MATH 3220-3 HOMEWORK 3

DUE MARCH 6

(1) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove the following statements:

(a) f is continuous on $\mathbb{R}^2$.
(b) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on all of $\mathbb{R}^2$ and are bounded.
(c) At $(0,0)$ the directional derivatives $D_v f$ exist for all unit vectors $v \in \mathbb{R}^2$.
(d) f is not differentiable at $(0,0)$.

(2) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Prove, directly from the definition of differentiability, that f is differentiable at $(0,0)$, and find its derivative $d_{(0,0)}f$.
(b) Show that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on all of $\mathbb{R}^2$. Observe that this gives another proof of the differentiability of f at $(0,0)$.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = (x^2 - y^2, 2xy) = (u,v)$$

(a) Observe that $f(-x,-y) = f(x,y)$, so f is not (globally) injective.
(b) Use the Inverse Function Theorem to prove that if $(x_0,y_0) \neq (0,0)$, then (x_0,y_0) has a neighborhood U with the property that f maps U bijectively to its image $V = f(U)$.
(c) Prove that $(0,0)$ has no such neighborhood.
(d) Find explicit formulas for a local inverse of $f|_U$ where U is a neighborhood of $(1,0)$.

(4) (Rudin Chap 9, Ex 16) Let

$$f(t) = \begin{cases} t + 2\,t^2 \sin(\frac{1}{t}) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Show

(a) f is differentiable.
(b) $f'(0) = 1$.
(c) f' is bounded on $(-1, 1)$.
(d) f is not one-to-one in any neighborhood of 0. Thus the continuity of f' is needed in the inverse function theorem.

(5) Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be of class $\mathcal{C}^1$ (continuously differentiable). Recall that if $p \in U$ and v is a unit vector, the *directional derivative of f at p in direction v*, $(D_v f)(p)$ is defined to be

$$D_v f(p) = (\frac{d}{dt} f(p + tv))|_{t=0} = d_p f(v) = \nabla_p f \cdot v$$

the second equality by the chain rule, the third the definition of the gradient. A point $p \in U$ is called a *critical point of f* if $d_p f = 0 \iff \nabla_p f = 0$

(a) Prove that if p is a local maximum of f, then it is a critical point of f. Same for a local minimum.
(b) (This is a quick explanation of the Lagrange multiplier method. More details later in class)
If $g : U \to \mathbb{R}$ is also $\mathcal{C}^1$, if $G = \{p \in U \ : \ g(p) = 0\}$ and $d_p g \neq 0$ for all $p \in G$, then the implicit function theorem can be used to rigorously define critical points of the restriction $f|_G$ and to prove that a local maximum or minimum of this restriction is a critical point. Moreover, there is a useful criterion for $p_0 \in G$ to be critical for $f|_G$, the *Lagrange multiplier method*:

$$p_0 \in G \text{ is critical for } f|_G \iff \exists \lambda \in \mathbb{R} \text{ s.t. } \nabla_{p_0} f = \lambda \nabla_{p_0} g.$$

Since, by assumption, $\nabla_p g \neq 0$ for all $p \in G$, the orthogonal complement $\nabla_p g^\perp$ is the *tangent space to G at p* and the Lagrange multiplier condition is equivalent to

$$\nabla_{p_0} f \text{ is perpendicular to } (\nabla_{p_0} g)^\perp$$

or, briefly, $\nabla_{p_0} f$ is perpendicular to G at p_0.
Let's take all this for granted.
(c) Let $x_0 \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be square distance from x_0:

$$f(x) = |x - x_0|^2 = (x - x_0) \cdot (x - x_0)$$

where $u \cdot v$ is the usual dot product of vectors in $\mathbb{R}^n$. Find $\nabla_x f$.
Suggestion: Expand $(x + h - x_0) \cdot (x + h - x_0)$ and compute $d_x f(h)$ directly from the definition of $d_x f$.
(d) As above, let $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$, let $G = \{g = 0\}$ and suppose $\nabla_p g \neq 0$ for all $p \in G$. Suppose $x_0 \notin G$ and suppose $x_1 \in G$ minimizes the distance $|x - x_0]$ for $x \in G$. Prove that $x_1 - x_0$ is perpendicular to G.
Comment: We have used this in the past for g a linear function.