NONPOSITIVELY CURVED SURFACES IN $\mathbf{R}^3$

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ABSTRACT. We consider complete nonpositively curved surfaces with one end twice continuously differentiably immersed in Euclidean three space. If such a surface is embedded near infinity and has square integrable second fundamental form then it must lie between two parallel planes.

We generalize some results obtained by H. Chan [2], [3] about asymptotically totally geodesic, complete, nonpositivly curved surfaces C^2 immersed in $\mathbb{R}^3$. We show that if such a surface has one end which is embedded near infinity and satisfies the condition that the second fundamental form is square integrable, then it must be contained in a slab, the closed region between two parallel planes. Moreover, the surface must contact the bounding planes of the narrowest slab along locally nonconvex sets heading to infinity. This implies H. Chan's result that a one ended surface with nonpositive curvature having only isolated parabolic points admits no isometric immersion of this type at all [2], [3].

An example of such a surface is the solution of the equation

$$(1+z)x^2 - (1-z)y^2 = 2z(1-z^2)$$

in three space which is topologically the punctured torus. This surface lies in the slab $|z| \leq 1$ and for $x^2 + y^2 > 2$ is the graph of a single valued function $z = u(x, y)$. The curvature is negative except at the lines $z - 1 = x = 0$ and $z + 1 = y = 0$ where it vanishes. Also both the curvature and the square length of the second fundamental form $|\mathcal{A}|^2$ decay at the order of ρ^{-4} as the distance from a point $\rho \to \infty$.

White[24] studied immersed surfaces with square integrable second fundamental form. He showed that if such a surface is nonpositively

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curved, then its ends are properly immersed and the Gauss map converges at the end. Thus ends contribute integer multiples of 2π to the total curvature. Other classes of nonpositively curved surfaces have also been studied. Using a method of Verner[23], Chan studied surfaces whose curvature is strictly negative, whose Gauss map is one to one and convergent near each end and whose total curvature is -2π outside every peripheral closed geodesic [2], [4].

The best known nonexistence results about immersing nonpositively curved surfaces make no extrinsic hypotheses other than regularity. Efimov proved that there are no C^2 immersions of simply connected surfaces with sufficiently negative curvature. This means curvature is bounded above by a negative constant [5], [13], or is slowly varying but allowed to decay inversely with the distance squared [6]. Perelman generalized Efimov's theorem to halfspaces [16]. See Rozendorn's [19] survey. Existence results are obtained by Hong [7] who constructs a global isometric immersion of a simply connected surface with sufficiently regular metric whose negative curvature decays like $\rho^{-2-\epsilon}$ and whose first and second derivatives decay correspondingly, where ρ is the distance from a point. His constructions improve previous results on isometric immersions for strips [12], [17], [20], [21], [22].

One motivation for this study comes from the general relativity question about the possibility of constructing embedding diagrams for initial metrics for the Cauchy problem for Einstein's equations. Misner [14] constructed a complete Riemannian metric $d\sigma^2$ on $\mathbb{S}^1 \times \mathbb{S}^2 - \{(p_0, q_0)\}$ with vanishing scalar curvature. This manifold and its double cover represent a stationary initial time slice of a spacetime with a wormhole or with a pair of black holes. The metric is invariant under rotations about q_0 in the second factor. If $\mathbb{S}' = \mathbb{S}^1$ is a great circle of $\mathbb{S}^2$ through q_0 then the restriction to $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}'$ is the metric

$$\begin{aligned} ds_M^2 &= d\sigma^2\big|_{\mathbb{T}^2} = a^2\phi^4\left(dx^2+dy^2\right) \\ \phi(x,y) &= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{\cosh(x+2\mu n) - \cos y}} \end{aligned}$$

where a, μ are constants and (x,y) are coordinates in the flat torus $\mathbb{R}^2/(2\mu\mathbb{Z} \oplus 2\pi\mathbb{Z})$ and $(p_0,q_0) = (0,0)$ is the end. An embedding diagram is a C^2 isometric immersion of the Misner surface $(\mathbb{T}^2 - \{(p_0,q_0)\}, ds_M^2)$ into $\mathbb{R}^3$. The curvature of ds_M^2 is strictly negative [18] and the end is asymptotic to an end of the $t=0$ slice of the Schwarzschild metric $(\mathbb{R} \times \mathbb{S}^2, d\sigma_S^2)$. In fact $K = \mathbf{O}(\rho^{-3})$ so the Efimov type nonexistence theorems don't apply. Lifting the closed geodesics around the ends to the universal cover gives directions in which the curvature fails to decay, thus the Hong type existence theorems don't obviously apply either.

If one one restricts the Schwarzschild metric to a great circle $\mathbb{S}' \subset \mathbb{S}^2$, then the resulting complete Riemannian metric on $U = \mathbb{R} \times \mathbb{S}'$, *i.e.*, $ds_S^2 = d\sigma_S^2|_U$ admits an isometric embedding into Euclidean space as a parabola of revolution. In particular both K and $|\mathcal{A}|^2$ are $\mathbf{O}(\rho^{-3})$ as $\rho \to \infty$ and so the second fundamental form is in $L^2(U, ds_S^2)$. Since the metric $ds_{\tilde{M}}^2$ away from the black holes is asymptotic to the metric ds_S^2 away from the single Schwarzschild hole, it is natural to postulate that if an embedding diagram for the Misner surface existed, it would be asymptotic to the embedding of the Schwarzschild surface. Price and Romano [18] tried constructing such an embedding diagram by propogating Schwarzschild data toward the black hole numerically, but the solution developed a shock suggesting that no diagram exists. Chan's theorem confirms nonexistence of an embedding diagram with nice end behavior.

Theorem 3. [2], [3] *There is no C^2 isometric immersion in Euclidean three space of the Misner wormhole surface $\mathcal{W} = (\mathbb{T}^2 - \{(p_0, q_0)\}, ds_M^2)$ under the hypothesis that it is embedded near the end and that it has square integrable mean curvature*

$$\int_{\mathcal{W}} H^2 \, da < \infty.$$

Proof. The Misner surface is negatively curved with finite total curvature. Mean curvature is $H = h_{11} + h_{22}$ in a local orthonormal frame. Hence $\int |\mathcal{A}|^2 \, da = \int H^2 - 2K \, da < \infty$ so Theorem 1 and Theorem 2 apply. $\square$

Chan [4] showed nonexistence of C^3 immersions of other physically interesting negatively curved surfaces such as the double cover of the Misner surface and the Brill Lindquist surfaces assuming square integrable second fundamental form and one to one Gauss map near each end.

1. Preliminaries

Let M be a complete, oriented, nonpositively curved surface C^2 immersed in Euclidean space $\mathbb{R}^3$. We assume that M satisfies

$$\int_M |\mathcal{A}|^2 \, da < \infty \tag{1}$$

where the length of the second fundamental form in local orthonomal frame is given by $|\mathcal{A}|^2 = \sum_{i,j=1}^{2} h_{ij}^2$. Hence M has finite total curvature, thus, by Huber's theorem [11], is conformal to a closed Riemann surface with finitely many punctures $\Sigma - \{q_1, \dots q_r\}$. It follows by a theorem of B. White [24] that M is properly immersed near each end and the Gauss map extends to a continuous function $\mathcal{G} : \Sigma \to \mathbb{S}^2$. We orient our surface so that as a point approaches the end q_1, the Gauss map tends to vertical $(0, 0, 1)$. If the surface is embedded near the q_1 end, then in a sufficiently small disk neighborhood of q_1, the surface M is given as the graph of a function $z = u(x, y)$ where $(x, y) \in \mathbb{R}^2$ and $|(x, y)| \geq R$ some $0 < R < \infty$.

Let γ be a round circle in the plane, $D(\gamma)$ the closed disk bounded by γ, $r(\gamma)$ its radius and $s(\gamma)$ its center. Let $a(\gamma) = (a_1, a_2)$ and $c(\gamma)$ be the coordinates of the plane P_γ which we identify with the affine function $P_\gamma(x, y) := a_1x + a_2y + c$ that best approximates the surface in $D(\gamma)$. That is, a and c are uniquely determined by

$$
\begin{aligned}
m(\gamma) :&= \inf_{b_1, b_2, d \in \mathbb{R}} \left\{ \max_{(x,y)\in\gamma} |u(x, y) - b_1x - b_2y - d| \right\} \\
&= \max_{(x,y)\in\gamma} |u(x, y) - P_\gamma(x, y)|.
\end{aligned} \tag{2}
$$

Thus, P_γ is the unique best affine approximation in sup norm to u restricted to γ and $m(\gamma)$ is the error. Let

$$\operatorname{Can}(\gamma) = \{(x, y, z) : (x, y) \in D(\gamma) \text{ and } |z - P_\gamma(x, y)| \leq m(\gamma)\}.$$

Let $P_\gamma^\pm$ denote the upper and lower planes of $\operatorname{Can}(\gamma)$, namely

$$P_\gamma^\pm(x, y) = a(\gamma) \cdot (x, y) + c^\pm(\gamma)$$

where $c^\pm(\gamma) = c(\gamma) \pm m(\gamma)$. The following is an easy special case of Chebychev's Theorem [1].

Lemma 1.1. *A linear function P_γ which best approximates the continuous function u on the circle γ in supremum norm is uniquely characterized by the condition that there are (at least) four points in γ, in order around the circle $\beta_1(\gamma)$, $\beta_2(\gamma)$, $\beta_3(\gamma)$, $\beta_4(\gamma)$ so that β_1 and β_3 are in P_γ^+ and β_2 and β_4 are in P_γ^-. In other words $u - P_\gamma$ alternates between its extreme values $+m(\gamma)$ and $-m(\gamma)$ at these points.*

The following is a simple case of de la Vallée-Poussin's Theorem [1].

Lemma 1.2. *If for some linear function P the difference $u-P$ assumes the values λ_1, $-\lambda_2$, λ_3, $-\lambda_4$ at four consecutive points around the circle γ, where all $\lambda_i > 0$, then the deviation of the best linear approximation to $u \in C(\gamma)$ satisfies $m(\gamma) \geq \min\{\lambda_1, \ldots, \lambda_4\}$.*

The following is proved by S. Bernstein, E. Hopf [9] and R. Osserman [15] using the maximum principle.

Lemma 1.3. *Let $G \subset \mathbb{R}^2$ be a open set and $u \in C^2(G) \cap C(\overline{G})$ such that $u_{xx}u_{yy} - u_{xy}^2 \leq 0$ in G. If $u \geq 0$ on ∂G and G is bounded, then $u \geq 0$ on G. If $u \geq 0$ on ∂G, $u > 0$ somewhere in G and G is unbounded but can be placed in a sector of angle $0 \leq \theta < \pi/2$, say $G \subset \{(x,y) : |y| < x \tan\theta\}$, then there are constants $c > 0$ and large r_0 so that for all $r \geq r_0$ there holds $\sup\{u(x,y) : (x,y) \in G, x^2+y^2 = r^2\} \geq cr$. If $M \subset \mathbb{R}^3$ is C^2 immersed surface with nonpositive curvature and if $R \subset M$ is a bounded subset, then R is contained in the convex hull of ∂R.*

E. Hopf [8] proved the following accessibility lemma to fill a gap in Bernstein's proof of his theorem about entire nonpositively curved functions.

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\Omega' \subset \Omega$ be a connected open subset whose boundary has at least one point in common with $\partial\Omega$. Then the set $C = \partial\Omega \cap \partial\Omega'$ contains points which are accumulation points of a Jordan curve $J : [0,\infty) \to \Omega$ as $t \to \infty$. If C is the union of two disjoint closed sets, then each contains accumulation points of such J.*

2. Construction of arcs to infinity

If the error made by the best approximating plane for a fixed circle increases as the radius increases, then it must grow at least linearly. First we wish show that there is a circle γ such that the planes $P_\gamma^\pm$ are transverse to $u(x,y)$ at least at one of the extreme points β_i.

Lemma 2.1. *Let $A = \{(x,y) \in \mathbb{R}^2 : r_1 \leq |(x,y)| \leq r_2\}$ be an annulus and let $\Gamma(r) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = r^2\}$ and Γ_1 and Γ_2 be the inner and outer boundary circles of A, resp. Suppose that M is a compact piece of a C^2 surface with $K \leq 0$ immersed in the closed solid cylinder $\Sigma = D(\Gamma_2)\times\mathbb{R}$ of $\mathbb{R}^3$ so that $\partial M \subset \partial\Sigma$. Further suppose that the surface is a graph given by $z = u(x,y)$ in the region $A \times \mathbb{R}$ where $u \in C^2(A)$. Let $m(r) = m(\Gamma(r))$ be the error function (2). Choose $\eta > 0$. Then whenever $r_1 < r < r_2$ such that m is differentiable at r and $m'(r) > 0$,* *then one of the points $\beta_i \in \Gamma(r)$ is a point where u is not tangent to $P^{\pm}_{\gamma}$. That is, $\nabla u(\beta_i) \neq a(\Gamma(r))$ for some $i \in \{1,2,3,4\}$. If $m(\Gamma_1) < m(\Gamma_2)$ then there is such $r \in (r_1, r_2)$. In fact, r can be chosen so that*

$$|a(\Gamma_1) - a(\Gamma(r))| < \eta. \tag{3}$$

Proof. The idea of the proof is that if $P^{\pm}$ is tangent to u at all minima and maxima then $m(\Gamma(r))$ would be constant.

First observe that $m(\Gamma(r))$ is a nondecreasing function. This follows from the maximum principle. For small $h > 0$ the $\mathrm{Can}(\Gamma(r+h))$ contains the surface within $\Gamma(r)$, *i.e.*, $M \cap D(\Gamma(r)) \times \mathbb{R} \subset \mathrm{Can}(\Gamma(r+h))$. Similarly $m(\gamma') \leq m(\gamma'')$ whenever the disks bounded by the circles satisfy $B(\gamma') \subset B(\gamma'')$.

Let $\|u\|_{C^2(A)} < k/3 < \infty$ be a bound on second derivatives in the compact set A. So for contradiction, suppose for all $r \in (r_1, r_2)$ that $\nabla u(\beta) = \nabla P_{\Gamma(r)}(\beta)$ whenever $\beta \in \Gamma(r)$ is extremal, that is, such that $u(\beta) \in P^{+}_{\Gamma(r)}(\beta) \cup P^{-}_{\Gamma(r)}(\beta)$. We prove that left derivate of m exists and vanishes at r. By subtracting the affine function $a(\Gamma(r)) \cdot (x,y) + c(\Gamma(r))$ from u we may suppose that $P^{+}(\Gamma(r)) = c = m(r)$ and that the gradient vanishes. Thus $|u(\beta+\delta) - u(\beta)| \leq k|\delta|^2$ whenever β is a critical point for u such as at one of the extrema. It follows for any $0 < h < r - r_1$ that if $\beta_i' \in \Gamma(r-h)$ denote the four points radially inward from the $\beta_i \in \Gamma(r)$ then $u(\beta_i') \geq c - kh^2$ for $i = 1,3$ and $u(\beta_i') \leq -c + kh^2$ for $i = 2,4$. It follows from de la Vallée Poussin's Theorem, Lemma 1.2, and the fact that m is nondecreasing that $m(r) - kh^2 \leq m(r-h) \leq m(r)$. Together with the fact that m is Lipschitz so differentiable almost everywhere, one concludes that the derivative vanishes wherever it exists, proving the first part of the assertion.

If $m(r) = m(r_1)$ for $r_1 \leq r < r_2$ then $a(r)$ is constant because $\mathrm{Can}(r_1)$ lies between the planes $a(r) \cdot (x,y) + c(r) \pm m(r_1)$ for all r. If we choose $r_3 = \sup\{r \in [r_1, r_2) : m(r) = m(r_1)\}$ then by continuity and constancy of $a(r)$, we have $a(r_3) = a(r_1)$. For $\epsilon < r_2 - r_3$ sufficiently small, we have (3) whenever $|r - r_3| < \varepsilon$. Now just apply the first argument to the interval $r_1 = r_3$ and $r_2 = r_3 + \varepsilon$. $\square$

The circle just obtained may be modified so that the best approximating plane is not horizontal.

Lemma 2.2. *Let A, $\Gamma(r)$ and Σ as in Lemma 2.1. Suppose that M is a compact piece of a C^2 surface, immersed in the closed solid cylinder Σ with $\partial M \subset \partial\Sigma$ and $K \leq 0$. Further suppose that the surface is a* *graph given by $z = u(x,y)$ in the region $A \times \mathbb{R}$ where $u \in C^2(A)$. Then whenever $r_1 < r_2$ and $m(r_1) < m(r_2)$ then there is a circle $\gamma \subset A(r_1, r_2)$ such that one of the $\beta_i \in \gamma$ is not a point of tangency between $P^{\pm}_{\gamma}$ and u. That is, $\nabla u(\beta_i) \neq a(\gamma)$ for some $i \in \{1,2,3,4\}$. Moreover, γ can be chosen so that $a(\gamma) \neq 0$*

Proof. If some circle from Lemma 2.1 satisfies $a(\Gamma(r)) \neq 0$ then we are done. The idea is to show that by translating a circle it is possible to find one such that $a(\gamma') \neq 0$. Then with $\eta = |a(\gamma')|$ we apply Lemma 2.1 to circles concentric to γ' to find one satisfying the transversality.

Choose $r_1 < r_3 < r_4 < r_2$ so that $m(r_1) < m(r_3) < m(r_4) < m(r_2)$ and $\varepsilon > 0$ small. As before, let $r \in (r_3, r_4)$ so that u is transverse to $P^+(\Gamma(r))$ at, say, $\beta_1 \in \Gamma(r)$. For $\delta > 0$ small enough we have $r_3 < r - \delta < r + \delta < r_4$ and $u(x,y) > u(\beta_i) - \varepsilon$ for $(x,y) \in B(\beta_i, \delta)$, $i = 1,3$ and $u(x,y) < u(\beta_i) + \varepsilon$ for $(x,y) \in B(\beta_i, \delta)$, $i = 2,4$. Now choose $r_3 < r_5 < r$ and let $\gamma = \Gamma(r_5) + t_1\beta_1$ for small $t_1 > 0$ be the translate of the circle toward β_1 which remains in the annulus $\gamma \subset A(r_3, r) \cup B(\beta_1, \delta)$. Thus $L = B(\beta_1, \delta) - B(\Gamma(r)) \subset A(r_3, r_2)$ is a crescent shaped region outside γ. By construction, $\varepsilon_2 = \sup\{u(x,y) - P^+_{\Gamma(r)}(x,y) : (x,y) \in L\} > 0$. By continuity, there is a smallest $0 < t_2 < t_1$ so that $\sup\{u(x,y) - P^+_{\Gamma(r)}(x,y) : (x,y) \in \partial B(t_2\beta_1, r_5)\} = \varepsilon_2/2$. Observe that for this circle $\gamma' = \partial B(t_2\beta_1, r_5)$ we have a maximum of $u(x,y) - P^+_{\Gamma(r)}$ attained in L but nowhere else. In particular, near β_2, β_4 on γ' we have $u(x,y) - P^-_{\Gamma(r)} < \varepsilon$ and near β_3, $u(x,y) - P^+_{\Gamma(r)} \leq 0$ since $\gamma' - B(\beta_1, \delta) \subset B(0, r)$. Thus $a(\Gamma(r)) \neq a(\gamma')$ since $P^+_{\Gamma(r)}$ does not satisfy Chebychev's criterion for optimality. Notice also that $m(r_1) < m(\gamma') < m(r_2)$ and $B(0, r_3) \subset B(\gamma') \subset B(0, r) \cup B(\beta_1, \delta) \subset B(0, r_4)$ so the lemma may be applied to circles parallel to γ'. $\square$

Our argument now follows the middle of E. Hopf's proof of Bernstein's theorem about nonpositively curved entire graphs [9].

Lemma 2.3. *Let M be a complete surface with nonpositive curvature and one end. Suppose that $M \subset \mathbb{R}^3$ is a C^2 immersion such that the surface is embedded near infinity as the graph of a function, i.e. for some $r_6 < \infty$, M is the graph of a function $z = u(x,y)$ outside the disk $B(0, r_6)$. Suppose that u has sublinear growth. That is, there exists a continuous function $\epsilon : [r_6, \infty) \to \mathbf{R}$ so that $\sup\{|u(x,y)| : (x,y) \in D(r)\} < r\epsilon(r)$ for all $r \geq r_6$ and $\epsilon(r) \to 0$ as $r \to \infty$. Let $P = P^+_{\gamma}$ be upper best approximating plane for a circle γ surrounding $B(0, r_6)$ such that $z \leq P(x,y)$ for all $(x,y,z) \in M \cap D(r_6) \times \mathbb{R}$ and $\nabla P \neq 0$. Further suppose that $U = \{(x,y) \in \mathbb{R}^2 : u(x,y) > P(x,y)\} \neq \emptyset$. Then* *each connected component $\Omega \subset U$ contains a Jordan curve $J : \mathbb{R} \to \Omega$ such that $J(t) \to \infty$ as $t \to \pm\infty$ and $\lim_{t\to\pm\infty} J(t)/|J(t)| = \pm v$ where v is a unit vector perpendicular to ∇P.*

Proof. By enlarging r_6 and translating a distance δ we may suppose $\gamma = \Gamma(r_6)$. We can adjust $\varepsilon(r)$ accordingly by replacing it with $(r+\delta)\varepsilon(r+\delta)/r$. And we may assume that ϵ is nonincreasing by replacing it by $\sup\{\epsilon(\rho) : \rho \geq r\}$ if necessary. By rotation and vertical translation, we arrange $P(0,0) = 0$ and $\nabla P = (0, q_0)$ for some $q_0 > 0$ so that $P(x,y) = q_0 y$. Let $\xi(x,y) := u(x,y) - P(x,y)$. Observe that M is a graph where $\xi > 0$ because $z \leq P^+_\gamma(x,y)$ for $(x,y,z) \in M \cap D(r_6) \times \mathbf{R}$.

Choose $r_8 > r_6$ so large that $\epsilon(r) < q_0$ whenever $r \geq r_8$. Define a region $S = B(0,r_8) \cup \{(x,y) \in \mathbb{R}^2 : |y| < r\epsilon(r)/q_0\}$. S is bounded by the continuous curves $L^\pm$ where $L^+ = L^+_{-1} \cup L^+_0 \cup L^+_1$. L^+_1 is the curve $y = +r\epsilon(r)/q_0$ for $r > r_8$ and $x > 0$. This is simply in polar coordinates $\sin(\theta) = y/r = \epsilon(r)/q_0$ which tends to zero as $x \to \infty$. The other curves L^-_1 is the reflection across the x-axis and $L^\pm_{-1}$ are the reflections of $L^\pm_1$ across the y-axis. $L^\pm_0$ are the circular arcs $r = r_8$ in the upper and lower halfplanes, resp. Now $\xi \leq u - q_0 y < r\epsilon - r\epsilon = 0$ on $L^+_{\pm 1}$ and $\xi > 0$ on $L^-_{\pm 1}$. On L^+_0 the same holds since it holds at $L^+_0 \cap L^+_1$ and y is larger on the rest of L^+_0 whereas $|u| \leq r_8\epsilon(r_8)$ in $B(0,r_8)$. Indeed $\xi < 0$ above L^+ since the argument holds for arcs concentric to L^+_0 of larger radius. Thus the nodal set $\xi^{-1}(0)$ is contained in S.

Now to prove that Ω contains J. If Ω has a point in common with L^- then it contains all of L^-. Of course it contains no point of L^+. Thus assume that Ω does not meet $L^+ \cup L^-$ so $\Omega \subset S$. Let Ω_s be a nonempty component of the set $\xi^{-1}(s,\infty) \cap \Omega$ for some small regular value $s > 0$. The closure is thus $\overline{\Omega_s} \subset \Omega$. Let $\tilde{\mathcal{D}}$ denote the closed disk which is the geometric compactification of $\mathbb{R}^2$. Let tilde denote compactification in $\mathcal{D}$ and $\mathcal{D}(\infty)$ the circle at infinity. The sets Ω_s and Ω can have common boundary points in the compactification only at $\pm v \in \mathcal{D}(\infty)$. Both $\pm v$ are common boundary points. If $+v$ or $x = +\infty$ is not a boundary point of Ω_s then Ω_s would lie in a halfplane $x < x_0$. $S \cap \Omega_s$ would then lie in a sector and the function $\xi - s$ would be zero on the boundary of Ω_s and positive inside. By Lemma 1.3, Ω_s is unbounded and ξ would have linear growth as $x \to -\infty$ contrary to hypothesis. It follows from E. Hopf's accessibility Lemma 1.4 that both points $\pm v$ are limit points of Jordan arcs from within Ω. □

Even if the error increases with radius so that by Lemma 2.1 there are points where the surface and the upper planes meet transversally, it may happen that for no circle are there points where the lower plane is actually transverse to the surface also. In this case the lower contact set

$$F^- = \{(x,y) \in \mathbb{R}^2 : u(x,y) = P_\gamma^-(x,y)\} \tag{3}$$

has a neighborhood in which points at infinity are accessible.

Lemma 2.4. *Let M be a complete surface with nonpositive curvature and one end. Suppose that $M \subset \mathbb{R}^3$ is a C^2 immersion such that for some $0 < r_1 < r_2 \le \infty$, M is the graph of a function $z = u(x,y)$ outside the cylinder $B(0,r_1) \times \mathbb{R}$. Let $\gamma = \Gamma(r_1)$ and $t_0 \in \gamma \cap F$ where F is the lower contact set* (3). *Let $P = P_\gamma^-$ be lower best approximating plane such that $z \ge P(x,y)$ for all $(x,y,z) \in M \cap (D(r_1) \times \mathbb{R})$. Further suppose that $u \ge P$ for all $(x,y) \in A(r_1,r_2)$. Then for all $\eta > 0$ the set $F_\eta := \{(x,y) \subset \mathbb{R}^2 : u(x,y) < P(x,y) + \eta\}$ contains a Jordan arc $J : [r_1,r_2] \to F_\eta - B(0,r_1)$ such that $J(r_1) = t_0$, $J(r_2) \in \Gamma(r_2)$ if r_2 finite and J is unbounded such that*

$$\lim_{t\to\infty} \frac{J(t)}{|J(t)|} = v \tag{4}$$

converges to a unit vector v if $r_2 = \infty$.

Proof. Write the complementary sets as the disjoint union of open connected components

$$\mathbb{R}^2 - D(r_1) - F = \textstyle\coprod_{i\in I} U_i, \qquad \mathbb{R}^2 - F = \coprod_{k\in K} V_k, \tag{5}$$

where $I, K \subset \mathbb{Z}$ are index sets. For each $i \in I$ there is a unique $k(i) \in K$ such that $U_i \subset V_{k(i)}$ although each V_k may contain several U_i's. We arrange indices so that $i \le 0$ whenever $V_{k(i)} \cap D(r_1) = \emptyset$. Call $\mathcal{T} = \partial(\gamma \cap F)$ the set of boundary points in γ.

First we show that V_k are convex. In Euclidean space it suffices to show that the boundary is locally convex. If this were not the case for some $k \in K$, then there would be a open disk slice region (one of the pieces cut by a secant) $X \subset \mathbb{R}^2$ whose boundary consists of a closed circular arc and an open line segment $\partial X = A \amalg S$ such that $A \subset V_k$ and a point $q \in F \cap X$. There is a plane λ such that $\lambda = P$ over S and which cuts off a cap R from M containing q. Thus we have a contradiction since $\partial R \subset \lambda$ and $q \notin \lambda$ violates the convex hull property Lemma 1.3.

Second, all $\overline{U_i}$ with $i \le 0$ meet $\Gamma(r_2)$. Note that in this case $U_i = V_{k(i)}$. This means U_i with $i \le 0$ are unbounded if $r_2 = \infty$. If not, since

$u - P = 0$ on ∂U_i, by the maximum principle $u - P \equiv 0$ on U_i which contradicts its definition.

We prove the statement in several cases. If t_0 is an interior point of $F \cap \gamma$ then we begin J with an arc in $F \cap \gamma$ to its boundary. Thus we may assume $t_0 \in \mathcal{T}$.

Suppose t_0 is a boundary point of any of the $\overline{U_i}$ which meet both γ and $\Gamma(r_2)$. Since $\partial(U_i \cap D(r_2))$ is locally convex at all points outside $D(r_1)$, a subarc along ∂U_i from γ to $\Gamma(r_2)$ is a Lipschitz path and thus furnishes J. If U_i is unbounded, then (4) converges by convexity. If $\gamma - F$ consists of finitely many intervals, then by proceeding in turn through bounded arcs of $\partial U_i - D(r_1)$ or arcs of interior components of $F \cap \gamma$ we can assemble a path $J \subset F$. If this path reaches a $\overline{U_{i'}}$ which meets $\Gamma(r_2)$ then we continue along this $\partial U_{i'}$ to $\Gamma(r_2)$. If all of the U_i are bounded then eventually the succession of arcs reaches a point where $\max\{|q| : \ q \in \partial U_i \text{ for some } i\}$ is realized, and then the radial path outward from this point connects to $\Gamma(r_2)$ via F. Thus we suppose for the rest of the argument that $\gamma - F$ has infinitely many components.

Thus in the remaining cases there is $s_\infty = \lim_{j\to\infty} s_{i_j}$, a limit point of some pairwise distinct boundary points $s_{i_j} \in \mathcal{T} \cap \partial U_{i_j}$ for some subsequence $\{i_j\} \subset I$. Since the set of limit points is closed in γ, we may assume that s_∞ is a closest limit point to t_0 of F in γ. We may suppose, by restricting to a subsequence if necessary, that U_{i_j} form a strictly monotone sequence ($U_i's$ are pairwise disjoint) and that s_{i_j} is the closest point of $\overline{U_{i_j}}$ to s_∞ along γ. If $s_\infty \neq t_0$, let α be a shortest arc of γ between t_0 and s_∞. We may assume that all $s_{i_j} \in \alpha$, *i.e.* s_{i_j} accumulates from the α side. If not, then there are only finitely many U_i which meet α. By following the corresponding ∂U_i allows us to connect t_0 to either $\Gamma(r_2)$ or to s_∞ through a path in F. If $s_\infty = t_0$ take α to be an arc ending at s_∞ containing the limiting sequence. Let $\rho_i = \sup\{|q| : q \in \partial U_i\}$. Let $\rho' = \sup\{\rho_i : i > 0\}$.

Consider now the case the that $r_2 < \infty$. This case is handled using compactness. Choose any $t_\infty \in \Gamma(r_2) \cap F$. This set is nonempty for otherwise $u - P > 0$ on $\Gamma(r_2)$ so $F \cap D(r_2) = \emptyset$ by the convex hull property. Since $D(r_2)$ is compact, there are only finitely many of the U_i such that $U_i \cap A(r_1, r_2) \neq \emptyset$ which are not contained in F_η. That is because $k = ||u||_{C^1(A(r_1,r_2))} < \infty$ so for any point $q_i \in U_i \cap A(r_1, r_2) - F_\eta$ has $u(x, y) > s - dk$ by the mean value theorem, where d is the distance from q_i to (x, y). Thus $E_i = B(q_i, \eta/k) \cap A(r_1, r_2) \subset F_\eta$ so each E_i is disjoint for different i and has uniformly positive area. Thus only finitely many such i, say $i_1, \ldots, i_m$ can have all E_i fit inside the annulus.

J is constructed as follows. Let ν be a minimum length curve among curves in $A(r_1, r_2)$ from t_0 to t_∞. ν might follow γ for a while and then be a straight line. ν meets finitely many U_{i_j}, say in the order U_{i_1} first, U_{i_2} second and so on. For each U_{i_j} in turn, replace $\nu \cap U_{i_j}$ by an arc of ∂U_{i_j}. If it reaches $\Gamma(r_2)$ we stop. Continue through the j and call the result J. This replacement can happen at most m times along γ and at most m times along the straight line. The perimeters of the sets $U_{i_j} \cap A(r_1, r_2)$ are bounded by $2\pi r_1 + 2\pi r_2$ thus the length of ν is increased by at most $4m\pi(r_1 + r_2)$. We can make it a Jordan arc by omitting loops. We have actually shown how to connect t_0 to $\Gamma(r_2)$ or to t_∞ whichever comes first by a path in F_η.

For the next case we suppose that $r_2 = \infty$ and $r_3 = \rho' < \infty$. This case is handled similarly. Choose $r_3 < r_7 < \infty$. If there are no U_i with $i \le 0$ then choose $t_\infty = r_7 s_\infty \in \Gamma(r_7)$. Then define μ to be the radially outward semiinfinite ray from t_∞ which lies in F. If $U_{i'} \ne \emptyset$ for some $i' \le 0$ choose in addition a $t_\infty \in \partial U_{i'}$, which may require r_7 to be increased, and let μ be an arc of $\partial U_{i'}$ heading to infinity. Since $D(r_7)$ is compact, there are only finitely many of the U_i such that $U_i \cap A(r_1, r_7) \ne \emptyset$ which are not contained in F_η as before.

The next part of J is constructed as follows. Let ν be a minimum length curve $A(r_1, r_7)$ from t_0 to t_∞. Then ν meets finitely many $\overline{U_{i_j}}$. For each U_{i_j} replace $\nu \cap \overline{U_{i_j}}$ by an arc of $\partial U_{i_j} - \gamma$. If it reaches infinity we stop. Continue through the j and call the result ν'. This replacement can happen at most $2m$ times. The perimeters of the sets $U_{i_j} \cap A(r_1, r_7)$ are bounded by $2\pi r_1 + 2\pi r_7$ thus the length of ν is increased by at most $4m\pi(r_1 + r_7)$ whenever a finite arc is added. Thus the desired path is obtained by concatenating $J = \nu' \cdot \mu$. We can make it a Jordan arc by omitting loops. We have actually shown how to connect t_0 to infinity or to t_∞ whichever comes first by a path in F_η. Again, the limit exists as before.

In the last case we have $r_2 = \rho' = \infty$. There is a sequence of positive $\{i'\} \in I$ so that $\operatorname{diam}(U_{i'})$ is unbounded as $i' \to \infty$. There may be a subsequence with $\rho_{i_j} = \infty$ for all j. If not, choose a subsequence so that $\rho_{i_j} > r_1 + j$ is monotonically increasing to infinity. Choose points s_j, t_j in $\partial U_{i_j} \cap \mathcal{T}$ such that $s_j = s_{i_j} \in \gamma$ and the arc $s_j t_j$ of ∂U_{i_j} intersects both bounding circles of $A(r_1, r_1 + j)$. Let $w_j = \overline{s_j t_j} / |\overline{s_j t_j}|$ be a unit vector. We may also arrange that the subsequence $\{i_j\}$ satisfy $|t_j| \ge |t_{j-1}|$. Then choose a subsequence $\{\iota\}$ so $w_\iota \to w_\infty$ converges monotonically. $s_\iota \to s_\infty$ converges monotonically by construction. Let κ_ι denote the ray $\tau \mapsto s_\iota + \tau w_\iota$ and κ_∞ the ray $\tau \mapsto t_0 + \tau w_\infty$. Since $s_\iota \in \overline{U_\iota}$ are disjoint we have $\kappa_\iota([0, \iota]) \subset U_\iota$ so κ_ι converge to κ_∞ uniformly on compact subsets and from one side. We chose s_ι to be the closest point in $\gamma \cap \partial U_\iota$ to s_∞.

Now we shall choose the path inductively which approximates κ_∞. Choose $\ell = \ell(1) > 1$ so that $\kappa_k([0, 3 + 2r_1]) \subset F_\eta \cap N$ for all $k \geq \ell$ where N is the tubular neighborhood of points of distance less than η to κ_∞. In fact any other U_i from the original collection which lies between $U_{\ell(1)}$ and κ_∞ will satisfy $\partial U_i \cap A(r_1, r_1 + 1) \subset F_\eta$. Using the first part of the argument, there are only finitely many of the original U_i which contain points in $A(r_1, r_1 + 1) - F_\eta$. Let ν be a shortest path in $A(r_1, r_1+1)$ from t_0 to a point of the arc $s_\ell t_\ell$ in $\partial U_\ell \cap \Gamma(r_1+1)$. Now deform ν outward along these finitely many ∂U_i. If any of the U_i are unbounded we may stop. Call the resulting path J_1 and parameterize it on $[r_1, r_1+1]$ or $[r_1, \infty)$ if unbounded. It may poke outside $\Gamma(r_1+1)$.

Assume that $J_1, \dots, J_\iota$ have been chosen as finite paths. Choose $\ell = \ell(\iota + 1) > \ell(\iota)$ such that $\kappa_k([0, \iota + 3 + 2r_1]) \subset F_\eta \cap N$ for all $k \geq \ell$. Let $\sigma_{\iota+1}$ be a path in $\partial U_\ell \subset F$ from $\Gamma(r_1+\iota)$ to $\Gamma(r_1+\iota+1)$ between κ_ℓ and κ_∞. Such must exist since the U_ι are disjoint so boundary subarcs must be between $\kappa_{\ell(\iota)}$ and $\kappa_{\ell(\iota+1)}$ and since $|t_{\ell(\iota)}| > r_1 + \iota + 1$. Now let $\nu_{\iota+1}$ be a path along $\Gamma(r_1 + \iota)$ from J_ι to $\sigma_{\iota+1}$ which is in $F_\eta \cap N$ by construction. Let the concatenation $J_{\iota+1} = \nu_{\iota+1} \cdot \sigma_{\iota+1}$ parameterized on $[r_1 + \iota, r_1 + \iota + 1]$. Concatenating $J = J_1 \cdot J_2 \cdot \dots$ gives the desired path in F_η. Omitting loops will make it a Jordan arc. Since $J_i \subset N$ for $i > 1$ also (4) follows with $v = w_\infty$. $\square$

3. Proof of the slab theorem

The ocsillation of u is defined to be

$$\omega(r) = \sup_{(x,y)\in\Gamma(r)} u(x,y) - \inf_{(x,y)\in\Gamma(r)} u(x,y).$$

The following theorem is sharp for complete nonpositively curved surfaces in $\mathbb{R}^3$ in the sense that omitting any hypothesis will allow an example not contained in a slab. The catenoid does not have one end. The Enneper's surface does not have its end embedded. And flat $K = 0$ surfaces, which are generalized cylinders [10] can have arbitrary $\omega(r)$ growth, but don't have $|\mathcal{A}| \in L^2$.

Theorem 1. *Let M be a complete, nonpositively curved Riemannian two manifold with one end. Suppose that M is C^2 immersed in $\mathbb{R}^3$ such that the surface is embedded near infinity and has square integrable second fundamental form* (1)*. Then M lies between two planes, i.e., suppose the surface is oriented so that* $(0,0,1)$ *is the limiting normal* *vector to* M *at infinity, then* $c^+ = \sup\{z : (x,y,z) \in M\}$ *and* $c^- = \inf\{z : (x,y,z) \in M\}$ *are finite.*

Proof. White showed that under (1) the Gauss map of a nonpositively curved surface converges as a point tends toward the end and the end is properly immersed. Thus it is possible to rotate so that the limit of the Gauss map is $(0,0,1)$. Since M is also embedded near the end, for some $r_1 < \infty$, M is the graph of a function $z = u(x,y)$ outside the cylinder $B(0,r_1) \times \mathbb{R}$. Furthermore, by the mean value theorem, the oscillation cannot satisfy the condition that there is a $k > 0$ and $R > r_1$ such that $\omega(r_j) \geq k r_j$ for a sequence $R \leq r_j \to \infty$ without violating convergence of the Gauss map. Thus, u has sublinear growth. Details are in [2], [3].

Let $\Gamma_1 = \partial D(r_1)$ and let $P_1^{\pm}$ be the upper and lower planes of $\operatorname{Can}(\Gamma_1)$, where $P_1^+ = a_1 x + a_2 y + c_1$. If we show that $m(r)$ is constant for $r \geq r_1$ then we are done. In fact, then $a(r) = 0$ and so $p_\infty^{\pm} = c_1 \pm m(r)$. To see this, first observe that $m(r)$ being constant implies $a(r)$ and $c(r)$ are constant. This is simply because the only way that $\operatorname{Can}(\Gamma(r))$ can contain $M \cap (D(r_1) \times \mathbb{R})$ is if $P_{\Gamma(r)}^{\pm} = P_1^{\pm}$ since the minimal planes are unique. Then $a(r) = 0$. If not, along the ray $\zeta(\tau) = \tau a(r)$ we have $|u(\zeta(\tau)) - |a(r)|^2\tau - c(r)| \leq m(r)$ thus by the mean value theorem there is a sequence $\tau_i \to \infty$ along which $d/d\tau\, u(\zeta(\tau_i)) = |a(r)|^2$ so that in particular, the normal $\mathcal{G}(\zeta(\tau_i)) \cdot (0,0,1) \leq |a(r)|^2(1 + |a(r)|^4)^{-1/2} < 1$. It follows that $\mathcal{G}(\zeta(\tau_i))$ does not converge to $(0,0,1)$, a contradictioin.

The rest of the argument rules out the possibility that $m(r)$ is increasing. Thus suppose that $m(r_2) > m(r_1)$ for some $r_2 > r_1$. By Lemma 2.2, for $r_1 < r_3 < r_4 < r_2$ such that $m(r_1) < m(r_3) < m(r_4) < m(r_2)$ there is a circle $\gamma \subset A(r_3, r_4)$ such that u is transverse to one of the $P_\gamma^{\pm}$ at one of the $\beta_i \in \gamma$ and $a(\gamma) \neq 0$. By renumbering and replacing u by $-u$ if necessary, we may suppose that $\nabla u(\beta_1) \neq \nabla P_\gamma^+(\beta_1)$. Let its radius be $r_6 = r(\gamma)$. By translation and rotation we can arrange that the center of γ is the origin, $c(\gamma) = 0$, $a(\gamma) = (0, q_0)$ where $q_0 > 0$. Thus there are $r_5 < r_6 < r_7$ so that the new $A(r_5, r_7)$ is contained in the old $A(r_1, r_2)$. Thus $P_6^+(x,y) = q_0 y + c$ for $c = m(r_6) > 0$. Let Ω be the component of $\{(x,y) \in \mathbb{R}^2 : (x,y,z) \in M \text{ and } z > P_6^+(x,y)\}$ which contains β_1 in its boundary. By transversality there is only one such component. By definitions $\Omega \cap D(r_6) = \emptyset$. By Lemma 2.3, there is a path $J_1 : \mathbb{R} \to \Omega \cup \{\beta_1\} \subset S$ which tends to infinity for both directions $x \to \pm\infty$, where S is the set defined in the proof of Lemma 2.3. Also $\lim_{\tau\to\pm\infty} J_1(\tau)/|J_1(\tau)| = \pm v_1$ for some unit vector v_1 such that $v_1 \cdot \nabla P_6 = 0$, or $J_1 \to \pm v_1$ as $\tau \to \pm\infty$ in the geometric compactification $\mathcal{D}$. By splicing in a path we may assume that $J_1(0) = \beta_1$. There we also proved that $u > P_6^+$ on the lower boundary component L^- of S and $u < P_6^+$ on the upper boundary component L^+.

Consider the sets

$$\begin{aligned} G_s =& \{(x,y) \in \mathbb{R}^2 : (x,y,z) \in M \text{ and } z < q_0 y - c + s\}, \\ F^- =& \{(x,y) \in \mathbb{R}^2 : (x,y,z) \in M \text{ and } z \le q_0 y - c\}. \end{aligned}$$

If G_0 is nonempty, then let $\rho = \inf\{|t| : t \in G_0 - B(0, r_6)\}$ and $t_1 \in \partial G_0$ where $|t_1| = \rho$. Let $\eta = c/2$. If $\rho > r_6$ then as in the proof of Lemma 2.4 we may choose a path in $G_\eta - B(0, r_6)$ connecting β_2 to t_1 and then by Lemma 2.3, extend it to a path $J_2 : (-\infty, \infty) \to G_\eta$ such that $J_2(0) = \beta_2$ and $\lim_{\tau\to\pm\infty} J_2(\tau)/|J_2(\tau)| = \pm v_1$ for the same v as for J_1. If $G_0 = \emptyset$ then we use Lemma 2.4 instead. This time there is a path $J_2 : [0, \infty) \to G_\eta$ such that $J_2(0) = \beta_2$ and (4) converges to the unit vector v_2.

Similarly we may construct a $J_4 : [0, \infty) \to G_\eta$ such that $J_4(0) = \beta_4$ and $J_4/|J_4| \to v_4$ as $\tau \to \infty$. Let

$$\begin{aligned} H_s =& \{(x,y) \in \mathbb{R}^2 : (x,y,z) \in M \text{ and } z > q_0 y + c - s\}, \\ F^+ =& \{(x,y) \in \mathbb{R}^2 : (x,y,z) \in M \text{ and } z \ge q_0 y + c\}. \end{aligned}$$

Note that $H_\eta \cap G_\eta = \emptyset$. By replacing u by $-u$ in Lemma 2.4 we also obtain a path $J_3 : [0, \infty) \to H_\eta$ such that $\lim_{\tau\to\infty} J_3(\tau)/|J_3(\tau)| = v_3$ and $J_3(0) = \beta_3$.

Now observe that these four paths must be disjoint. J_2, J_4 are in G_η which is disjoint from H_η which contains J_1, J_3. To see the pairs are disjoint, suppose that J_1 and J_3 coincide at a point. Thus there is a simple path $\psi : [0,1] \to H_\eta - B(0, r_6)$ connecting β_1 to β_3. Complete the circuit with the straight line segment σ from β_3 to β_1 and call the enclosed region R. z is larger than P_6^- on the loop $\psi \cdot \sigma$. Thus by the convex hull property $z \ge P_6^-$ for all $(x,y,z) \in M \cap (R \times \mathbb{R})$. It follows that $G_0 \cap R = \emptyset$. However G_η meets $R - B(0, r_6)$ since one of β_2 or β_4 are in this set by the Jordan curve theorem, say β_2. This is a contradiction since J_2 can't reach infinity since it can't cross $\psi \subset H_\eta$. Similarly J_1 is disjoint from J_3.

We conclude the argument by analyzing two possibilities. Consider the region R^- bounded by the closed Jordan arc $L^- - J_1$ in $\mathcal{D} \cup \{\pm v_1\}$. Either one of the β_2 or β_4 lies in R^- or neither does. Say $\beta_2 \in R^-$. Since $u - P^- < -c/2$ on J_2 but > 0 on $J_1 \cup L^-$, we must have J_2 an unbounded path between L^- and J_1 and thus $v_2 = \pm v_1$, say $v_2 = v_1 = (1, 0)$. Now consider vertical lines $x = x_0$ for large x_0. The vertical must intersect J_2 at some point y_2. Note that $(x_0, y_2) \subset R^-$. For large and small y the point $(x_0, y) \notin R^-$. Therefore, there must be y_1 and y_3, one greater, the other less than y_2 such that $(x_0, y_1) \in J_1$ and $(x_0, y_3) \in L^-$. Now $u - P_6$ is positive at (x_0, y_i) for $i = 1, 3$ and negative at (x_0, y_2). By the intermediate and mean value theorems, there is y_4 between y_1 and y_3 where $\partial u/\partial y(x_0, y_4) = \partial P_6/\partial y(x_0, y_4) = q_0$. In particular the Gauss map is uniformly bounded away from $(0, 0, 1)$ for a sequence tending to v_1, *i.e.* $|\mathcal{G}(x_0, y_4, u(x_0, y_4)) - (0, 0, 1)|^2 \geq 1 - (1 + q_0^2)^{-1/2}$ for all x_0. This means that a sequence of Gauss maps doesn't converge to $(0, 0, 1)$ as $x_0 \to \infty$ and we have reached a contradiction.

The other case is if none of the β_2, β_4 are in R^-. Consider instead the region R^+ bounded by the closed Jordan arc $J_1 - L^+$ in $\mathcal{D} \cup \{\pm v_1\}$. Since J_3 is disjoint from $J_1 \cup L^+$, (4) must converge to one of the $\pm v_1$, say $J_3 \to +v_1$ as $\tau \to \infty$. Again, one of β_2 or β_4, say β_2, lies in the region bounded by $J_1' - J_3 + \sigma$ where J_1' is the subarc of J_1 from β_1 to v_1 and σ is a secant of $\Gamma(r_6)$ from β_3 to β_1. J_2 is a Jordan arc which connects β_2 to v_1 since it can't access other points of $\mathcal{D}(\infty)$ being bounded by $J_1' \cup J_3$. Now consider vertical lines $x = x_0$ for large x_0. The vertical must intersect J_2 at some point y_2. Note that for large and small y the point $(x_0, y) \notin R^+$. Therefore, there must be y_1 and y_3, one greater, the other less than y_2 such that $y_i \in J_i$ for $i = 1, 3$. Now $u - P_6$ is positive at (x_0, y_i) for $i = 1, 3$ and negative at (x_0, y_2). By the intermediate and mean value theorems, there is y_4 between y_1 and y_3 where $\partial u/\partial y(x_0, y_4) = \partial P_6/\partial y(x_0, y_4) = q_0$. In particular the Gauss map is uniformly bounded away from $(0, 0, 1)$ at some $(x_0, y_4) \in S$. This means that a sequence of Gauss maps doesn't converge to $(0, 0, 1)$ as $x_0 \to \infty$ and we have reached a contradiction. □

We now formulate a theorem which describes the lower contact set established by Theorem 1. It describes the upper contact set as well by reflecting $z \mapsto -z$.

Theorem 2. *Let M be a complete, nonpositively curved Riemannian two manifold with one end. Suppose that M is C^2 immersed in $\mathbb{R}^3$ such that the surface is embedded near infinity and has square integrable second fundamental form* (1). *Suppose the surface is oriented so that* $(0, 0, 1)$ *is the limiting normal vector to M at infinity and so that M is the graph of a function $z = u(x, y)$ for $x^2 + y^2 \geq r_1^2$. Let $c^- = \inf\{z : (x, y, z) \in M\}$ and $F^- = \{(x, y) \in \mathbb{R}^2 : (x, y, c^-) \in M\}$ be the lower contact set. Suppose we decompose the complements into connected components as in* (5). *Then*

(1) *There is at most one* V_i *with* $V_i \subset \mathbb{R}^2 - D(r_1)$ *i.e. so that* $i \le 0$. *Such a* V_0 *must be a halfplane.*
(2) *There is a path* $J : [0, \infty) \to F^-$ *such that* (4) *converges.*
(3) *If* $(x_0, y_0) \in F^-$ *then* $K(x_0, y_0, c^-) = 0$. (*The Gauss curvature vanishes on* F^-.)

Proof. Let V_i be one of the domains with $i \le 0$, thus M is a graph over V_i. Since V_i is convex and not the whole plane, it must lie in a convex sector of angle $\theta \in (0, \pi/2]$. By Lemma 1.3 this is a contradiction to sublinear growth unless $\theta = \pi/2$ and V_i is a halfplane. Thus there are at most two disjoint V_i in the plane for $i \le 0$, call them V_0, V_{-1} which must be parallel. Let $S = \mathbb{R}^2 - V_0 - V_{-1}$ be the strip between them. Note that the boundary of the region of M in the cylinder $S \times \mathbf{R}$ lies in the plane $z = c^-$. Since $\nabla u \to 0$ as distance from the origin $r \to \infty$ we have $\zeta(r) = \sup\{z : (x, y, z) \in M \cap ((\Gamma(r) \cap S) \times \mathbf{R})\} \to c^-$ as $r \to \infty$. It follows by the convex hull property that $(D(r) \cap S) \times \mathbf{R} \cap M \subset (D(r) \cap S) \times [c^-, \zeta(r)]$ which tends to a plane as $r \to \infty$, which is contradiction, proving (1).

If $U_0 \ne \emptyset$ then let J be a ray of ∂U_0. Otherwise, the construction in Lemma 2.4 constructed a path for every $\eta > 0$ in F_η in two steps. The first part was to connect some $t_0 \in F^- \cap \Gamma(r_1)$ to a point of $\Gamma(r_1 + 1) \cap F$ by a path in F_η which may poke out of $D(r_1 + 1)$ and which may be unbounded as $\eta \to 0$. If any of the replacement ∂U_i's are unbounded, then a subarc of ∂U_i is the desired J. If $\rho' < \infty$ then a radial ray lies in F. For $\rho' = \infty$, the second part of the construction was to follow segments of ∂U_i between κ_∞ and κ_ℓ. This part of the path converged to κ_∞ uniformly as $\eta \to 0$, thus in this case, by continuity, $\kappa_\infty \in F$ which is the desired path.

Since $z \ge c^-$ for all $(x, y, z) \in M$ near $Q_0 = (x_0, y_0, c^-)$, it follows that at Q_0 the normal vector to M is $(0, 0, 1)$ and the second fundamental form is nonnegative definite. The only way this can happen for nonpositive curvature is if $K(q_0) = 0$. □

We expect every point of F may be connected to ∞ by a path in F satisfying (4). However, we cannot rule out the possibility that F is not locally path connected.

We end by constructing examples of arbitrary genus, genenralizing the surface mentioned in the second paragraph. In cylindrical coordinates of $\mathbb{R}^3$, for $n \ge 2$ an integer, the surface

$$r^n (z - \cos n\theta) = z - z^3.$$

has genus $n-1$ and lies in the slab $|z| \leq 1$. Its upper contact set F^+ consists of n equally spaced rays emanating from the origin. The curvature vanishes along the rays in the intersection with the planes $z = 0, \pm 1$. Solving for r as a function of z and θ, we express the curvature in terms of derivatives of r and check that the sign of the resulting expression is nonpositive. The computation is facilitated using the symbolic computational package MAPLE.

As with the proofs of Bernstein's theorem, it would be desirable to find another proof here which is purely analytic.

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