Foundations of Analysis II
Week 7

Domingo Toledo

University of Utah

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Two Theorems in one-variable Calculus

- $f : [a, b] \to \mathbb{R}$ differentiable and there exists a constant M such that $|f'(x)| \leq M$ for all $x \in [a, b] \Rightarrow |f(b) - f(a)| \leq M(b - a)$

- Usually proved from the mean value thm:
  $f : [a, b] \to \mathbb{R}$ differentiable $\Rightarrow$ there exists $c \in (a, b)$ such that
  $$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Then $|f(b)-f(a)| = |f'(c)|\,|b-a|$

$\leq n\,|b-a|$

- $I \subset \mathbb{R}$ an open interval, $f : I \to \mathbb{R}$ differentiable and for some $x_0 \in I$, $f'(x_0) \neq 0$.
- Then there exists open interval J with $x_0 \in J \subset I$ such that
  - $f|_J$ is invertible,
  - its image is an open interval $J \subset \mathbb{R}$, and
  - $f^{-1} : J' \to J$ is differentiable.

(Inverse function theorem)

Pf ① $f'(x_0) \neq 0 \Rightarrow \exists$ interval J,

$x_0 \in J \subset I$

s.t. $f'(x) \neq 0 \ \forall x \in J$.

Say $f'(x) > 0$ on $J \Rightarrow f$ increasing $\Rightarrow$ injective

So f is injective

rest later

Change of topic in response to a question:

Jacobian Matrix

$f : \mathbb{R}^m \to \mathbb{R}^n$

$f = (f_1, \dots, f_n) = (f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m))$

n functions of m variables.

$\mathbb{R} \to \mathbb{R}^n$ "vector function of 1-variable"

$\mathbb{R}^m \to \mathbb{R}$

f diff at x

$$\boxed{f(x+h) - f(x) = (d_x f)(h) + o(|h|)}$$

$d_x f : \mathbb{R}^n \to \mathbb{R}^n$ linear

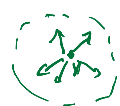
$\updownarrow$ ← standard basis

matrix

that's the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_3}{\partial x_1} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \end{pmatrix}$$

$f(x+h) - f(x)$ ~~~~ linear in h

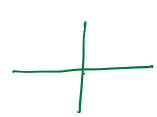


$f(x)$

$\frac{\partial f}{\partial x_1}$ exists at x $\quad f(x + te_1) - f(x) = \frac{\partial f}{\partial x_1}(x)\, t + o(t)$

$$f(x_1+t, x_2, \dots, x_n) - f(x_1, \dots, x_n) = \frac{\partial f}{\partial x_1}(x_1, \dots, x_n)\, t + o(t)$$

eg $\quad f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (0,0) \end{cases}$



$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{\overset{0}{f(h,0)} - \overset{0}{f(0,0)}}{h} = 0$$

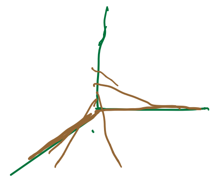
$$\frac{\partial f}{\partial y} = 0$$

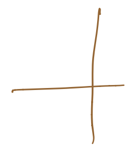
$$f(h,k) = \boxed{0} + o\left(\sqrt{h^2+k^2}\right)$$

$$\frac{\frac{hk}{h^2+k^2}}{\sqrt{h^2+k^2}} = \frac{hk}{\left(\sqrt{h^2+k^2}\right)^3} \not\to 0$$

$h = k = t$

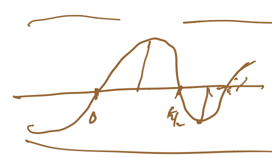
$$\frac{t^2}{|t|^3} = \frac{1}{|t|} \to \infty \text{ as } t \to 0$$

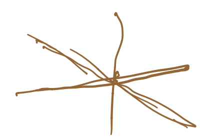




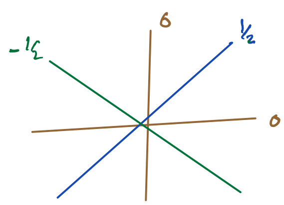
$x = r\cos\theta$
$y = r\sin\theta$

$$\frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta = \tfrac{1}{2}\sin 2\theta = \boxed{z = \frac{xy}{x^2+y^2}}$$





$$f(tx, ty) = f(x,y)$$



"directional derivatives" exist in every direction

$$\lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

$v = e_1$ $\quad \frac{\partial f}{\partial x_1}$

$v = e_2$ $\quad \frac{\partial f}{\partial x_2}$

v arb.

Diff $\Rightarrow$ Existence of partial derivs

$\Leftarrow$

Diff $\Leftarrow$ Continuous diff at x

$\Updownarrow$

All partials (exist and are continuous at x

$f(x,y) = \frac{xy}{x^2+y^2}$

$\frac{\partial f}{\partial x}$

If $f : \mathbb{R}^2 \to \mathbb{R}$ diff at $(0,0)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

$\vec{v} = (\cos\theta, \sin\theta)$

direc deriv $\underline{\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta}$

$(d_{x_0} f)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$

Direc der in $\cos\theta, \sin\theta$

$\cancel{(\cos\theta\ \sin\theta)} =$

$\cos\theta\sin\theta$

$f(h,k) - f(0,0) = \frac{hk}{h^2+k^2}$

$\cancel{\frac{\partial f}{\partial x}} =$

f diff at $(0,0)$ & $\frac{\partial f}{\partial x}(0,0) = 0$

$\frac{\partial f}{\partial y}(0,0) = 0$

$\Rightarrow$ all direc derivs at $(0,0)$ $\underline{=} 0$

diff at x $\Rightarrow$ all partials exist at x

$\Rightarrow$ Matrix of $d_x f$

is the Jacobian matrix

Ex $f(x,y) = \frac{xy}{x^2+y^2}$

Jacobian matrix at $(0,0)$ $= (0\ \ 0)$

if diff

$\Rightarrow \boxed{d_{(0,0)} f} = \underline{0}$

$f(h,k) - 0 = o\left(\sqrt{h^2+k^2}\right)$

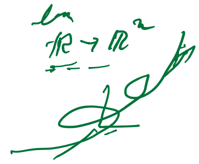
$\Rightarrow \lim_{(h,k)\to(0,0)} f(h,k)$ exists & $= 0$

vs

$\mathbb{R}^n \to \mathbb{R}^2$

"big jump" from $\mathbb{R} \to \mathbb{R}$ to $\mathbb{R}^2 \to \mathbb{R}$

$\mathbb{R} \to \mathbb{R}^2$



$\mathbb{R}^2 \to \mathbb{R}$

$\mathbb{R}^3 \to \mathbb{R}$

$\vdots$

$\mathbb{R}^{100} \to \mathbb{R}$

$\mathbb{R}^{501} \to \mathbb{R}^{2050}$

Def (X, d_X), (Y, d_Y) metric spaces

$$f : (X, d_X) \to (Y, d_Y)$$

is called Lipschitz

$\iff \exists C > 0$ s.t.

$$\boxed{d_Y(f(x), f(y)) \le C\, d_X(x,y)}$$

If f is Lipschitz, C is called a Lipschitz constant for f.

"the" Lipschitz const = inf {C : C a Lipschitz const}

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f distorts distance by at most a factor of C

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one variable calc

$f : [a,b] \to \mathbb{R}$, diff, $|f'(x)| \le M$

$\Rightarrow |f(b) - f(a)| \le M|b-a|$

Pf

$I \subset \mathbb{R}$ $\quad f: I \to \mathbb{R}$ diff $\quad |f'(x)| \leq M$

$\forall x, y \in I \quad |f(x) - f(y)| \leq M|x-y|$

$\Rightarrow |f'(x)| \leq M \Rightarrow f$ is Lipschitz with ~~con~~ M

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f is Lipschitz $\Rightarrow f$ is uniformly continuous

$(f$ diff $|f'(x)| \leq M) \Rightarrow f$ Lip. cont.

$f(x) = x^2$ unif cont $[-c, c]$

not on $(-\infty, \infty)$

Equicont $\quad \mathcal{F}$ of Lipschitz by same const M

$\Rightarrow \mathcal{F}$ is equicont

$|f(x) - f(y)| \leq M|x-y| \quad \left.\begin{matrix}\forall f \\ \forall x, y\end{matrix}\right\} \Rightarrow$ equicont

How to get equicont func?

$\{\varphi\}$ on $\mathbb{R}$ $\quad |\varphi(x)| \leq M$

$\psi(x) = \int_0^x \varphi(t)\,dt$

$|\psi'(x)| \leq M$

$\Rightarrow \{\psi\}$ equicont

Higher dimension:

$f: U \to \mathbb{R}^n$, $\quad U \subset \mathbb{R}^n$

U open & convex

$f: U \to \mathbb{R}^n$ differentiable

Suppose $\exists M$ s.t. $\|d_x f\| \leq M \quad \forall x \in U$.

$\Rightarrow |f(x) - f(y)| \leq M|x-y| \quad \forall x, y \in U$.

Jacobian Matrix

$f(r,\theta) = (r\cos\theta, r\sin\theta)$

$\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

$\begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix}$

$f\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$\det = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$r(\cos^2\theta + \sin^2\theta) = r$

$I \subset \mathbb{R} \xrightarrow{f} \mathbb{R} \quad |f'(x)| \le M$

(a,b), (a,∞), $(-\infty, \cdot)$

$\Rightarrow |f(y) - f(x)| \le M|y-x|$

$\forall x, y \in I$

Proof:

$\begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$2x\sin(1/x) - x^2\cos(1/x)\frac{1}{x^2}$

0

---

false

$\tan x$

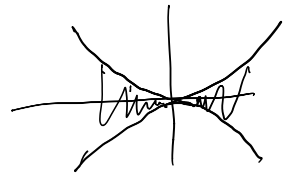
$|\tan(y) - \tan(x)| \le M|x-y|$

$x, y \in (-\pi/2, \pi/2)$

$\exists c$ between x, y

$\frac{f(y) - f(x)}{y - x} = f'(c)$

$x^2 \sin(1/x)$



$$\underbrace{2x \sin\left(\tfrac{1}{x}\right)}_{\to 0} + x^2 \cos\left(\tfrac{1}{x}\right)\left(\tfrac{-1}{x^2}\right)$$

$$= \underbrace{2x \sin(1/x)}_{\to} - \underbrace{\cos(1/x)}_{\text{osc } [-1,1]}$$

$$\frac{f(0+h) - f(0)}{h}$$

$$\frac{h^2 \sin(1/h)}{h} = h \sin(1/h) \to 0$$

Comments on HW 2

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \pi^2/8$$

$$\sum_{n} 1/n^2 = \pi^2/6$$

$$\sum_{n \text{ odd}} \frac{1}{n^2} \qquad \underbrace{\sum_{n}}_{\pi^2/6} = \sum_{n \text{ odd}} + \underbrace{\sum_{\text{even}}}_{\pi^2/8}$$

$\sum_{\text{even}}$

$\sum_{\text{odd}}$

$$\sum_k \frac{1}{(2k)^2} = \frac{1}{4} \sum_k 1/k^2$$

$$\pi^2/6 + \frac{1}{4}\pi^2/6 = \pi^2/8$$

$$\sum_{n} = \sum_{\text{even}} + \sum_{\text{odd}}$$

$$? \quad \pi^2/6 = \tfrac{1}{4}\pi^2/6 + \pi^2/8$$

$$\boxed{\alpha/\pi \text{ irrational} \longrightarrow \frac{1}{N} f(x+n\alpha)} \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$$

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try: $\underline{e^{ikx}}$

$k=0$ ok

$$\frac{N}{N} \to \frac{1}{2\pi} 2\pi \quad \checkmark$$

$k \neq 0$
$\in \mathbb{Z}$

$$\frac{\sum e^{ik(x+n\alpha)}}{N} = \frac{e^{ikx}\sum e^{ikn\alpha}}{N}$$

$$= \frac{e^{ikx}}{N}\,\frac{1-(e^{ik\alpha})^N}{\left|1-e^{ik\alpha}\right|} \longrightarrow 0$$

$$\boxed{\alpha/\pi \text{ irrational} \Leftrightarrow e^{ik\alpha} \neq 1 \quad \forall k \in \mathbb{Z}.}$$

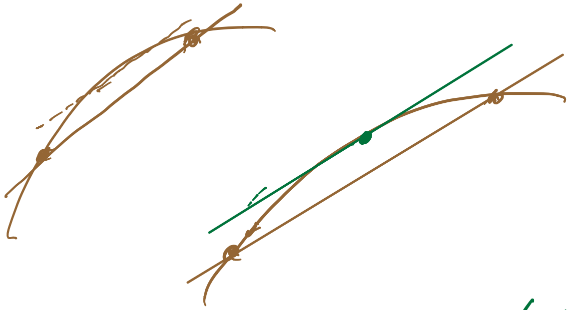
Several Variable Generalizations

- If $U \subset \mathbb{R}^m$ is open and convex and $f : U \to \mathbb{R}^n$ differentiable.
- Let $x, y \in U$ and let $\gamma : [0, 1] \to U$ be the straight line segment from x to y
- $\gamma(t) = (1 - t)x + ty, 0 \leq t \leq 1.$
- Can we say that there is $c \in [0, 1]$ such that

$$f(y) - f(x) = d_c f(y - x)$$

$\exists\, c$ between x & y s.t.

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$



$$|f(y) - f(x)| = |f'(c)||y - x| \le M|y - x|$$

$$\gamma(t) = (1-t)x + ty$$

$$\gamma'(t) = -x + y = y - x$$

$$f(y) - f(x) = \cancel{\frac{d}{dt} f(\gamma(t))}$$

I

$$|f'(t)| \le M \implies |f(y) - f(x)| \le M|y - x|$$

$f : [a,b] \to \mathbb{R}$ conti, diff on (a,b)

$$\exists\, c \in [a,b] \text{ s.t. } \frac{f(b) - f(a)}{b - a} = f'(c)$$

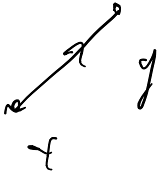
$$|f(b) - f(a)| = \underbrace{|f'(c)|}_{\le M}\,|(b - a)|$$

in more variables

$U \subset \mathbb{R}^m$ convex, $f : U \to \mathbb{R}^n$

(cont diff

given $x, y \in U$



$\alpha_{x,y}(t) = (1-t)x + ty$

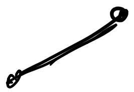
$\varphi(t) = f(\alpha_{x,y}(t))$

$\varphi(1) - \varphi(0) = \varphi'(c) \quad 0<c<1$

$$\boxed{f(y) - f(x) = df_{(c)}\,(y-x)}$$

$\mathbb{R}^m \to \mathbb{R}$ OK 1-variable

t



$\mathbb{R}^m \to \mathbb{R}^n$ $n > 1$

?

$$\boxed{f(y)-f(x)} = \int_0^1 \frac{d}{dt} f(\gamma(t))\,dt$$

$$= \int_0^1 (d_{\gamma(t)} f)\,(y-x)\,dt$$

$$\boxed{\le \left| \left(\int_0^1 (d_{\gamma(x)} f)\,dt \right) (y-x) \right|}$$

$$|f(y)-f(x)| \le \int_0^1 \|(d_x f) f\|\;|y-x|$$

$$\leq \int_0^1 \underbrace{\| d_\gamma f \|}_{\leq M} dt \; |y-x|$$

$$|f(y) - f(x)| \leq M |y-x|$$

$$|f(y) - f(x)|^2 = (f(y) - f(x)) \cdot (f(y) - f(x))$$

$$(f(y) - f(x)) \cdot f(\gamma(t)) \qquad \gamma(t) = \gamma_{x,y}(t)$$

$$\varphi(t) = (f(y) - f(x)) \cdot f(\gamma(t))$$

$$\varphi(1) - \varphi(0) = \varphi'(c)$$

$$(f(y) - f(x)) \cdot f(y) - (f(y) - f(x)) \cdot f(x)$$

$$= |f(y) - f(x)|^2$$

$$|f(y) - f(x)|^2 = (f(y) - f(x)) \cdot (d_{\gamma(c)} f)(y-x)$$

$$\text{Sch} \to \leq |f(y) - f(x)| \, |d_{\gamma(c)} f (y-x)|$$

$$\leq |f(y) - f(x)| \, \underbrace{\| d_{\gamma(c)} f \|}_{\leq M} |y-x|$$

$$|f(y) - f(x)| \leq M |y-x|$$

Moral Remember

$$\| d_x f \| \leq M$$

$\Rightarrow |f(y) - f(x)| \leq M |y - x|$

$f : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$

$\uparrow$ convex

$\Uparrow$

$$\boxed{f(y) - f(x) = (d_{z(t)} f)(y - x)}$$

lin $\mathbb{R} \to \mathbb{R}$

$\underbrace{d_{(x)} f}$ "total der", "differential", "derivative"

<u>Thm</u> $f : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$, suppose all $\frac{\partial f}{\partial x_i}$, $i = 1, \ldots, m$

are defined in U (near x enough)

and continuous (at x)

$\Rightarrow$ f is differentiable at x

$\boxed{\text{Cont'ble partials}}$ $\Rightarrow$ diff $\Rightarrow$ existence of partials

$\not\Leftarrow$ $\not\Leftarrow$

$\uparrow$ look mostly at this class called C^1

C^1 = continuous partial ders

$C^1 \Rightarrow$ diff $\Rightarrow$ partial derivs exist

$\not\Leftarrow$ $\not\Leftarrow$

Pf $U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

Jump 1 – 2 in domain.

$\mathbb{R}^2 \to \mathbb{R}^m$

$\hookrightarrow$ m functions $\mathbb{R}^2 \to \mathbb{R}$

e_1, e_2

$h = h_1 e_1 + h_2 e_2$

$f(x+h) - f(x)$

$$= \underbrace{f(x+h_1e_1+h_2e_2) - f(x+h_1e_1)}_{\frac{\partial f}{\partial x_2}(x+he_1+\xi e_2)h_2} + \underbrace{f(x+h_1e_1) - f(x)}_{\frac{\partial f}{\partial x_1}(x+he_1)h_1}$$

$\xi(h)$ bet 0 & h_1

$$\frac{\partial f}{\partial x_2}(x)h_2 + \left(\frac{\partial f}{\partial x_2}(x+he_1+\xi e_2)h_2 - f(x)h_2\right) \quad \frac{\partial f}{\partial x_1}(x)h_1$$

$$h_1\left(\frac{\partial f}{\partial x_1}(x+\xi e_1) - \frac{\partial f}{\partial x_1}(x)\right)$$

$$= \boxed{\frac{\partial f}{\partial x_2}(x)h_2 + \frac{\partial f}{\partial x_1}(x)h_1} \quad (\partial_x f)(h)$$

$$+ \varphi_2(x,h)h_2 + \varphi_1(x,h)h_1$$

$$\varphi_1(x,h) = \left(\frac{\partial f}{\partial x_1}(x+he_1) - \frac{\partial f}{\partial x_1}(x)\right)h_1$$

Schwarz ineq $\le \underbrace{\sqrt{\varphi_2^2+\varphi_1^2}\,\sqrt{h_2^2+h_1^2}}$

This is $o(|h|)$

$$\varphi_2(x,h) = \underbrace{\frac{\partial f}{\partial x_2}(x+h_1e_1+\xi e_2) - \frac{\partial f}{\partial x_2}(x+h_1e_1)}_{\to 0}$$

because
$$\frac{\sqrt{\varphi_1^2+\varphi_2^2}\,\sqrt{h_1^2+h_2^2}}{\sqrt{h_1^2+h_2^2}} = \sqrt{\varphi_1^2+\varphi_2^2} \to 0$$

by continuity of $\frac{\partial f}{\partial x_1}, \partial f/\partial x_2$

Class $C^1(U)$: cont $\frac{\partial f}{\partial x^i}$ $i=1,\dots,m$ on U.

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diff $\Leftrightarrow$ "close to linear"

<u>Inverse function thm</u> $\quad$ $d_x f$ cont & invertible at x_0 $\Rightarrow$ f "invertible near x_0"

$U \subset \mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$

f is C^1 $\quad$ $d_x f$ cont func of x.

Suppose $\exists\, x_0 \in U$ s.t. $d_{x_0} f$ is invertible in $L(\mathbb{R}^n)$

$\Rightarrow \exists$ nbhds N_{x_0} of x_0, N_{y_0} of $y_0 = f(x_0)$

s.t. $f: N_{x_0} \to N_{y_0}$

is bijective,

$f^{-1}: N_{y_0} \to N_{x_0}$ is C^1.

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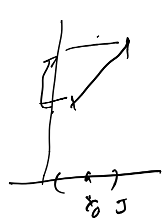
familiar in one variable

$f: I \subset \mathbb{R} \to \mathbb{R}$

$x_0 \in I \quad f'(x_0) \neq 0$

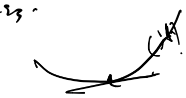


$\exists J \quad x_0 \in J \subset I$

$f'(x_0) \neq 0 \Rightarrow > 0$
< 0

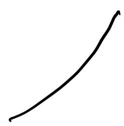
$\Rightarrow \exists J \ni x_0$ s.t. $f'(x) > 0 \quad \forall x \in J$

$\Rightarrow f|J : J \to \mathbb{R}$ increasing.



x^2

$(1+\varepsilon)^2 = 1+2\varepsilon$



$f|J$ injective

$f'(x) > 0$ on J

$\Rightarrow \forall x, y \in J, \ x < y,$
$f(x) < f(y)$

$f(J) =$ interval

because $y_1 = f(x_1)$
$y_2 = f(x_2)$
$\forall y, \ y_1 < y < y_2$
$\exists x \quad x_1 < x < x_2$ intermediate value thm.
$f(x) = y$

in other words
$\forall \ y_1 < y_2 \in f(J), \forall y \ y_1 < y < y_2 \Rightarrow y \in f(J)$

$\Rightarrow f(J)$ an interval, say J'

Next prove $f^{-1} : J' \to J$ is
diff (next time)