MATH 3220-3 HOMEWORK 2

DUE FEBRUARY 11

(1) (This problem and the next are part of problems 12,13 of Rudin,Chapter 8). Find the Fourier coefficients $\{c_n\}$ and Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$ of the following two functions:

(a) For fixed $\delta > 0$ let

$$f(x) = \begin{cases} 1 & \text{if } |x| < \delta, \\ 0 & \text{if } \delta < |x| < \pi. \end{cases}$$

on $[-\pi, \pi]$ and extend to a periodic function of period 2π to all of $\mathbb{R}$.

(b) Let $f(x) = x$ on $[0, 2\pi)$ and extend to a function periodic of period 2π to all of $\mathbb{R}$.

(2) (a) Write down Parseval's identity (Rudin, Theorem 8.16)

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

for each function.

(b) Check that for the first function it gives

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2},$$

and for the second

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(c) Put $\delta = \frac{\pi}{2}$ in the identity for the first function. Write down explicitly what it gives, and check that it is compatible with the answer for the second function.

(3) (Rudin, Chapter 8, Problem 16) A variation on Fejer's theorem: Prove that if f is Riemann integrable and there is an x with the property that the one-sided limits $f(x^+), f(x^-)$ exist, then

$$\lim_{N\to\infty} \sigma_N(f;x) = \frac{1}{2}(f(x^+) + f(x^-)),$$

where

$$\sigma_N(f;x) = \frac{s_0(f,x) + s_1 f(;x) + \cdots + s_N(f;x)}{N+1}$$

is the sequence of Cesaro means of the Fourier sums $s_N(f,x)$.

Suggestion: Accept all the details of the proof of Fejer's theorem in Rudin, Chap 8, Exercise 15. In particular, accept that

$$\sigma_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt$$

where

$$K_N(t) = \frac{1}{N+1}\frac{1-\cos((N+1)t)}{1-\cos(t)}$$

has properties (a),,(b),(c). The proof of $\sigma_N(f,x) \to f(x)$ uniformly is based on the identity

$$f(x) - \sigma_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x) - f(x-t))K_N(t)dt$$

and the continuity of f. Find a suitable substitute for this identity that will give the result we want when we only know the existence of one-sided limits.

(4) (Rudin, Chp 8, ex 19): Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(x+2\pi) = f(x)$, and suppose $\alpha \in \mathbb{R}$ and $\frac{\alpha}{\pi}$ is irrational. Prove that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

Suggestion: Do it first for $e^{imx}, m \in \mathbb{Z}$.

(5) Recall the Γ-function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

One way to compute $\Gamma(\frac{1}{2}) = \int_0^\infty t^{\frac{1}{2}}e^{-t}dt$ is to make the substitution $t = s^2$, which gives $\Gamma(\frac{1}{2}) = \int_{-\infty}^\infty e^{-s^2}ds$. This can be computed by the trick of relating it to an integral over $\mathbb{R}^2$ and using polar coordinates:

$$(\int_{-\infty}^\infty e^{-x^2}dx)^2 = \int_{-\infty}^\infty e^-(x^2+y^2)dxdy = \int_0^{2\pi}\int_0^\infty e^{-r^2}rdrd\theta = \pi$$

Use this same idea to compute

- s_{n-1} = the volume of $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$
- b_n = the volume of the unit ball $B^n = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| \leq 1\}$

in terms of the Gamma function.

Suggestion: Start from

$$\pi^{\frac{n}{2}} = \int_{-\infty}^\infty \dots \int_{-\infty}^\infty e^{-(x_1^2+\dots+x_n^2)}dx_1\dots dx_n = \int_{S^{n-1}}\int_0^\infty e^{-r^2}r^{n-1}drd\Theta$$