Foundations of Analysis II
Week 9

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HW

$(x^2-y^2, 2xy) = (u,v)$

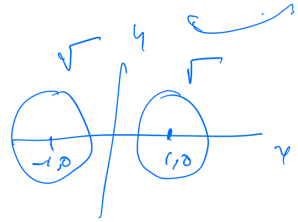
Note on HW: $\mathbb{R}^2 \to \mathbb{R}^2$

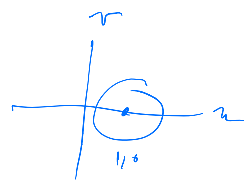
$\pm\sqrt{}$

Since $(x,y) \to (u,v)$ is 2:1, reasonable to have $\sqrt{}$ in formulas

$(1,0) \to (1,0)$

$(-1,0) \to (1,0)$





$f(x,y) = f(-x,-y)$

Note: this is the same as the complex function $z \to z^2$

its inverse is $\sqrt{w}$

$r,\theta \to r^2, 2\theta$

in polar coords

Critical Points

$U^{\text{open}} \subset \mathbb{R}^n$

$f : \underline{U} \longrightarrow \mathbb{R}$

$f(x+h) - f(x)$

Differentiable Functions

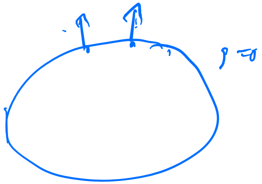
- $f : U \to \mathbb{R}$, where U open.

- Situation that can be reduced to that.

non-singular hypersurface

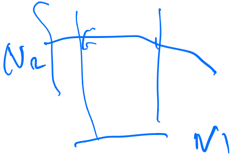
$\{g = 0\}$ $\quad g : U \to \mathbb{R}$ C^1 fun

$\nabla_p g \neq 0$ when $g(p) = 0$



Impl func thm: every $p \in \{g=0\}$ has a nbd

$$x_i = \text{func}(x_1, \dots, \hat{x_i}, \dots, x_n)$$



$$\{g=0\} \cap (N_1 \times N_2)$$

$$= (x, \varphi(x))$$

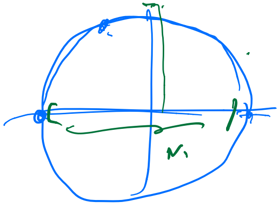
$$\varphi: N_1 \to N_2$$

N_1 open $\subset \mathbb{R}^{n-1}$

Lagrange Multipliers

$x^2+y^2-1=0$

any pt $\neq (\pm 1, 0)$



$(x^2+y^2-1) \cap (-1,1) \to (0, 1+\epsilon)$

$x \to (x, \sqrt{1-x^2}) = \{ (x,y) \in N_1 \times N_2 :$
$x^2+y^2-1=0\}$

f on $x^2+y^2-1=0$ is diff

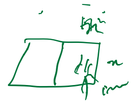
$\Leftrightarrow$ $f(x, \sqrt{1-x^2})$ diff

& $f(x, -\sqrt{1-x^2})$ diff

$$\begin{cases} f(\sqrt{1-y^2}, y) & \text{cl. cc} \\ f(-\sqrt{1-y^2}, y) & \text{diff} \end{cases}$$

"differentiable manifold"

$\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$



“Surfaces”

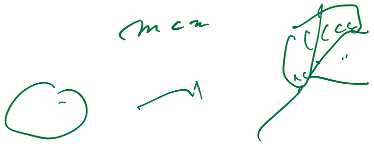
$x^2+y^2+z^2-1=0$

- Equations

- Parametric

U open $\subset \mathbb{R}^m$ $\quad f : U \to \mathbb{R}^n$

$m < n$



Curves $(\cos t, \sin t)$ Image of circle.

$I \xrightarrow{\gamma} \mathbb{R}^n$ parametrized curve

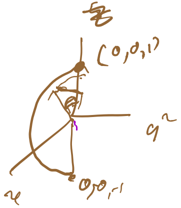
Examples

Equations:

$x^2+y^2+z^2-1=0$

parametric:

Spherical coordinates



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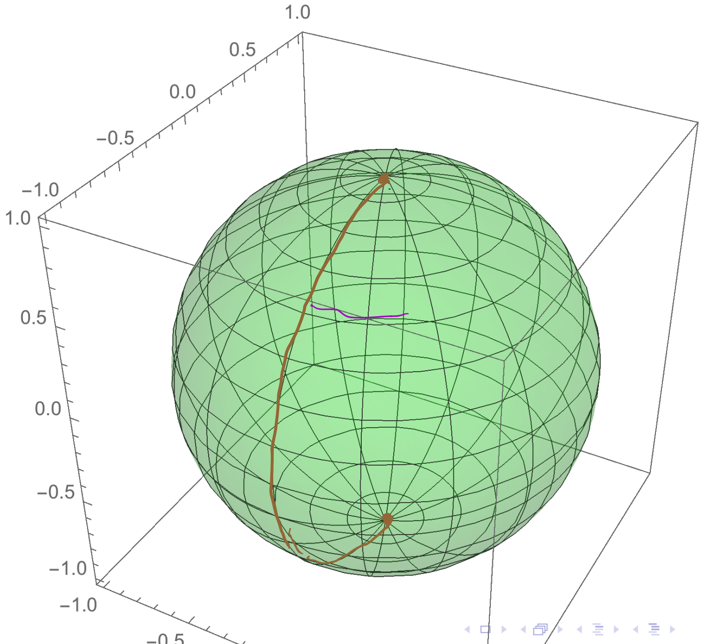


$x = \sin\varphi \qquad 0 \le \varphi \le \pi$
$z = \cos\varphi$

rotate about z axis

$$x = \sin\varphi \cos\theta$$
$$y = \sin\varphi \sin\theta$$
$$z = \cos\varphi$$

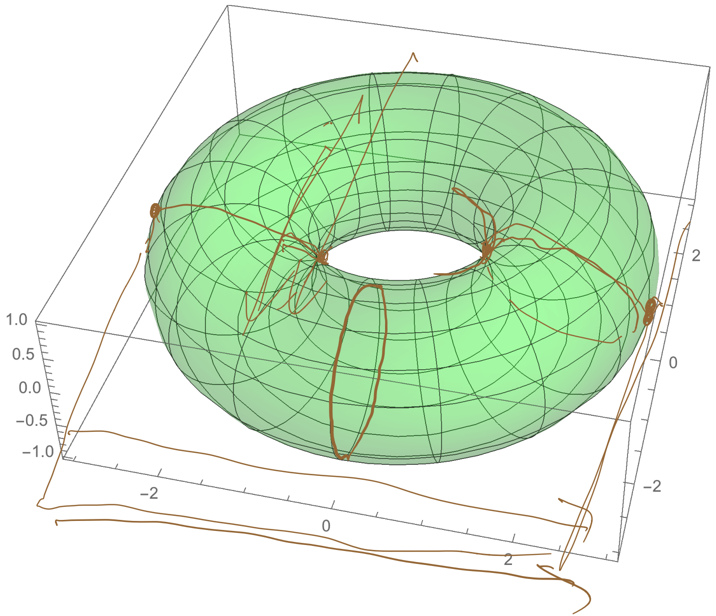
$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$

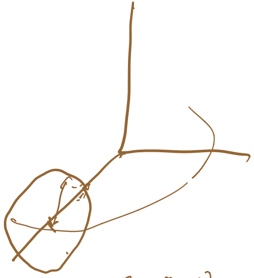
$0 \times [0, 2\pi] \to N = (0, 0, 1)$
$\pi \times [0, 2\pi] \to S = (0, 0, -1)$

$$[0, \pi] \times [0, 2\pi] \xrightarrow{\Phi} \mathbb{R}^3$$

$$\text{image } \Phi = \{x^2 + y^2 + z^2 = 1\}$$

Out[•]=





center $(2, 0, 0)$

radius 1

in x, z plane

$$x = 2 + \cos\varphi$$
$$y = 0$$
$$z = \sin\varphi$$

rotate about z-axis

$$\left\{\begin{array}{l} \varphi + 2m\pi,\ \theta + 2n\pi \\ \varphi,\ \theta \end{array}\right.$$

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$$

$$\varphi, \theta \mapsto \cdot$$

$$\varphi + 2\pi, \theta + 2\pi \mapsto \cdot$$

$$\left\{\begin{array}{l} x = (2 + \cos\varphi)\cos\theta \\ y = (2 + \cos\varphi)\sin\theta \\ z = \sin\varphi \end{array}\right.$$

Higher Derivatives

$f : U \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}$
same ($\mathbb{R}^m$)

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

$d(df)$

f $\mathbb{R}$

df $\mathbb{R}^n$-vld

$d(df)$ $\mathbb{R}^{n \times n}$-vdu

C^1: all $\frac{\partial f}{\partial x_i}$ exist & cont

$\Rightarrow$ diff

C^2: all $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are cont.

C^k — $\dfrac{\partial^\ell f}{\partial x_{i_1} \cdots \partial x_{i_\ell}}$ exist $\ell \le k$ & are cont

C^∞ all exist & cont.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

Symmetry of Second Derivatives

Thm $f \in C^2$, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

$\forall\, i,j$

In fact if for fixed i,j

$\frac{\partial^2 f}{\partial x_i \partial x_j}$ & $\frac{\partial^2 f}{\partial x_j \partial x_i}$ both

exist & are cont

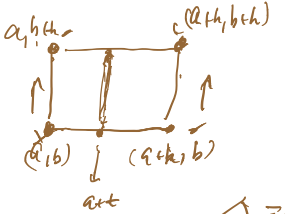
$\Rightarrow$ equal

Rudin: suppose $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists & is cont at (a)

$\Rightarrow \quad \frac{\partial^2 f}{\partial x_j \partial x_i}(a)$ exists & $=$

Lemma $(U \subset \mathbb{R}^2) \xrightarrow{f} \mathbb{R}$ $\quad \forall (a,b) \in U$

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$



$$\boxed{u(t) = f(a+t, b+k) - f(a+t, b)}$$

$$\Delta = u(h) - u(0)$$

$$\boxed{\begin{aligned} &= f(a+h, b+h) - f(a+h, b) \\ &- (f(a, b+h) - f(a, b)) \end{aligned}}$$

MVT $u(h) - u(0) = u'(\xi) h$ for some ξ, $0 < \xi < h$

$$u'(t) = \frac{\partial f}{\partial x}(a+t)\Big|^{b+h} - \frac{\partial f}{\partial x}(a)\Big|^{b}$$

$$u(h) - u(0) = \frac{\partial f}{\partial x}(a+\xi, b+h) - \frac{\partial f}{\partial x}(a+\xi, b)$$

$$u(t) = f(a+t, b+h) - f(a+t, b)$$

$$u'(t) = \frac{\partial f}{\partial x}(a+t, b+h) \boxed{\frac{d(a+t)}{dt}} + \frac{\partial f}{\partial y}(a+t, b+h) \underbrace{\frac{\partial(b+h)}{\partial t}}_{=0}$$

$$- \frac{\partial f}{\partial x}(a+t)(1) - C)0$$

$$= \frac{\partial f}{\partial x}(a+t, b+k) - \frac{\partial f}{\partial x}(a+t, b)$$

$$u(h) - u(0) = u'(\xi) h$$

$$= \Big(\underbrace{\frac{\partial f}{\partial x}(a+\xi, b+h) - \frac{\partial f}{\partial x}(a+\xi, b)}\Big) h$$

$0 < \xi < h$ or $h < \xi < 0$?

$$= \left(\frac{\partial^2 f}{\partial y \, \partial x}(a+\xi, b+\eta) k\right) h$$

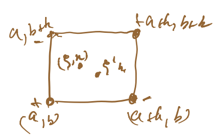
$0 < \eta < k$, $k < \eta < 0$

$$\Delta = \left(\frac{\partial^2 f}{\partial y \, \partial x}(a+\xi, b+\eta)\right) hk$$

for some ξ bet 0 & t, η bet 0 & k

Compute Δ in opposite order

$$\Delta = \frac{\partial^2 f}{\partial x \, \partial y}(a+\xi', b+\eta') hk$$



$$\frac{\partial^2 f}{\partial x \, \partial y}(a+\xi', b+\eta')$$

$$= \frac{\partial^2 f}{\partial y \, \partial x}(a+\xi, b+\eta)$$

$\varepsilon > 0$ $\exists \delta$ s.t. $|h|, |k| < \delta$

$$\rightarrow \left| \frac{\partial^2 f}{\partial x \, \partial y}(a, b) - \frac{\partial^2 f}{\partial x \, \partial y}(a+\alpha, b+\beta) \right| < \varepsilon$$

$|\alpha| < \delta$, $|\beta| < \delta$

$$\left| \frac{\partial^2 f}{\partial y \, \partial x}(a, b) - \frac{\partial^2 f}{\partial x \, \partial y}(a+\alpha, b+\beta) \right| < \varepsilon$$

$$\rightarrow \left| \frac{\partial^2 f}{\partial x \, \partial y}(a, b) - \frac{\partial^2 f}{\partial y \, \partial x}(a, b) \right| < 2\varepsilon$$

$$\Rightarrow =$$

$\mathbb{R}^2 \to \mathbb{R}$

$\Downarrow$ $\mathbb{R}^m \to \mathbb{R}^3$

$f_1(x_1, \ldots, x_m)$
$\vdots$
$f_n(x_1, \ldots, x_m)$

Taylor's Formula

$f \in C^3$

to order 2

$f : (U \subset \mathbb{R}^n) \to \mathbb{R}$

$a \in U$

$a = (a_1, \dots, a_n)$

$h = (h_1, \dots, h_n)$

$$f(a+h) - f(a) = \frac{\partial f}{\partial x_1}(a) h_1 + \frac{\partial f}{\partial x_2}(a) h_2 + \dots + \frac{\partial f}{\partial x_n}(a) h_n + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) h_i h_j + o(|h|^2)$$

$$(h_1, \dots, h_n) \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = h^T A h$$

$A = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)$

symmetric matrix.

1-variable Taylor's

$\varphi(t)$

$$\varphi(t)-\varphi(0)=\varphi'(0)h+\frac{1}{2}\varphi''(0)h^2+o(|h|^2)$$

Apply to $\varphi(t)=f\left(a+t\frac{h}{\|h\|}\right)$

Critical Points

a = critical pt

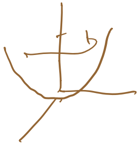
$$\frac{\partial f}{\partial x_i}(a) = 0$$

Taylor $f(a+h) - f(a)$

$$= \frac{1}{2} \sum \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\, h_i h_j$$

$$+ o(|h|^2)$$

= quadratic function + higher order.



$\mathbb{R}^2 \to \mathbb{R}$

a is critical pt

and matrix $\frac{\partial^2 f}{\partial x_i \partial x_j}(a)$ is invertible

$\Rightarrow f \sim$ quadratic function

but in $\mathbb{R}^2$

min $\to x^2+y^2$, $-(x^2+y^2) \leftarrow$ max

x^2-y^2

$\uparrow$ saddle pt.

Taylor's Formula to order 2

- U convex nbhd of $0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ of class C^2.
- Then

$$f(x) - f(0) = \boxed{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0)\, x_i} = \nabla_0 f \cdot x$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(0)\, x_i x_j$$
$$+ \boxed{o(|x|^2)}$$

- (Same for $f : U \to \mathbb{R}^m$)

$f \in C^2$

$f(x) - f(0) =$ linear in x

$+$ quad in x

$+ \to 0$ faster than quadratic

$r(f,x) =$ "remainder"

$$\frac{r(f,x)}{|x|^2} \to 0 \quad \text{as } x \to 0.$$

Proof

of Taylor's formule

$t = 0$ $f(0)$

- Start from

$$\boxed{f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\, dt}$$

- Apply chain rule

$$\frac{\partial}{\partial t} f(tx) = \sum \frac{\partial f}{\partial x_i}(tx) \frac{d(t x_i)}{dt} = \sum \frac{\partial f}{\partial x_i}(tx)\, x_i$$

$$f(x) - f(0) = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) x_i dt$$

$$\sum_{i=1}^n \int \left(\frac{\partial f}{\partial x_i}(0) + \left(\frac{\partial f}{\partial x_i}(tx) - \frac{\partial f}{\partial x_i}(0) \right) \right) dt\, x_i$$

$f(y) - f(0) = \sum \frac{\partial f}{\partial x_i}(0)$

$\int_0^1 \frac{\partial f}{\partial x}(0)\, dt$

- Rewrite

$x=0$

$\int_0^1 \frac{d}{ds} \frac{\partial f}{\partial x_i}(st)\, ds$

$$f(x) - f(0) = \sum_{i=1}^{n} \left(\int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \right) x_i$$

$f(s\,t\,x)\Big|_{s=0}^{s=1} = f(x) - f(0)$

- In other words,

$$f(x) - f(0) = \sum_{i=1}^{n} a_i(x) x_i$$

where $a_i : U \to \mathbb{R}$ are $\mathcal{C}^1$ and $a_i(0) = \frac{\partial f}{\partial x_i}(0)$

$\frac{\partial f}{\partial x_i}(0)$

- Write $a_i(x) = a_i(0) + (a_i(x) - a_i(0))$
- Next write $a_i(x) - a_i(0)$ expliicitly

$$
\begin{aligned}
a_i(x) - a_i(0) &= \int_0^1 \boxed{\left(\frac{\partial f}{\partial x_i}(tx) - \frac{\partial f}{\partial x_i}(0)\right)} dt \\
&= \int_0^1 \left(\int_0^1 \frac{d}{dt}\frac{\partial f}{\partial x_i}(stx)ds\right)dt \\
&= \int_0^1 \int_0^1 \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(stx)\, t x_j\, dsdt \\
&= \sum_{j=1}^{n} \left(\int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_j \partial x_i}(stx)\, t\, dsdt\right) x_j
\end{aligned}
$$

$\frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i}(0)\, x_i x_j$

$\frac{\partial^2 f}{\partial x_j \partial x_i}(0) + \frac{\partial^2 f}{\partial x_j \partial x_i}(stx) - \frac{\partial^2 f}{\partial x_j \partial x_i}(0)$

$r(f,x) = o(|x|^2)$ as 0

$$\int_0^1\int_0^1 \frac{\partial^2 f}{\partial x_j \partial x_i}(0)\, t\, ds\, dt$$

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(0)\left(\int_0^1 ds\right)\left(\int_0^1 t\,dt\right)$$

$1 \quad 1/2$

$$\sum \int_0^1\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(0)\, t\, ds\, dt$$

$$\int_0^1\int_0^1 t\, ds\, dt$$

$\frac{1}{2}$

$$\left(\int_0^1 ds\right)\left(\int_0^1 t\,dt\right)$$

$1 \quad 1/2$

$$\iint \left\{\frac{\partial^2 f}{\partial x_j \partial x_i}(stx) - \frac{\partial^2 f}{\partial x_j \partial x_i}(0)\right\} t\, ds\, dt$$

$$\int \frac{d}{dr} \frac{\partial^2 f}{\partial x_j \partial x_i}(rstx)\, dr$$

st

$$\int_0^1\int_0^1\int_0^1 st^2\, dr\, ds\, dt$$

$$\frac{s^2}{2}\Big|_0^1 \frac{t^3}{3}\Big|_0^1 = \frac{1}{6} = 1/3!$$

- Thus $a_i(x) - a_i(0) = \sum_{j=1}^{n} b_{ji}(x) x_j$ where
  - $b_{ji} : U \to \mathbb{R}$ are continuous.
  - $b_{ji}(0) = \frac{1}{2} \frac{\partial^2 f}{\partial x_j \partial x_i}(0)$
- Put together, gives Taylor's formula

$$f(x) - f(0) = \sum_{i=1}^{n} a_i(0) x_i + \frac{1}{2} \sum_{i,j=1}^{n} b_{ji}(0) x_j x_i + r(f, x)$$

- The "remainder" term

$$r(f, x) = \frac{1}{2} \sum_{j,i=1}^{n} (b_{ji}(x) - b_{ji}(0)) x_j x_i$$

is $o(|x|^2)$ by the continuity of b_{ji}.

Critical Points

$$f(x) - f(0) = \underbrace{\left[\ \ \right]}_{=0} + \frac{1}{2}\left(\sum \frac{\partial^2 f}{\partial x_i \partial x_j}(0)\, x_i x_j\right) + o(|x|^2)$$

Hessian of f at 0:

$$H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)$$ symm $n \times n$ matrix

at a crit. pt. $\quad f(x) - f(0) = \frac{1}{2} x^t \left(H(f)(0)\right) x + o(|x|^2)$

Non-degenerate critical pt:

Hessian matrix is invertible.

$\frac{\partial^2 f}{\partial x_i}(0) \quad 0;$

Morse Lemma

$$\left(\frac{\partial^2 f}{\partial x_i\, \partial x_j}(0)\right) \quad \text{invertible}.$$

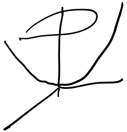
called a non-degenerate critical point.

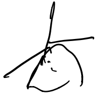
easy examples:

$f(x,y) = x^2 + y^2$

$f(x,y) = x^2 - y^2 = (x+y)(x-y)$

$f(x,y) = -(x^2+y^2)$





Morse Lemma: if 0 is a non-deg crit pt of f, then there is a change of variables, valid in some nbd of 0, $x_1, \dots x_n \to u_1, \dots, u_n$

$$\ni \quad f(u_1, \dots, u_n) = \pm u_1^2 \pm u_2^2 \pm \dots \pm u_n^2$$

Meaning of change of variables: there is a

$$\boxed{\begin{aligned} x_1 &= x_1(u_1, \dots, u_n) \\ x_2 &= x_2(u_1, \dots, u_n) \\ &\vdots \\ x_n &= x_n(u_1, \dots, u_n) \end{aligned}} \quad \text{i.e.} \quad x = \Phi(u)$$

Φ: nbd of 0 in $(u_1, \dots, u_n)$ for nbd of 0 in $(x_1, \dots, x_n)$

So that $f(\Phi(u))$

$$= f(x_1(u_1, \dots, u_n), \dots, x_n(u_1, \dots, u_n)) = \pm u_1^2 \pm \dots \pm u_n^2$$

~~Determinants~~

Prove Morse lemma for $n=2$

$$\boxed{f(x,y) = \underline{a(x,y)}\, x^2 + 2\underline{b(x,y)}\, xy + \underline{c(x,y)}\, y^2}$$

$a(0,0) = \frac{\partial^2 f}{\partial x^2}(0,0)$

$b(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0)$

$c(0,0) = \partial^2 f/\partial y^2 \,(0,0)$

$= a(0,0)x^2 + 2b(0,0)xy + c(0,0)y^2 + \underbrace{(a(x,y) - a(0,0))}_{\to 0} x^2 + \dots$

$|r(x)| \le \underbrace{\|(\begin{smallmatrix} a & b \\ b & c\end{smallmatrix})\|}_{\to 0} \|(x,y)\|^2$

$$\begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}$$

non-deg

$ac - b^2 \neq 0$ at $(0,0)$

$\Rightarrow$ in a nbd of $(0,0)$

where from?

Start $\to$ $\underline{f(x) - f(0)}$ $\int_0^1 \frac{d}{dt} f(tx)\,dt$ $\to$ $\sum_i \left(\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt \right) x_i$

$-\frac{\partial f}{\partial x_i}(0)$

$$= \sum \left(\int_0^1 \underline{\frac{\partial f}{\partial x_i}(tx)}\, dt \right) x_i$$

$$\frac{\partial f}{\partial x_i} = 0 \quad \text{for } i = 1, \dots, n$$

$$\sum \left(\int_0^1 \left(\int_0^1 \underbrace{\frac{d}{ds} \frac{\partial f}{\partial x_i}(stx)}\, ds \right) dt \right) x_i$$

$$\sum_i \int_0^1 \left(\sum_j \frac{\partial^2 f}{\partial x_j \partial x_i}(stx)\, t\, x_j\, ds \right) dt$$

$$\cdots \sum_{i,j} \underbrace{\left(\int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_j \partial x_i} (st x)\, t \, ds \, dt \right)} x_i x_j$$

$$x = 0$$

$$\sum \frac{\partial^2 f}{\partial x_i \partial x_j}(0) \, \frac{1}{2} x_i x_j$$

$$f(x) - \overset{0}{f(0)} = \sum a_{ij}(x)\, x_i x_j \qquad a_{ij}(x)$$

$$a_{ij}(0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0)$$

$f(0,0) = 0$

For $n = 2$

$$\boxed{f(x,y) = a(x,y)\, x^2 + 2b(x,y)\, xy + c(x,y)\, y^2}$$

$$a(0,0) = \frac{\partial^2 f}{\partial x^2} \ldots$$

u, v

$\pm u^2 \pm v^2$

$$\begin{pmatrix} a(0) & b(0) \\ b(0) & c(0) \end{pmatrix} \text{ invertible}$$

$$\boxed{a(0)\, c(0) - b(0)^2 \neq 0}$$

$$\Downarrow$$

$$a(x,y)\, c(x,y) - b(x,y)^2 \neq 0$$

for $(x,y) \in N$ = nbhd of $(0,0)$

harmless assumption:

$a(0) \neq 0$ (by rotating coord. st. $x^2 \neq 0$)

$a(x,y) \neq 0$ on N $\quad > 0$ / < 0

Case ① $a > 0$

$$u_1 = \sqrt{a(x,y)} \left(x + \frac{b(x,y)}{a(x,y)}\, y \right)$$

$$v_1 = y$$

~~Want~~ why $\quad u_1^2 = a(x,y)\left(x^2 + 2\frac{b}{a} xy + \frac{b^2}{a^2} y^2 \right)$

$$= a x^2 + 2bxy + \frac{b^2}{a} y^2$$

$\underbrace{ax^2 + 2ab\,xy}+ cy^2$

$= u_1^2 + \left(c - \frac{b^2}{a}\right) y^2$

$= u_1^2 + \frac{ac - b^2}{a} v_1^2$

need x, y are diff fns of u_1, v_1

Inverse Func thm

$u_1 = \sqrt{a}\left(x + \frac{b}{a} y\right)$

$\frac{\partial u_1}{\partial x}$ ~~$(0,0) =$~~ $\sqrt{a}$

$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix} (0,0)$

$=$

$\frac{\partial u_1}{\partial x} = \frac{\partial}{\partial x}(\sqrt{a}) \left(x + \frac{b}{a} y\right) + (\sqrt{a})(1 + (-1) + 0)$

$\frac{\partial u_1}{\partial x}(0,0) = \sqrt{a}\,(0,0) \neq 0$

$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \\ 0 & \frac{\partial u_2}{\partial y} \end{pmatrix}$ $\begin{pmatrix} \neq 0 & ? \\ 0 & 1 \end{pmatrix}$ invertible

Apply Inverse Func thm: can solve for x, y as func of u, v: $x = x(u,v)$, $y = g(u,v)$

$u_1^2 + \frac{ac - b^2}{a} v_1^2$ $ac - b^2 > 0$

$ac - b^2 < 0$

$u = u_1$

$v = \sqrt{\frac{ac - b^2}{a}}\, v_1$ > 0

< 0

$v = \sqrt{\frac{b^2 - ac}{a}}\, v_1$

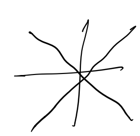
IFT

$a > 0$

$a < 0$

$\boxed{f(x,y) = u^2 \pm v^2}$

$= u^2 \pm v^2$



2 ingredients. choose u_1, v_1

as fns of x, y

$f(x,y)$ (x,y) $\cancel{f(x)} = u_1^2$

$= f(x(u,v), g(u,v))$ $(2,2)$ ~ $u_1^2 + c_1 v_1^2 = f(x)$

$$\begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix} \quad \pm u_1^2 \pm u_2^2 \ldots$$

C^∞

2   02   11   20   30   12   21   03

Morse thry

$v_1 \wedge \cdots \wedge v_n$ A $\big|\ \big|\ \big|$ $v_1 \cdots v_n$

$\det A$

$v_1 \wedge \cdots \wedge v_n$

$|\det A| = \text{vol}$

$v_1 v_2$ $\mathbb{R}^4$

v_1

$(t = 2)$

Exterior Algebra

Grassman Algebra

$A = \sqrt{\ldots}$

Summary 0 a critical pt

$\Leftrightarrow \dfrac{\partial f}{\partial y_i}(0) = 0$ for $i = 1, \dots, n$.

Def Non-deg crit pt.

$\left(\dfrac{\partial^2 f}{\partial y_i \partial y_j}(0) \right)$ is invertible.

a symmetric $n \times n$ matrix

H_{ij} = Hessian matrix of f at 0

Morse Lemma (C^∞)

if 0 is a non-degenerate critical pt of f,

then $\exists$ nbd N_1, N_2 of 0

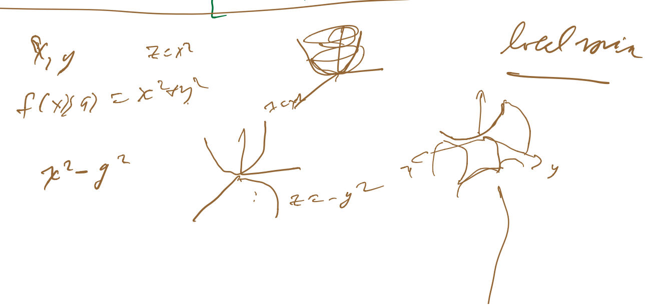
$\Phi : N_2 \xrightarrow{\approx} N_1$ C^∞, invertible, C^∞ inverse

$\Phi(u_1, \dots, u_n) = (x_1(u_1, \dots, u_n), \dots, x_n(u_1, \dots, u_n))$

s.t. $f \circ \Phi = \pm u_1^2 \pm u_2^2 \pm \cdots \pm u_n^2$

$n = 2$ $\exists \Phi$ s.t. $f \circ \Phi(u_1, u_2)$

$$= \begin{cases} u_1^2 + u_2^2 \\ -u_1^2 + u_2^2 \\ u_1^2 - u_2^2 \\ -u_1^2 - u_2^2 \end{cases}$$



$-x^2+y^2$

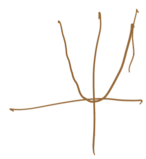
saddle

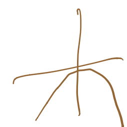
$-x^2-y^2$

local max

$n=1$ $\quad f'(0)$ & $f''(0)\neq 0 \implies$ nondeg

Most of the func





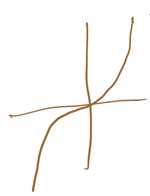
x^2

$-x^2$

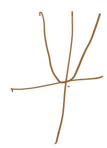
---

deg: $f''(0)=0$

exceptions



x^3



x^4

$-x^4$

$n=2$ $\quad x^2+y^3$ $\quad x^2+y^4$

x^2-y^3

non deg $\Leftrightarrow$ "quad terms dominate"

---

n - variables

$x_1^2+\dots+x_n^2$

$x_1^2+\dots+x_{n-1}^2-x_n^2$

$x_1^2+\dots+x_{n-2}^2-x_{n-1}^2-x_n^2$

x^2+y^2

$x^2-y^2 \Leftrightarrow -x^2+y^2$

$-x^2-y^2$

$x_1^2-x_2^2-\dots-x_n^2$

$-x_1^2-\dots-x_n^2$

$n+1$

$n=1$

$n=2$

NEXT TOPIC:

INTEGRATION over "k-dimensional surfaces" in $\mathbb{R}^n$

($k \le n$)

$k=1$ curve in $\mathbb{R}^n$

$k=2$ surface in $\mathbb{R}^n$

$n=3$ ⋮

n n

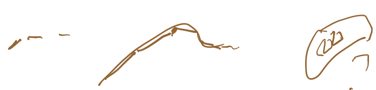
k-dimensional volume in $\mathbb{R}^n$

start from simplest figures

$k=1$ line segments

$k=2$ rectangles → parallelograms

$k=3$ → →



Problem define → integrands

→ integrals

integrate an "integrand"

$\int_a^b \boxed{f(x)\,dx}$ ← integrand over some k-dim surface

$\underset{a\quad b}{\vdash\!\!\dashv}$ ← dregion of integr.

$f(x)$

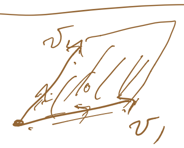
$\boxed{f(x)\,dx}$?

Today vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$

$(k \le n)$

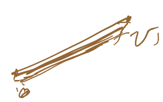
linearly independent

"parallelipiped" $= \{ t_1 v_1 + t_2 v_2 + \cdots + t_k v_k \mid 0 \le t_i \le 1 \}$



$k=2$ in $\mathbb{R}^3$

$k=1$



$k=3$

~~how big~~ · length $k=1$

k-dim. "volume" area $k=2$

Book-keeping mechanism

Grassman algebre also called

Exterior algebra

of $\mathbb{R}^n$

a vector space

$\mathbb{R}^{\binom{n}{0}} \oplus \mathbb{R}^n \oplus \mathbb{R}^{\binom{n}{2}} \cdots \oplus \mathbb{R}^{\binom{n}{3}} \cdots \mathbb{R}^{\binom{n}{n}}$

$\oplus$: direct sum

(cartesian product

$n=3$

$\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$

Λ = capetil Lambda λ Lamda

$\Lambda^0 = \mathbb{R}$

$\Lambda^1 = \mathbb{R}^3$

$\Lambda^2 = \mathbb{R}^{\binom{n}{2}}$

$\Lambda^3 = \mathbb{R}^{\binom{n}{3}}$

$\vdots$

$\Lambda^{n-1} = \mathbb{R}^n$

$\Lambda^n = \mathbb{R}$

to define them,

~~take~~

$\mathbb{R}^n$, inner product (dot product

~~three~~ $e_1, \ldots, e_n$ on ON basis for $\mathbb{R}^n$

ex $e_1 = (1, 0 \ldots 0)$

$e_2 = (0, 1, 0 \ldots 0)$

ex

$\mathbb{R}^3 \quad \Lambda^0 \quad \mathbb{R}$

$e_1, e_2, e_3 \; : \; \Lambda^1(\mathbb{R}^3)$.

Λ^2

$e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3$

$e_1 \wedge e_2 \wedge e_3 \qquad \Lambda^3$

$\mathbb{R}^4 \qquad \mathbb{R} \qquad \Lambda^0$

$\mathbb{R}^4 \qquad \Lambda^1$

e_1, e_2, e_3, e_4

$\mathbb{R}^{6 = \binom{4}{2}} \qquad \Lambda^2$

$e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_1 \wedge e_4$

$e_2 \wedge e_3,\ e_2 \wedge e_4$

$e_3 \wedge e_4$

$\Lambda^3 \qquad \mathbb{R}^4$

$e_1 \wedge e_2 \wedge e_3 \qquad e_1 \wedge e_2 \wedge e_4$

$e_1 \wedge e_3 \wedge e_4$

$e_2 \wedge e_3 \wedge e_4$

$\Lambda^4 \quad e_1 \wedge e_2 \wedge e_3 \wedge e_4$

---

define volume of $\{ \sum t_i v_i \; : \; 0 \le t_i \le 1 \}$

$v_1 = a_1 e_1 + a_2 e_2 + \cdots a_n e_n$

$|v_1| = \sqrt{a_1^2 + \cdots a_n^2}$

in $\mathbb{R}^n$

$v_1 = a_{11} e_1 + a_{21} e_2 + \cdots a_{n1} e_n$

$v_2 = a_{12} e_1 + a_{22} e_2 + \cdots + a_{n2} e_n$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \\ a_{n1} & a_{n2} \end{pmatrix}$

defn $v_1 \wedge v_2 = \cancel{(a_{11} e_1 + a_{21} e_2 \cdots a_{n1} e_n)} \wedge (a_{12} e_1 \cdots a_{n2}$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

$n=3$

$$(a_{11}e_1 + a_{21}e_2 + a_{31}e_3) \wedge (a_{12}e_1 + a_{22}e_2 + a_{32}e_3)$$

$$= a_{11}a_{12}\,\cancel{e_1 \wedge e_1}^{\,0} + a_{11}a_{22}\,\underline{e_1 \wedge e_2} + a_{11}a_{32}\,e_1 \wedge e_3$$

$$a_{21}a_{12}\,\underline{e_2 \wedge e_1} + a_{21}a_{22}\,\cancel{e_2 \wedge e_2}^{\,0} + a_{21}a_{32}\,e_2 \wedge e_3$$

$$\underline{a_{31}a_{12}\,e_3 \wedge e_1} + a_{31}a_{22}\,e_3 \wedge e_2 + a_{31}a_{32}\,\cancel{e_3 \wedge e_3}^{\,0}$$

$$e_i \wedge e_j = -e_j \wedge e_i$$

$$e_i \wedge e_j \wedge e_k = -e_j \wedge e_i \wedge e_k$$

$$=$$

$$e_i \wedge e_i = 0$$

$i \; j \; k$

6

$$(a_{11}a_{22} - a_{21}a_{12})\,e_1 \wedge e_2$$

$$a_{11}a_{32}\,e_1 \wedge e_3 \quad a_{31}a_{12}\,e_3 \wedge e_1$$

$$(a_{11}a_{32} - a_{31}a_{12})\,e_1 \wedge e_3$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad e_1 \wedge e_2$$

$+$

$$\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{e_1 \wedge e_2}, \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}_{e_1 \wedge e_3}, \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{e_2 \wedge e_3} \right)$$