# MATH 3220-3 HOMEWORK 4

DUE APRIL 3

Fix $0 < b < a$ and define $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ by $\Phi(\phi, \theta) = (x(\phi,\theta), y(\phi,\theta), z(\phi,\theta))$ where

$$\begin{aligned} x &= (a + b\ \cos\phi)\cos\theta \\ y &= (a + b\ \cos\phi)\sin\theta \\ z &= b\ \sin\phi \end{aligned}$$

This is a parametrized surface in $\mathbb{R}^3$. Let $T$ denote its image, which is a *torus*. By periodicity, $T = \Phi(D)$ where $D = I_1 \times I_2$ for any two intervals $I_1, I_2$ of length $2\pi$. We have many choices for $D$. In each problem we may make a different choice as convenient.

(1) (a) Sketch $T$. Show the meaning of $a, b, \phi, \theta$ in your sketch.
 (b) Prove that $T$ is the same as the set of solutions of $g(x, y, z) = 0$ where

$$g(x, y, z) = (x^2 + y^2 + z^2 - a^2 - b^2)^2 - 4a^2(b^2 - z^2)$$

 (c) Show that the gradient $\nabla_{(x,y,z)} g \neq 0$ at every $(x, y, z)$ with $g(x, y, z) = 0$
 (d) What would happen if $0 < b = a$? Would the last statement still be true?
 (e) What would happen if $0 = b < a$?
 (f) $g(x, y, z)$ is an equation of degree $4$ in $x, y, z$. Is it possible to define $T$ by equations of smaller degree?

(2) With the same meaning of $\Phi, D$ and $T$, let $f : T \to \mathbb{R}$ be the function $f(x, y, z) = x$ for $(x, y, z) \in T$. (See Rudin, exercise 9.12, with different notation).
 (a) Use the parametrization $\Phi$ to find the critical points of $f$ on $T$. In other words, find the points in $D$ (say, take $D = [0, 2\pi] \times [0.2\pi]$) where the gradient of $f \circ \Phi$ vanishes.
 (b) For each critical point that you found, show that it is non-degenerate and decide if it is a local maximum, local minimum, or saddle point.
 (c) Find the critical points of $f$ by applying the method of Lagrange multipliers to $f$ and the equation $g = 0$ above. Check that you got the same critical points as those that you found using the parametrization.

(3) With the same meaning of $\Phi, D, T$:
 (a) For each $p = (\phi, \theta) \in D$, Find

$$\frac{\partial \Phi}{\partial \phi}(p) \wedge \frac{\partial \Phi}{\partial \theta}(p) \in \Lambda^2(\mathbb{R}^3)$$

(b) Find the norm $\left|\frac{\partial\Phi}{\partial\phi} \wedge \frac{\partial\Phi}{\partial\theta}\right|$.

(c) Use the formula (to be discussed in class) for the area of a parametrized surface $\Phi : D \to \mathbb{R}^n$

$$Area = \int_D \left|\frac{\partial\Phi}{\partial\phi} \wedge \frac{\partial\Phi}{\partial\theta}\right| d\phi d\theta$$

to find the area of $T$. Any surprises? Any remarks?

(d) Take $D = [-\pi/2, 3\pi/2] \times [-\pi/2, 3\pi/2]$, and subdivide $D$ into $D_1$ and $D_2$ by the value of the first coordinate $\phi$, that is, define

$$D_1 = D \cap \{-\pi/2 \leq \phi \leq \pi/2\}, \quad D_2 = D \cap \{\pi/2 \leq \phi \leq 3\pi/2\}$$

Let

$$T_1 = \Phi(D_1), \;\; T_2 = \Phi(D_2)$$

so that $T = T_1 \cup T_2$ with $T_1 \cap T_2$ one-dimensional, hence area zero.
Find the areas of $T_1, T_2$, check that they add to the area of $T$. Finally check with your sketch of $T$ to see if the one you found to be of larger area corresponds to what you see in your sketch.

(4) Same $\Phi, T, D_1, D_2, T_1, T_2$ as in last problem. Find

(a) $\Phi^*(dy \wedge dz)$

(b) $\int_T \Phi^*(dy \wedge dz)$

(c) For $i = 1, 2$ subdivide $D_i = D_{i,1} \cup D_{i,2}$ by the value of the second coordinate $\theta$:

$$D_{i,1} = D_i \cap \{-\pi/2 \leq \theta \leq \pi/2\} \text{ and } D_{i,2} = D_i \cap \{\pi/2 \leq \theta \leq 3\pi/2\}$$

Let $T_{i,j} = \Phi(D_{i,j})$ be the corresponding decomposition of $T$:

$$T = T_{1,1} \cup T_{1,2} \cup T_{2,1} \cup T_{2,2}$$

with all intersections zero or one-dimensional, hence area zero.

(i) Find the four values of

$$\int_{T_{i,j}} \Phi^*(dy \wedge dz) \quad \text{for } i, j = 1, 2$$

by computing the integrals of the form you found in part (a) over $D_{i,j}$

(ii) Let $p_{y,z} : T \to \mathbb{R}^2$ be projection on the $(y, z)$-plane: $p_{y,z}(x, y, z) = (y, z)$, Prove (briefly, say by looking at the sketch) that $p_{y,z}|_{T_{i,j}} : T_{i,j} \to \mathbb{R}^2$ is bijective onto its image, with smooth inverse.

(iii) From the general theory it follows that

$$\int_{D_{i,j}} \Phi^*(dy \wedge dz) = \int_{T_{i,j}} (p_{y,z})^*(dy \wedge dz) = \pm Area(p_{y,z}(T_{i,j}))$$

Find these projections $p_{y,z}(T_{i,j})$ (look at your sketch) and check this last equation by using elementary geometry, then compare with your answer in (i).