X = \operatorname{Proj} R$ is

$$\dim_K H^0(X, \mathcal{O}_X) = 1 + \dim_K [H^1_{\mathfrak{m}}(R)]_0\,,$$

where R is an $\mathbb{N}$-graded *reduced* ring of positive dimension, which is finitely generated over an algebraically closed field $R_0 = K$. The point is that in this case the Frobenius is bijective on $[H^1_{\mathfrak{m}}(R)]_0$. To see this, let

$$\eta = \left[\left(\frac{y_1}{x_1}, \ldots, \frac{y_d}{x_d}\right)\right] \in [H^1_{\mathfrak{m}}(R)]_0$$

be an element with $F(\eta) = 0$, where $H^1_{\mathfrak{m}}(R)$ is computed as the cohomology of a suitable Čech complex. Then there exists a homogeneous element $r \in R$ with $(y_i/x_i)^p = r$ in R_{x_i} for all $1 \le i \le d$. Such an element r must have degree zero, and hence must be an element of K. But then $r^{1/p} \in K$, and, since R is reduced, $y_i/x_i = r^{1/p}$ for all i. It follows that

$$\eta = [(r^{1/p}, \ldots, r^{1/p})] = 0.$$

To complete the argument, note that $[H^1_{\mathfrak{m}}(R)]_0$ is a finite dimensional K-vector space, and that if $\eta_1, \ldots, \eta_n \in [H^1_{\mathfrak{m}}(R)]_0$ are linearly independent, then so are $F(\eta_1), \ldots, F(\eta_n)$. It follows that $F\colon [H^1_{\mathfrak{m}}(R)]_0 \to [H^1_{\mathfrak{m}}(R)]_0$ is surjective.

4. F-purity

A ring homomorphism $\varphi\colon R \to S$ is *pure* if $\varphi \otimes 1\colon R \otimes_R M \to S \otimes_R M$ is injective for every R-module M. If R is a ring containing a field of characteristic $p > 0$, then R is *F-pure* if the Frobenius homomorphism $F\colon R \to R$ is pure. The notion was introduced by Hochster and Roberts in the course of their study of rings of invariants in [HR1, HR2].

Examples of F-pure rings include regular rings of positive characteristic and their pure subrings. If $\mathfrak{a}$ is generated by square-free monomials in the variables $x_1, \ldots, x_n$ and K is a field of positive characteristic, then $K[x_1, \ldots, x_n]/\mathfrak{a}$ is F-pure.

Goto and Watanabe [GW] classified one-dimensional F-pure rings: let $(R, \mathfrak{m})$ be a local ring of positive characteristic such that $R/\mathfrak{m} = K$ is algebraically closed, $F\colon R \to R$ is finite, and $\dim R = 1$. Then R is F-pure if and only if

$$\hat{R} \cong K[[x_1, \ldots, x_n]]/(x_i x_j \mid i < j)\,.$$

Two-dimensional F-pure rings have attracted a lot of attention: Watanabe [Wat1] proved that F-pure normal Gorenstein local rings of dimension two are either rational double points, simple elliptic singularities, or cusp singularities. He also classified two-dimensional normal $\mathbb{N}$-graded rings R over an algebraically closed field R_0, in terms of $\mathbb{Q}$-divisors on the curve $\operatorname{Proj} R$, [Wat2]. In [MS] Mehta and Srinivas obtained a classification of two-dimensional F-pure normal singularities in terms of the resolution of the singularity. Hara completed the classification of two-dimensional normal F-pure singularities in terms of the dual graph of the minimal resolution of the singularity, [HaN].

The results of Section 3 imply that over separably closed fields, F-pure domains of dimension two are Cohen-Macaulay. The point is that if R is an F-pure ring, then the Frobenius action $F\colon H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is an injective map.

Corollary 4.1. *Let R be a local ring with* $\dim R \geq 2$*, which contains a field of positive characteristic. If R is F-pure and the punctured spectrum of R is formally geometrically connected, then* $\operatorname{depth} R \geq 2$*.*

In particular, if R is a complete local F-pure domain of dimension two, with a separably closed coefficient field, then R is Cohen-Macaulay.

Proof. An F-pure ring is reduced, so $H^0_{\mathfrak{m}}(R) = 0$. By Theorem 3.1, $H^1_{\mathfrak{m}}(R)$ is F-torsion. Since R is F-pure, it follows that $H^1_{\mathfrak{m}}(R) = 0$. □

In the graded case, we similarly have:

Corollary 4.2. *Let R be an $\mathbb{N}$-graded ring with* $\dim R \geq 2$*, which is finitely generated over a field R_0 of positive characteristic. If R is F-pure and* $\operatorname{Proj} R$ *is geometrically connected, then* $\operatorname{depth} R \geq 2$*.*

The ring R below is a graded F-pure domain of dimension two, and depth one. The issue is that $\operatorname{Proj} R$ is connected though not geometrically connected.

Example 4.3. Let K be a field of characteristic $p > 2$, and $a \in K$ an element such that $\sqrt{a} \notin K$. Let $R = K[x, y, x\sqrt{a}, y\sqrt{a}]$. The domain R has a presentation

$$R = K[x, y, u, v]/(u^2 - ax^2, v^2 - ay^2, uv - axy, vx - uy)\,,$$

and if K^{sep} denotes the separable closure of K, then

$$R \otimes_K K^{\mathrm{sep}} \cong K^{\mathrm{sep}}[x, y, u, v]/(u - x\sqrt{a}, v - y\sqrt{a})(u + x\sqrt{a}, v + y\sqrt{a})\,.$$

Using a change of variables, $R \otimes_K K^{\mathrm{sep}} \cong K^{\mathrm{sep}}[x', y', u', v']/(x', y')(u', v')$. Since $(x', y')(u', v')$ is a square-free monomial ideal, $R \otimes_K K^{\mathrm{sep}}$ is F-pure and it follows that R is F-pure. However, R is not Cohen-Macaulay since x, y is a homogeneous system of parameters with a non-trivial relation

$$(x\sqrt{a})y = (y\sqrt{a})x\,.$$

Using the Čech complex on x, y to compute $H^1_{\mathfrak{m}}(R)$, we see that it is a 1-dimensional K-vector space generated by the element

$$\eta = \left[\left(\frac{x\sqrt{a}}{x}, \frac{y\sqrt{a}}{y}\right)\right] \in H^1_{\mathfrak{m}}(R)$$

corresponding to the relation above. Given $e \in \mathbb{N}$, let $p^e = 2k + 1$. Then

$$F^e(\eta) = a^k \eta\,,$$

which is a nonzero element of $H^1_{\mathfrak{m}}(R)$. Consequently $H^1_{\mathfrak{m}}(R)$ is not F-torsion, corresponding to the fact that $\operatorname{Proj} R$ is not geometrically connected.

The corollaries obtained in this section imply that over a separably closed field, a graded or complete local F-pure domain of dimension two is Cohen-Macaulay. We record an example which shows that this is not true for rings of higher dimension.

Example 4.4. Let K be a field of characteristic $p > 0$, and take

$$A = K[x_1, \ldots, x_d]/(x_1^d + \cdots + x_d^d)$$

where $d \geq 3$. Let R be the Segre product of A and the polynomial ring $B = K[s, t]$. Then $\dim R = d$, and the Künneth formula for local cohomology implies that

$$H^{d-1}_{\mathfrak{m}_R}(R) \;\cong\; [H^{d-1}_{\mathfrak{m}_A}(A)]_0 \otimes_K [B]_0 \;\cong\; K,$$

so R is not Cohen-Macaulay. If $p \equiv 1 \mod d$, then A is F-pure by [HR2, Proposition 5.21]; hence $A \otimes_K B$ and its direct summand R are F-pure as well.

5. Algorithmic aspects

Let R be an $\mathbb{N}$-graded ring, which is finitely generated over a finite field $R_0 = K$. We wish to determine the number of geometrically connected components of the scheme $\operatorname{Proj} R$, i.e., the number of connected components of $\operatorname{Proj}(R \otimes_K \overline{K})$, or, equivalently, of $\operatorname{Proj}(R \otimes_K K^{\mathrm{sep}})$. While primary decomposition algorithms such as those of [EHV], [GTZ], or [SY], may be used to determine the connected components of $\operatorname{Proj} R$, there is computationally no hope of "determining" the connected components over the algebraic closure, $\overline{K}$. However, simply finding their number is much easier: by Theorem 3.4, this is $1 + \dim_K([H^1_{\mathfrak{m}}(R)]_0)_{\mathrm{st}}$. Computing this number involves three steps.

(1) Finding a good presentation of $[H^1_{\mathfrak{m}}(R)]_0$;
(2) Determining the Frobenius action on $[H^1_{\mathfrak{m}}(R)]_0$ in terms of this presentation;
(3) Computing the dimension of the F-stable part, $([H^1_{\mathfrak{m}}(R)]_0)_{\mathrm{st}}$.

If $R = A/\mathfrak{A}$ for a polynomial ring A, we first replace $\mathfrak{A}$ by an ideal that has the same radical as $\mathfrak{A}$, but does not have the homogeneous maximal ideal $\mathfrak{M}$ as an associated prime. This can be done by saturating $\mathfrak{A}$ with respect to $\mathfrak{M}$; if desired, one may simply compute the radical of $\mathfrak{A}$, but this is often computationally expensive. Now, since $\mathfrak{M}$ is not associated to $\mathfrak{A}$, one can find a homogeneous system of parameters $x_1, \dots, x_d$ for R such that each x_i is a nonzerodivisor on R.

The length ℓ of $[H^1_{\mathfrak{m}}(R)]_0$ may be computed by computing the length of its graded dual $[\operatorname{Ext}^{n-1}_A(R, A(-n))]_0$, where $\dim A = n$. Of course, if this length is zero, then $X_{\overline{K}}$ is connected. Consider the Koszul cohomology modules

$$H^1(x_1^t, \dots, x_d^t; R) = \frac{\left\{(a_1, \dots, a_d) \in R^d \mid a_i x_j^t = a_j x_i^t \text{ for all } i < j\right\}}{\left\{(r x_1^t, \dots, r x_d^t) \mid r \in R\right\}}.$$

These modules have an $\mathbb{N}$-grading, where for homogeneous elements $a_i \in R$, we define the degree of $[(a_1, \dots, a_d)] \in H^1(x_1^t, \dots, x_d^t; R)$ as

$$\deg[(a_1, \dots, a_d)] = \deg a_i - \deg x_i^t\,,$$

which is independent of i. This ensures that for each t, the map

$$\begin{gathered} H^1(x_1^t, \dots, x_d^t; R) \to H^1(x_1^{t+1}, \dots, x_d^{t+1}; R) \\ [(a_1, \dots, a_d)] \longmapsto [(a_1 x_1, \dots, a_d x_t)] \end{gathered}$$

preserves degrees. The module $H^1_{\mathfrak{m}}(R)$ is the direct limit of these Koszul cohomology modules, and the assumption that the x_i are nonzerodivisors ensures that the maps in the direct limit system are injective. The modules $H^1(x_1^t, \dots, x_d^t; R)$ may be computed for increasing values of t, until we arrive at an integer N such that

$$\ell\left([H^1(x_1^N, \dots, x_d^N; R)]_0\right) = \ell\,.$$

This gives us a presentation for $[H^1_{\mathfrak{m}}(R)]_0 = [H^1(x_1^N, \dots, x_d^N; R)]_0$, in terms of which we now analyze the Frobenius map. Replacing the x_i by their powers if needed, assume that $N = 1$. Let

$$\alpha = [(a_1, \dots, a_d)] \in [H^1(x_1, \dots, x_d; R)]_0\,,$$

in which case, $F(\alpha) = [(a_1^p, \dots, a_d^p)] \in [H^1(x_1^p, \dots, x_d^p; R)]_0$. Since the map

$$[H^1(x_1, \dots, x_d; R)]_0 \to [H^1(x_1^p, \dots, x_d^p; R)]_0$$

coming from the direct limit system is bijective, it follows that $a_i^p \in x_i^{p-1}R$ for each $1 \le i \le d$. Setting $b_i = a_i^p / x_i^{p-1}$, we arrive at

$$F(\alpha) = [(b_1, \dots, b_d)] \in [H^1(x_1, \dots, x_d; R)]_0 .$$

Using this description of Frobenius action on the finite dimensional K-vector space $[H^1_{\mathfrak{m}}(R)]_0 = [H^1(x_1, \dots, x_d; R)]_0$, it is now straightforward to compute the ranks of the vector spaces

$$[H^1_{\mathfrak{m}}(R)]_0 \supseteq F([H^1_{\mathfrak{m}}(R)]_0) \supseteq F^2([H^1_{\mathfrak{m}}(R)]_0) \supseteq \dots ,$$

and hence of the F-stable part, $([H^1_{\mathfrak{m}}(R)]_0)_{\mathrm{st}}$.

6. Appendix: F-torsion modules and F-stable vector spaces

Let R be a commutative ring containing a field K of characteristic $p > 0$. A *Frobenius action* on an R-module M is an additive map $F\colon M \to M$ such that $F(rm) = r^p F(m)$ for all $r \in R$ and $m \in M$. In this case, $\ker F$ is a submodule of M, and we have an ascending sequence of submodules of M,

$$\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \dots .$$

The union of these is the *F-nilpotent* submodule of M, denoted $M_{\mathrm{nil}} = \bigcup_{e\in\mathbb{N}} \ker F^e$. We say M is *F-torsion* if $M_{\mathrm{nil}} = M$.

Proposition 6.1. *Let $(R, \mathfrak{m})$ be a local ring containing a field of positive characteristic, and let M be an Artinian R-module with a Frobenius action. Then there exists $e \in \mathbb{N}$ such that $F^e(M_{\mathrm{nil}}) = 0$.*

Hence an Artinian module M is F-torsion if and only if $F^e(M) = 0$ for some e.

Proof. This is proved in [HS, Proposition 1.11] under the hypothesis that R is a complete local ring with a perfect coefficient field. The general case may be concluded from this, but a more elegant approach is via Lyubeznik's theory of F-modules; see [Ly1, Proposition 4.4]. □

If R is a ring containing a perfect field K of positive characteristic and M is an R-module with a Frobenius action, then $F(M)$ is a K-vector space, and we have a descending sequence of K-vector spaces

$$F(M) \supseteq F^2(M) \supseteq F^3(M) \supseteq \dots .$$

The *F-stable* part of M is the vector space $M_{\mathrm{st}} = \bigcap_{e\in\mathbb{N}} F^e(M)$.

Proposition 6.2. *Let $(R, \mathfrak{m}, K)$ be a local ring of dimension d which contains a field of positive characteristic.*

1. *$H^0_{\mathfrak{m}}(R)$ is F-torsion if and only if $d > 0$.*
2. *$H^d_{\mathfrak{m}}(R)$ is not F-torsion.*
3. *If $d = 0$ and K is perfect, then $H^0_{\mathfrak{m}}(R)_{\mathrm{st}} = R_{\mathrm{st}} = K$.*

Proof. (1) If $d = 0$, then $H^0_{\mathfrak{m}}(R) = R$, which is not F-torsion. If $d > 0$, then $H^0_{\mathfrak{m}}(R)$ is contained in $\mathfrak{m}$. Since every element of $H^0_{\mathfrak{m}}(R)$ is killed by a power of $\mathfrak{m}$, it follows that each element is nilpotent. (See also [Ly2, Corollary 4.6(a)].)

(2) View $H^d_{\mathfrak{m}}(R)$ as the cohomology of a Čech complex on a system of parameters $x_1, \dots, x_d$ for R, and let $\eta = [1 + (x_1, \dots, x_d)] \in H^d_{\mathfrak{m}}(R)$. For all $e_0 \in \mathbb{N}$, the collection of elements $F^e(\eta)$ with $e > e_0$ generates $H^d_{\mathfrak{m}}(R)$ as an R-module. Hence $F^{e_0}(\eta)$ cannot be zero by Grothendieck's nonvanishing theorem.

(3) Since $\mathfrak{m}$ is nilpotent in this case, for integers $e \gg 0$ we have

$$F^e(H^0_{\mathfrak{m}}(R)) = F^e(R) = \{x^{p^e} \mid x \in R\} = \{(y+z)^{p^e} \mid y \in K, z \in \mathfrak{m}\} = K\,. \qquad \square$$

Theorem 6.3. *Let $(R, \mathfrak{m})$ be a local ring with a perfect coefficient field K of positive characteristic. Let M be an Artinian R-module with a Frobenius action. Then M_{st} is a finite dimensional K-vector space, and $F\colon M_{\mathrm{st}} \to M_{\mathrm{st}}$ is an automorphism of the Abelian group M_{st}.*

If K is algebraically closed, then there exists a K-basis $e_1, \dots, e_n$ for M_{st} such that $F(e_i) = e_i$ for all $1 \le i \le n$.

Proof. For the finiteness assertion, see [HS, Theorem 1.12] or [Ly1, Proposition 4.9]. It is easily seen that $F\colon M_{\mathrm{st}} \to M_{\mathrm{st}}$ is an automorphism whenever M_{st} is finite dimensional. The existence of the special basis when K is algebraically closed follows from [Di, Proposition 5, page 233]. $\square$

Theorem 6.4 ([HS, Theorem 1.13])**.** *Let $(R, \mathfrak{m})$ be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let L, M, N be R-modules with Frobenius actions such that we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\ & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \end{array}$$

with exact rows. If L is Noetherian and N is Artinian, then the F-stable parts form a short exact sequence

$$0 \to L_{\mathrm{st}} \to M_{\mathrm{st}} \to N_{\mathrm{st}} \to 0\,.$$

Proposition 6.5. *Let $(R, \mathfrak{m}, K)$ be a complete local ring with an algebraically closed coefficient field of positive characteristic. Let $\mathfrak{n}$ denote the nilradical of R. Then for all $i \ge 0$, the natural map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R/\mathfrak{n})$, when restricted to F-stable subspaces, gives an isomorphism*

$$H^i_{\mathfrak{m}}(R)_{\mathrm{st}} \xrightarrow{\cong} H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}\,.$$

Proof. Let k be an integer such that $\mathfrak{n}^{p^k} = 0$. The short exact sequence

$$0 \to \mathfrak{n} \to R \to R/\mathfrak{n} \to 0$$

induces a long exact sequence of local cohomology modules

$$\cdots \to H^i_{\mathfrak{m}}(\mathfrak{n}) \xrightarrow{\alpha} H^i_{\mathfrak{m}}(R) \xrightarrow{\beta} H^i_{\mathfrak{m}}(R/\mathfrak{n}) \xrightarrow{\gamma} H^{i+1}_{\mathfrak{m}}(\mathfrak{n}) \to \cdots\,.$$

Consider an element $\mu \in \ker(\beta) \cap H^i_{\mathfrak{m}}(R)_{\mathrm{st}}$. Then $\mu \in \mathrm{image}(\alpha)$, so $F^k(\mu) = 0$. The Frobenius action on $H^i_{\mathfrak{m}}(R)_{\mathrm{st}}$ is an automorphism, so $\mu = 0$, and hence the map $H^i_{\mathfrak{m}}(R)_{\mathrm{st}} \to H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}$ is injective.

To complete the proof it suffices, by Theorem 6.3, to consider an element $\eta \in H^i_{\mathfrak{m}}(R/\mathfrak{n})_{\mathrm{st}}$ with $F(\eta) = \eta$, and prove that it lies in the image of $H^i_{\mathfrak{m}}(R)_{\mathrm{st}}$. Now $\gamma(\eta) \in H^{i+1}_{\mathfrak{m}}(\mathfrak{n})$ so $F^k(\gamma(\eta)) = 0$, and therefore $F^k(\eta) = \eta \in \ker(\gamma)$.

Let $\eta = \beta(\mu)$ for some element $\mu \in H^i_{\mathfrak{m}}(R)$. Then $\beta(F(\mu) - \mu) = 0$, which implies that $F(\mu) - \mu \in \mathrm{image}(\alpha)$. Consequently $F^k(F(\mu) - \mu) = 0$, which shows that $F^{k+1}(\mu) = F^k(\mu)$, and hence that $F^k(\mu) \in H^i_{\mathfrak{m}}(R)_{\mathrm{st}}$. Since

$$\beta(F^k(\mu)) = F^k(\beta(\mu)) = F^k(\eta) = \eta,$$

we are done. $\square$

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