Foundations of Analysis II
Week 1

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Spaces of Continuous Functions

- X metric space
- $\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ bounded and continuous } \}$
- *Norm* $\boxed{||f||} = \sup_{x \in X}\{|f(x)|\}$

norm

normed vector space

V = vector space (over ℝ) (ex ℝⁿ)

$v \in V$ $\quad v + w$

$\alpha \in \mathbb{R}$ $\quad \alpha w$

$C(X)$ is a vector space. "vectors" $\Leftrightarrow f: X \to \mathbb{R}$

"scalar" constant func

$$\alpha, f \to (\alpha f) \in C(X), \quad \alpha \in \mathbb{R},\ f \in C(X)$$

$$f, g \to f + g$$

Def <u>Norm</u> on $V \to \mathbb{R}$

$$v \to \|v\|$$

1) $\|v\| \geq 0$, $= 0 \Leftrightarrow v = 0$

2) $\|\alpha v\| = |\alpha| \, \|v\|$

3) $\|v + w\| \leq \|v\| + \|w\|$

Ex: $\mathbb{R}^n$

$$\|(x_1,\dots,x_n)\|_1 = \sum |x_i| .$$

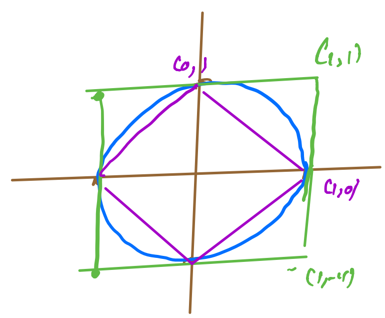
$$\|(x_1,\dots,x_n)\|_2 = \left(\sum |x_i|^2\right)^{1/2}$$

$$\|(x_1,\dots,x_n)\|_\infty = \max\{|x_1|,\dots,|x_n|\}$$

Visualize norms:

unit sphere (ball

$\{v : \|v\| = 1\}$ $\{\|v\| \le 1\}$



$\|v\|_2 = 1$

$\|v\|_\infty$

$\|v\|_1 = 1$

$(V, \|\ \|)$ normed vector space

$\Rightarrow$ metric space

(V, d)

$$d(u,v) = \|u - v\|$$

props norm $\Rightarrow$ props metric.

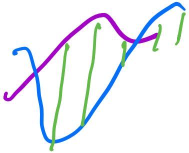
$\|(x_1, \dots, x_n)\| = \sum |x_i|$ $\qquad \mathbb{R}^2 \quad \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$

Back ▶ $d(f, g) = \|f - g\|$

Ex

$C(X)$

$$= \sup_{x \in X} |f(x) - g(x)|$$



$C(X)$

Theorem

$\mathcal{C}(X)$, d is a $\boxed{\text{complete}}$ metric space.

$d(f,g) = \|f-g\|$

Cauchy seq $\Rightarrow$ convergent

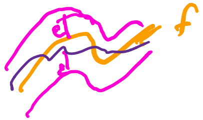
Cauchy: $\{f_n\}$ seq fns $\forall \varepsilon > 0\ \exists N$ s.t. $m,n \geq N \Rightarrow \|f_m - f_n\| < \varepsilon$

$\exists f \in C(X)$ s.t. $\forall \varepsilon > 0\ \exists N$ s.t. $n \geq N \Rightarrow \|f_n - f\| < \varepsilon$

$$\forall \varepsilon > 0 \ \exists N^{(\varepsilon)} \text{ s.t. } \sup_{x \in X} |f_n(x) - f(x)| < \varepsilon \quad \forall n > N$$

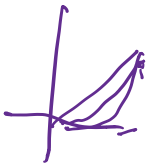
$$\Downarrow$$

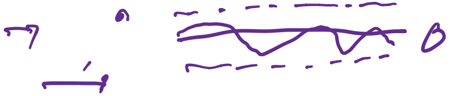
$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in X$$

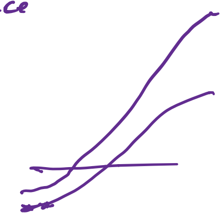


$\{x^n\}$ in $C[0,1]$

uniform convergence







$$\varphi(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

$f_n \to f$ in $\| \ \|$

$\iff f_n \to f$ uniformly on X

$X \xrightarrow{f} \mathbb{R}$

$\{f_n\}$ cont funcs $(\{f_n\} \in C(X))$

$f_n \to f$ uniformly on X

$\Rightarrow f$ is cont.

$f_n \to f \quad |f_n(x) - f(x)| < \varepsilon$?

$|f(x) - f(y)| < \varepsilon$

$|f(x) - f(y)| \qquad f_m(x) - f(x)$

$= |f(x) - f_n(x) + f_n(x) - f(y)|$

$|(f(x) - f_n(x)) + (f_n(x) - f_m(y)) + (f_n(y) - f(y))|$

$\varepsilon > 0 \ \exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \ \forall x$

cont of f at x_0

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to prove: $\forall \varepsilon > 0 \ \exists \delta \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\forall \varepsilon > 0 \ \exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \ \forall x$

$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f(x_0)$

$+ f(x_0) - f_n(x_0)|$

$$\le \boxed{|f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|}$$

$< \varepsilon$ $\forall n > N$ $< \varepsilon$

$\forall \varepsilon > 0 \; \exists \, \delta(\varepsilon, n)$ $|f_n(x) - f_n(x_0)| < \varepsilon$

$|x - x_0| < \delta(\varepsilon, n)$

given $\varepsilon > 0$ $\exists \, \delta_n$ s.t. $|f(x) - f(x_0)|$
$< \varepsilon + \varepsilon + \varepsilon$

$d(x, x_0) < \delta_n$

pick N s.t. $|f_n(x) - f(x)| < \varepsilon$ $\forall N > n$

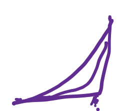
$\delta_{(N+1)}$

$|f(x) - f(x_0)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$

$|f(x) - f(x_0)| < \underbrace{\quad}_{< \varepsilon} + \quad + \underbrace{\quad}_{< \varepsilon}$

$\forall n$

$\{x^n\}$ on $[0,1]$



pt wise limit
~~not~~ cont

$\Rightarrow$ conv ~~not~~ unif

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f_n cont
$f_n \to f$ unif $[0,1]$

$\Rightarrow \int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$

$|\int f(x)dx| \le \int |f(x)|dx$

$\left| \int_0^1 f(x)dx - \int_0^1 f_n(x)\,dx \right|$ $< \varepsilon$.

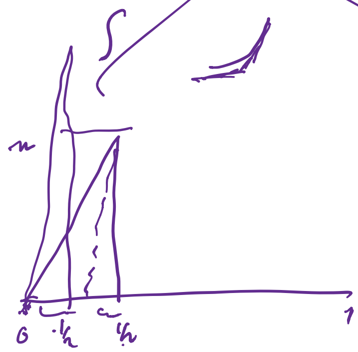
$\exists N$ s.t.

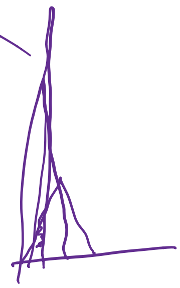
$\left| \int_0^1 (f(x) - f_n(x))\,dx \right| \le \int_0^1 \underbrace{|f(x) - f_n(x)|}_{< \varepsilon} dx$ $\forall x \in [0,1]$

$\le \varepsilon \int_0^1 dx = \varepsilon$

$\cancel{f_n(x) = x^n \text{ on } [0,1]}$

$\cancel{f_n \to f(x) = \begin{cases} 0 & \text{on } [0,1) \\ 1 & \text{on } 1 \end{cases}}$





$f_n \to 0$ p.w.

$\int_0^1 f_n(x)\,dx = 1$

$$\left| \int_0^1 (f_n(x) - f(x))\,dx \right|$$

$$\le \int_0^1 \underbrace{|f_n(x) - f(x)|}_{<\varepsilon} dx$$

$\forall \varepsilon > 0$

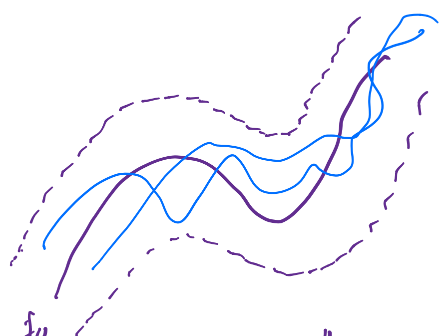
$\exists N$

s.t.

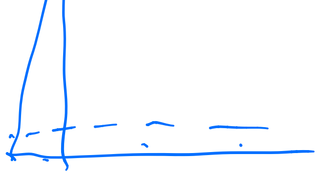
$|f_n(x) - f(x)| < \varepsilon$

$\forall x \in [a,b]$

$n > N \quad < \int_0^1 \varepsilon\,dx$



$\| f_0 - f \| < \varepsilon$

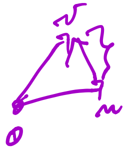


LastTime:

- Defined **Normed Vector Space**:
  - Vector space V (over $\mathbb{R}$) and a function $V \to \mathbb{R}$, written $v \to ||v||$ satisfying
    - $||v|| \geq 0$ and $||v|| = 0 \implies v = 0$
    - $||\alpha v|| = |\alpha|||v||$ for all $\alpha \in \mathbb{R}$ and $v \in V$.
    - For all $u, v \in V$

$$||u + v|| \leq ||u|| + ||v||$$

(triangle inequality)



- This gives a metric space V, d where

$$\boxed{d(u, v) = ||u - v||}$$

Examples

- $\mathbb{R}^n$ with any one of the following norms:
  - $$||(x_1, \ldots, x_n)||_1 = \sum_{i=1}^{n} |x_i|$$
  - $$||(x_1, \ldots, x_n)||_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$
  - $$||(x_1, \ldots, x_n)||_\infty = \max\{|x_1|, \ldots, |x_n|\}$$
- The space $\mathcal{C}(X)$ of bounded continuous functions on a metric space X, with norm

$$||f|| = \sup_{x \in X}\{|f(x)|\}$$

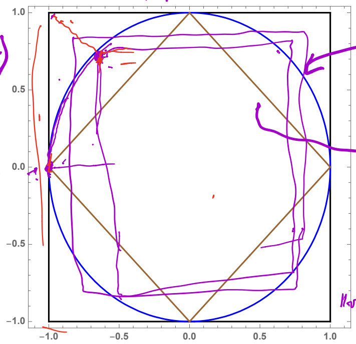
Visualize norms on $\mathbb{R}^2$

- Norms determined by the
  - Unit sphere $\{v : ||v|| = 1\}$ or
  - Unit ball $\{v : ||v|| \leq 1\}$
- Picture in $\mathbb{R}^2$:

$\|v\|_\infty$

All norms on $\mathbb{R}^n$ (say $\mathbb{R}^2$) are equivalent

Out[]=



$\|x\|_2 = 1$

$\|x\|_1 = 1$

$c\,\|x\|_2 \leq \|x\|_1 \leq C'\,\|x\|_2$

$\|v\|_2 = 1$

$\frac{1}{2} \le \|v\|_\infty \le 1$

$\|\alpha v\| = |\alpha|\,\|v\|$

$\frac{1}{\sqrt{2}} \le \|v\|_\infty \le 1$

$\frac{v}{\|v\|_2}$

$\frac{1}{\sqrt{2}} \le \left\|\frac{v}{\|v\|_2}\right\|_\infty \le 1$

$\frac{1}{\sqrt{2}}\|v\|_2 \le \|v\|_\infty \le \|v\|_2$

$\boxed{C\|\ \|' \le \|\ \| \le C'\|\ \|'}$

$C = \min_{\|v\|'=1} \|v\|$

$C' = \max_{\|v\|'\le 1} \|v\|$

any two norms on a vector space

in $\mathbb{R}^n$, any two norms are equiv

Pf. $\{\|v\| \le 1\} \subset \mathbb{R}^n$ is compact, $\|v\|'$ is cont $\Rightarrow$ min & max and max

$\frac{1}{\sqrt{2}}\sqrt{x^2+y^2} \le \max(|x|,|y|) \le \sqrt{x^2+y^2}$

$\sqrt{x^2+y^2} \le \sqrt{\max(|x|,|y|)^2 + \max}$

$\frac{x^2+y^2}{2}$ $\quad (\max(|x|,|y|))^2 \le x^2+y^2 \le 2(\max|x|,|y|)^2$

$x^2+y^2 \le 2(\max(|x|,|y|))^2$

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in $\mathbb{R}^n$ any two norms comp

finite dim vec sp

$C[0,1]$? are any two norms comparable?

$v \to \|v\|$ equivalent

$v \to \|v\|'$ $\Leftrightarrow$

$\exists$ constant $c, C' > 0$

$$c\|v\| \le \|v\|' \le C'\|v\|$$

$\forall v \in V$

$\|v_n\| \to 0$

$\Rightarrow \|v_n\|' \to 0$

$$0 \le \underbrace{\|v_n\|'}_{\downarrow 0} \le C' \underbrace{\|v_n\|}_{\downarrow 0}$$

$$\|v\| \le \frac{1}{c} \underbrace{\|v\|'}_{\downarrow 0}$$

0

Ex Find the best constants for any two of $\|v\|_1, \|v\|_2, \|v\|_\infty$ in $\mathbb{R}^2$ (in $\mathbb{R}^n$)

relate to the picture

Spaces of Continuous Functions

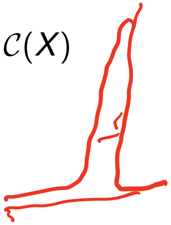
- X metric space
- $\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ bounded and continuous } \}$
- *Norm* $\boxed{||f|| = \sup_{x \in X}\{|f(x)|\}}$

- Theorem

  A sequence $\{f_n\}$ *in* $\mathcal{C}(X)$ *converges to* $f \in \mathcal{C}(X)$

$$\Longleftrightarrow$$

  f_n *converges to* f **uniformly** *on* X.



Proof

- $f_n \to f$ in the norm of $\mathcal{C}(X) \iff$
- For any $\epsilon > 0$ there exists N so that

$$||f_n - f|| < \epsilon \quad \text{for all } n > N \quad \iff$$

- For any $\epsilon > 0$ there exists N so that

$$\sup_{x \in X} |f_n(x) - f(x)| < \epsilon \quad \text{for all } n > N \quad \iff$$

- For any $\epsilon > 0$ there exists N so that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } n > N \text{ and for all } x \in X$$

which is the definition of uniform convergence.

$f_n \to f$

Theorem

$\mathcal{C}(X)$ is a complete metric space.

Proof:

- Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}(X)$.
- $\forall \epsilon > 0\ \exists N$ such that $m, n > N \Rightarrow |f_m(x) - f_n(x)| < \epsilon$. $\forall x \in X$
- In particular, for each $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$, has a limit $f(x)$.
- Get a function $f : X \to \mathbb{R}$ so that $f_n \to f$ **pointwise**
- Need to prove convergence is **uniform**.

Unif Cauchy + ptse conv $\Rightarrow$ unif conv.

N st $\forall m, n > N$, $\forall x \in X$

- $|f_n(x) - f(x)| \leq \underbrace{|f_n(x) - f_m(x)|}_{\epsilon/2} + |f_m(x) - f(x)|$
- Given $\epsilon > 0$:
- $\exists N = N(\epsilon)$ such that
  $m, n > N \Rightarrow |f_n(x) - f_m(x)| < \epsilon/2 \;\; \forall x \in X$
- $\exists M = M(x, \epsilon)$ such that $m > M \Rightarrow |f_m(x) - f(x)| < \epsilon/2$
- Given $x \in X$, choose
  $m = m(x, \epsilon) > \max(N(\epsilon), M(x, \epsilon))$.
- for this $m(x, \epsilon)$, the above inequlaity gives
  $|f_n(x) - f(x)| < \epsilon \;\; \forall n > N$ and $\forall\; x \in x$.
- Done!

indep of m

for x ∃ M
m > M(x)

N(ε)

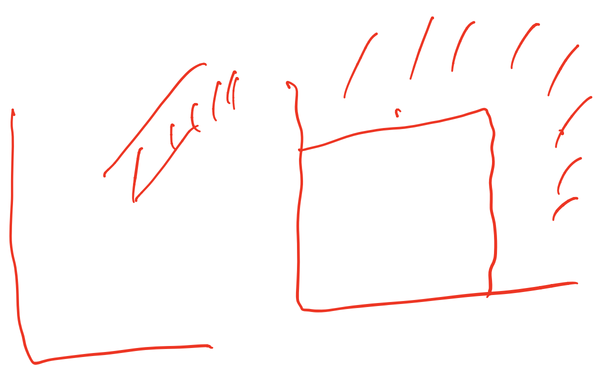
M(x,ε)

for each x

$$|f(x) - f_n(x)| \le \underbrace{|f_n(x) - f_m(x)|}_{< \varepsilon/2 \text{ for } m,n > N(\varepsilon)} + \underbrace{|f_m(x) - f(x)|}_{< \varepsilon/2 \text{ for } m > M(x,\varepsilon)}$$

for each x, if $m > M(x,\varepsilon)$

$$|f(x) - f_n(x)| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$



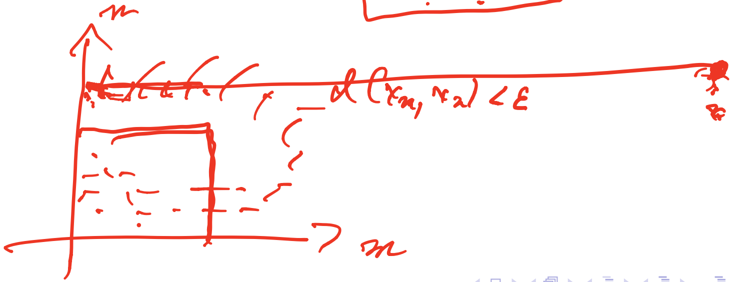
Remark

This proof shows how powerful Cauchy's condition is:

$\forall \epsilon > 0 \exists N$ such that $m, n > N \Rightarrow \boxed{d(x_m, x_n) < \epsilon}$

$\epsilon > 0$

N



Important Examples

- X *compact* metric space. Then

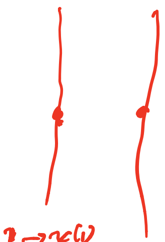
$$\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous } \}$$

(boundedness is automatic)

- $\boxed{X = [0, 1]}$ $[a, b]$

- $X = \{1, 2\}$

$\mathbb{R}^2$



$1 \to x(1)$
$2 \to x(2)$

$(x(1), x(2)) \in \mathbb{R}^2$

- $X = \{1, 2, \ldots, n\}$

$C(\{x_1, \ldots, x_n\}, \mathbb{R}) = \mathbb{R}^n, \|x\|_\infty$

Rudin

- $\mathcal{C}(X, \mathbb{R}), \ \mathcal{C}(X, \mathbb{C})$

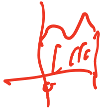
Space $\mathcal{C}(X, Y)$

- If Y is a metric space, can define $\mathcal{C}(X, Y)$
- If $f, g \in \mathcal{C}(X, Y)$, their distance is defined by

$$D(f, g) =$$

- Check this is a metric.

Functions on $\mathcal{C}([0,1])$



- Let $I : \mathcal{C}([0,1]) \to \mathbb{R}$ be defined by

$$I(f) = \int_0^1 f(x)\, dx.$$

- Is I continuous? cont at f

given $\varepsilon > 0$ $\exists\ \delta$ s.t. $\|g - f\| < \delta \Rightarrow |I(f)| < \varepsilon$

$\|f - g\| < \delta \Rightarrow |I(f) - I(g)| < \varepsilon$

$|I(f-g)|$

$|I(f) - I(g)| = \left| \int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx \right|$

Equiv Functions

$f_n \to f$ uniformly on $[0,1]$

$\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$

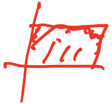
$$= \left| \int_0^1 (f(x)-g(x))\,dx \right|$$

$$\le \int_0^1 |f(x)-g(x)|\,dx$$

$$\le \int_0^1 \left(\sup |f(x)-g(x)|\right) dx$$

$$= \sup |f(x)-g(x)|$$

$$= \| f-g \|$$



$$|I(f)-I(g)| \le \| f-g \| \qquad \delta = \varepsilon$$

in $C([0,2])$ $\quad |I(f)-I(g)| \le 2\| f-g \|$

$$\varphi : X^{d_x} \to Y^{d_y} \qquad \boxed{\begin{array}{l} \exists C \text{ s.t.} \\ d_Y(\varphi(x),\varphi(y)) \le C\, d_X(x,y) \end{array}}$$

Cont

$\delta = \varepsilon / C$

- Define $\mathbf{I} : C([0,1]) \to C([0,1])$ by

$$\mathbf{I}(f) = \int_0^x f(t)\,dt. = \text{indef int of } f,\ f(0)=0.$$

- Is $\mathbf{I}$ continuous?

$$\|\mathbf{I}(f) - \mathbf{I}(g)\| = \sup_x \left| \int_0^x f(t)\,dt - \int_0^x g(t)\,dt \right|$$

$$= \sup_x \left| \int_0^x (f(t) - g(t))\,dt \right|$$

$x \in [0,1]$

$$\le \int_0^x |f(t) - g(t)|\,dt$$

$$\le \sup_x x\, \|f - g\|$$

$0 \le x \le 1$

$$\le \|f-g\|$$

$$\forall x, \quad |I(f)(x) - I(g)(x)| \le \|f-g\|$$

$$\Rightarrow \sup \{ \ldots \} \le \|f-g\|$$

$$\boxed{\|I(f) - I(g)\| \le \|f-g\|}$$

diff $\Rightarrow$ cont

$C[0,1]$

$C'([0,1]) \subset C$

- Let $\mathcal{C}^1([0,1]) \subset \mathcal{C}([0,1])$ be the subspace of continuously differentiable functions, that is,

$$\mathcal{C}^1([0,1]) = \{f : [0,1] \to \mathbb{R} : f' \text{ exists and is continuous } \}$$

norm on $\mathcal{C}^1$ = restriction of norm on $\mathcal{C}$.

- Define $\mathbf{D} : C^1([0,1]) \to \mathcal{C}([0,1])$ by

$$\mathbf{D}(f) = f'$$

$C([0,1])$

- Is $\mathbf{D}$ continuous?

C' continuous

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\|f\| < \delta \Rightarrow \|f'\| < \varepsilon$

$\Rightarrow |f'| < \varepsilon$

$|f| \le \varepsilon \Rightarrow |f'| \le ?$

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}$$

$$f_n'(x) = \frac{n \cos nx}{\sqrt{n}}$$

$$= \sqrt{n} \cos nx$$

$$\|f_n'(x)\| \to \infty$$

Change Norm

make it cont.

Norm on C^1: $\|f\|_1 = \sup\left(|f(x)| + |f'(x)|\right)$

Restatement

$\left.\begin{array}{l} f_n' \to f' \text{ unif} \\ f_n(0) \to f(0) \end{array}\right\} \Rightarrow f_n \to f \text{ unif}$

Exercise: comparing norms in $\mathbb{R}^2$

Look at Rudin pf of 7.17

~~use~~ f_n' cont and use ~~$f_n(x) = \int_0^x f_n'$~~

$g(x) = \int_0^x g'(t)\,dt$

if $g \in C^1$

Exercises

Draft

Official Exercise

posted next mon

Jan 14

Due Jan 21

f_n cont, diff

$f_n' \to$ converges uniformly on $[0,1]$

and $f_n(0) \to$ converges

(or: fix a pt x_0, $f_n(x_0) \to f(x_0)$)

$\Longrightarrow f_n \xrightarrow{\text{unif}} f$

$f_n' \to f'$

Rmk: if f_n' are assumed <u>continuous</u>

then shorter pf using

fund thm of calculus

$f_n' \xrightarrow{\text{unif}} g$

$$\boxed{\int_0^x f_n'(t)\,dt = f_n(x) - f_n(0)}$$

$\downarrow$ $\downarrow$

$\int_0^x g(t)\,dt$ $\qquad$ c_0

$\int_0^x |f_n'(t) - g(t)|\,dt$

$\varepsilon > 0 \ \exists N \quad n > N \Rightarrow |f_n'(t) - g(t)| < \varepsilon$

$\forall t$

$\cancel{\int_0^x f_n'(t)}$

f_n ~~conv~~ unif to $\boxed{\int_0^x g(t)\,dt}$

$f_n(x) = \underline{f_n(0)} + \underline{\int_0^x f_n'(t)\,dt}$

$\downarrow$ unif x

$\int_0^x g(t)\,dt$

f.

Later

Other norms on $\mathcal{C}([0,1])$

- $$||f||_1 = \int_0^1 |f(x)|\, dx$$

- $$||f||_2 = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}$$

- $||f||_\infty = \sup_{x \in X} |f(x)|$

The p-Norms, $1 \le p \le \infty$

- General formula, if $1 \le p < \infty$

$p=1$

$p=2$

$$||f||_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

- and for $p = \infty$

$$||f||_\infty = \sup_{x \in [0,1]} |f(x)|$$

- Similar formulas in $\mathbb{R}^n = \mathcal{C}(\{1, \ldots n\})$:
- Replace integrals by sums
- If $x = (x_1, \ldots, x_n)$

$$||x||_p = \left(\sum (|x_1|^p + \ldots |x_n|^p) \right)^{1/p}$$

- and

$$||X||_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

- Picture for $n = 2$

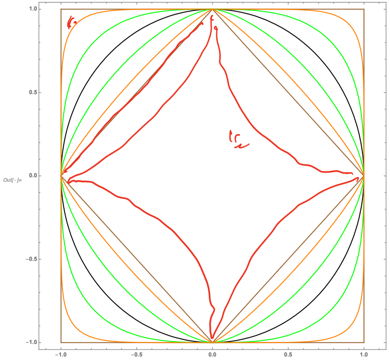
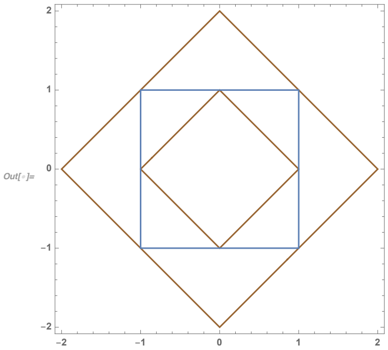


Figure: Unit Balls of p-norms in $\mathbb{R}^2$

- Figure shows, from inside out,
$p = 1, \;\; 7/6, \;\; 3/2, \;\; 2, \;\; 3, \;\; 7, \;\; \infty$

- In $\mathbb{R}^n$ (n an integer) all norms are equivalent.
- Example:



- Shows that $1 \leq ||x||_1 \leq 2$ on $||x||_\infty = 1$

- Same:

$$||x||_\infty \leq ||x||_1 \leq 2||x||_\infty$$

  on $\mathbb{R}^2$

- On $\mathcal{C}([0,1])$ have

$$||f||_1 \leq ||f||_\infty$$

- But no constant $C > 0$ such that

$$C||f||_1 \leq |f||_\infty$$

not OK

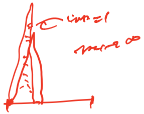
??

$C([0,1])$

$C\|f\|_\infty \le \|f\|_1 \le \|f\|_\infty$

OK

$\|f\|_1 \le (1)\|f\|_\infty$ $\quad$ $\|f\|_\infty \le \frac{1}{C}\|f\|_1$



estimate averages by max

estimate max by av.

$m \le \|v\|_\infty \le \max \|v\|_\infty$ on $\|v\|_1 = 1$

$m\|v\|_\infty \le \|v\|_1 \le \max \|v\|_1$

$\{f \in C[0,1] : \|f\|_\infty = 1\}$ unit sphere

$\le$ unit ball

not compact

$\{x^n\}$

$\|x^m - x^n\| = (m/n)^{n-m}$

$m < n$

$\mathbb{R}^n$ cpt $\Leftarrow$ closed & bdd

$C([0,1])$

in any metric sp

Compact $\Rightarrow$ closed
$\Rightarrow$ bounded

$\Leftarrow$ true in $\mathbb{R}^n$

not in general

---

in $C[0,1]$ compact $\Leftarrow$ closed
$\Rightarrow$ bdd
$\Rightarrow$ Equicontinuous

Def $\mathcal{F}$ equicont

$\forall \varepsilon > 0 \; \exists \delta > 0$ st ~~[illegible]~~

$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
$\forall f \in \mathcal{F}$
$\forall x, y \in X$

ε

$\mathcal{F}$ cpt

$\forall \varepsilon > 0 \; \exists$
$\Rightarrow f_1, \dots, f_n \in \mathcal{F}$ $\forall f \; \exists c.$
st. $\|f - f_i\| < \varepsilon$

est $|f(x) - f(y)|$ indep of f.

Thm $S \subset C[0,1]$ is compact
$\Leftrightarrow$ closed, bounded, equicont

$\Rightarrow$ HW
$\Leftarrow$ Arzela-Ascoli thm Next time

Completeness?

Equicontinuity

Compact subsets of $C(X)$

X cpct, metric.

- **Definition**

  A subset (family) $\mathcal{F} \subset \mathcal{C}(X)$ is **equicontinuous** $\iff$

$\{ f : X \to \mathbb{R} \}$

$\varepsilon, \delta, f, x, y$

$\forall \varepsilon > 0 \; (\quad) \; \exists \delta \; \forall f, \forall x, y$

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$\forall \varepsilon > 0 \; \exists \delta \; \forall x, y \; d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$

$\forall \varepsilon > 0 \; \forall x \; \exists \delta \; \forall y$