Foundations of Analysis II
Last Class

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Bott - Tu

Differential Forms in Algebraic Topology

Lebesgue Integral

$E \subset \mathbb{R}$ $\quad m(E) < \infty$

$f : E \to \mathbb{R}$ bdd func.

Simple functions

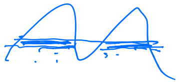
Thm f is measurable $\iff$

$$\sup_{\psi \le f} \int \psi \le \inf_{\varphi \ge f} \int_E \varphi$$

$$\sup \{ \quad \} = \inf \{ \quad \}$$

Simple: measurable function finitely many values

canonical decomposition $s = \sum a_i \chi_{A_i}$



$A_i = \{x : s = a_i\}$

A_i need not be intervals

Def $\mathcal{L}(\cdot)$: $\int_E f = \sup_{\varphi \le f} \{ \int \varphi \}$ $\left(= \inf_{\psi \ge f} \int \psi \right) \le U(f)$

~~if f is Riemann int~~

$\Leftarrow$ R-int integrable.

R-int $\Rightarrow$ $\mathcal{L}$-integrable.

$\mathcal{M}$ = Lebesgue-measurable sets σ-alg

Ex $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ in $[0,1]$

not R-int

but L-int.

$\int \chi_A = m(A)$

$\int \chi_{\mathbb{Q} \cap [0,1]} = m(\mathbb{Q} \cap [0,1]) = 0.$

Convergence theorems

$m(E)<\infty$ Monotone conv thm

$0 \le f_1(x) \le f_2 \le \cdots$ measurable $\to f$, pointwise.

f meas.

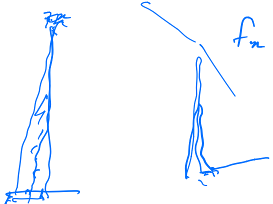
$$\int_E f_n \to \int_E f$$

Dominated Conv thm

$f_n \to f$ L-int $g \ge 0$

$|f_n| \le g$

$\Rightarrow \int f_n \to \int f$



$f_n \to 0$

$\int f_n = 1$

Littlewood's 3 principles

Royden's book Monoton Conv.

1) if $f_n \to f$ uniformly, obvious. $[0,\infty]$

2) $f_n \nearrow f$ where $\forall x \in E$, $f_n(x) \to f(x)$

given $\varepsilon > 0$, define $E_n = \{x \in E : |f_k(x) - f(x)| < \varepsilon \quad \forall k \geq n\}$

$f_k(x) \to f(x)$

given $x \in E$, $f_n(x) \to f(x)$

$\exists n$ s.t. $x \in E_n$

$\Rightarrow \quad E = \bigcup E_n$

$\Rightarrow \quad m(E_n) \to m(E)$

given $\varepsilon > 0$, n s.t. $m(E - E_n) < \varepsilon$

given $\varepsilon > 0$

$E = A \cup B$

$\exists n \quad |f_k(x) - f(x)| < \varepsilon \quad \forall x \in A$

$\forall \varepsilon > 0 \quad \exists \ E = A \cup B$

$\forall \varepsilon, \delta > 0$

1) $\exists n$ s.t. $|f_k(x) - f(x)| < \varepsilon \quad \forall x \in A, \ \forall k \geq n$

2) $m(B) < \delta$

$f_n \to f \qquad \forall \varepsilon > 0$

$\int f_n \to \int f \qquad \int_E |f_k - f| \leq \int_A |f_k - f| + \int_B |f_k - f|$

$\varepsilon \, m(A) \qquad < \delta \cdot M$

$k \geq n$

$|f_n| < M$

$\int f_k \to \int f$.

Compare to Riemann Integral

- $f : [a, b] \to \mathbb{R}$ bounded function.
- Riemann integrable $\Rightarrow$
  - Lebesgue integrable
  - Integrals agree

- Lebesgue's Theorem:

Theorem

$f : [a, b] \to \mathbb{R}$ bounded function. Then f is Riemann integrable on $[a, b]$ if and only if f is continuous almost everywhere on $[a, b]$.

- f continuous a.e. means $\exists E \subset [a, b]$ such that $m(E) = 0$ and $f|_{[a.b] \setminus E}$ is continuous.
- Follow proof of Rudin, Theorem 11.33 (b).

Royden

Upper and Lower Envelopes

- $\{P_k\}$ a carefully chosen sequence of partitions of $[a, b]$.

$a \qquad b$

$P_1 < P_2 < P_3 < \cdots$ 'mesh' $< \frac{b-a}{k}$

- $L(P_k, f)$, $U(P_k, f)$ lower and upper sums.
- $L(P_k, f) \to \mathcal{R}\underline{\int}_a^b f(x)dx$
- $U(P_k, f) \to \mathcal{R}\overline{\int}_a^b f(x)dx$

$[x_{i-1}, x_i] \to \inf f$, $\to \sup f$

- Define step functions $L_k, U_k : [a, b] \to \mathbb{R}$ such that
- $L(P_k, f) = \int_a^b L_k(x)dx$ and
- $U(P_k, f) = \int_a^b U_k(x)dx$.

- Moreover, for all $x \in [a, b]$

$$L_1(x) \leq L_2(x) \leq \cdots \leq f(x) \leq \cdots \leq U_2(x) \leq U_1(x)$$

$L_k(x) \to L(x) \leq f(x)$

$f(x) \leq U(x) \leftarrow U_k(x)$

- Limits $L(x), U(x)$ exist and
- $L(x) \leq f(x) \leq U(x)$

- Monotone convergence thm gives

$$\int_a^b L_k(x)dx \to \int_a^b L(x)dx.$$

- Thus

$$\boxed{\int_a^b L(x)dx = \mathcal{R}\underline{\int_a^b} f(x)dx}$$

- Similarly

$$\int_a^b U(x)dx = \mathcal{R}\overline{\int_a^b} f(x)dx$$

- For $x \notin \cup_1^\infty P_k$,

$$U(x) = \inf_{\delta>0} \sup_{|y-x|<\delta} f(y)$$

called the upper envelope of f at x.

- Similarly

$$L(x) = \sup_{\delta>0} \inf_{|y-x|<\delta} f(y)$$

$L(x) \le f(x) \le U(x)$

upper semi...

f cont at $x \iff U(x) = L(x)$

If $x \notin \cup P_k$, then f is continuous at x iff $L(x) = U(x)$

Coutable set

Measure $= 0$

R-int $\Longleftrightarrow \int U = \int L$

$$\int_a^b (U - L) = 0$$

$U - L \geq 0$ mean

$\int (U - L) = 0 \Longleftrightarrow U - L = 0$ a.e.

R-int $\Rightarrow$ f cont outside $\left(\begin{array}{c} \cup P_k \\ \{U - L > 0\} \end{array} \right)$

$$\boxed{\text{Cont at } x \iff L(x) = U(x)}$$

$L \leq f \leq U$

$\int L = \underline{R\int} f$

$\int U = \overline{R\int} f$

f cont at x & $\bigcup P_k$

$\Rightarrow U(x) = L(x).$