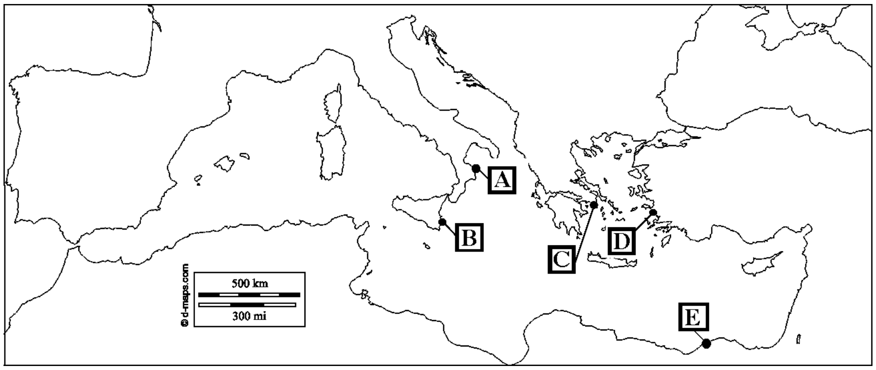
Math 3010 § 1. **First Midterm Exam** Name: Solutions
Treibergs February 7, 2018



1. *For each location, fill in the corresponding map letter. For each mathematician, fill in their principal location by number, and dates and mathematical contribution by letter.*

Mathematician	Location	Dates	Contribution
Archimedes	5	e	β
Euclid	1	d	δ
Plato	2	c	ζ
Pythagoras	3	b	γ
Thales	4	a	α

Locations		**Dates**	**Contributions**
1. Alexandria	*E*	a. 624–547 BC	α. Advocated the deductive method. First man to have a theorem attributed to him.
2. Athens	*C*	b. 580–497 BC	β. Discovered theorems using mechanical intuition for which he later provided rigorous proofs.
3. Croton	*A*	c. 427–346 BC	γ. Explained musical harmony in terms of whole number ratios. Found that some lengths are irrational.
4. Miletus	*D*	d. 330–270 BC	δ. His books set the standard for mathematical rigor until the 19th century.
5. Syracuse	*B*	e. 287–212 BC	ζ. Theorems require sound definitions and proofs. The line and the circle are the purest elements of geometry.

2. *Use the Euclidean algorithm to find the greatest common divisor of 168 and 198. Find two integers x and y so that* $\gcd(198, 168) = 198x + 168y$. *Give another example of a Diophantine equation. What property does it have to be called Diophantine? (Saying that it was invented by Diophantus gets zero points!)*

$$\begin{aligned} 198 &= 1 \cdot 168 + 30 \\ 168 &= 5 \cdot 30 + 18 \\ 30 &= 1 \cdot 18 + 12 \\ 18 &= 1 \cdot 12 + \boxed{6} \\ 12 &= 3 \cdot 6 + 0 \end{aligned}$$

So $\gcd(198, 168) = 6$.

$$\begin{aligned} 6 &= 18 - 12 \\ &= 18 - (30 - 18) && = 2 \cdot 18 - 30 \\ &= 2 \cdot (168 - 5 \cdot 30) - 30 && = 2 \cdot 168 - 11 \cdot 30 \\ &= 2 \cdot 168 - 11 \cdot (198 - 168) && = 13 \cdot 168 - 11 \cdot 198 \end{aligned}$$

Thus $x = \boxed{-11}$ and $y = \boxed{13}$.

The problem of finding Pythagorean triples is a Diophantine equation

$$x^2 + y^2 = z^2$$

An equation is called Diophantine when there are fewer equations than variables, but the solutions are restricted to be integers (or rational numbers). Another example is Pell's equation $y^2 - 2x^2 = 1$.

3. *Determine whether the following statements are true or false. Give a detailed explainantion of ONE of your answers (a)–(d).*

   (a) *There are five distinct regular polyhedra.*

   TRUE. There are five regular polyhedra: tetrahedron, cube. octohedron, dodecahedron and icosohedron.

   One considers the possible shapes of the faces (triangle, square, pentagon and so on) and then computes the possible number m of faces that meet at a vertex. This number has to be at least three for the polyherdon not to be two faces glued together. But it must satisfy $m\alpha_n < 360°$ where α_n is the interior angle of a regular n-gon for the polyhedron to be convex. The only possibilities for triangular faces when $\alpha_3 = 60°$ are $m = 3, 4, 5$. For square faces when $\alpha_4 = 90°$ the only possibilities are $m = 3$ and for pentagonal faces when $\alpha_5 = 108°$ is $m = 3$. $n \geq 6$ is not possible since $3\alpha_n \geq 360°$. That makes five possible figures $(n, m) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$. Finally, all of these polyhedra exist because they can be constructed, for example by computing the coordinates of their vertices in three space.

   (b) *There are only finitely many prime numbers.*

   FALSE. One argues by contradiction. Suppose that there are finitely many primes and the complete list of distinct primes is $\{p_1, p_2, \ldots, p_k\}$. Then one considers the number

   $$x = p_1 \cdot p_2 \cdots p_k + 1.$$

   It is larger than any of the primes in the list, so should be composite, a product of several of them. However, x is not divisible by any of the primes in the list, which is a contradiction. Therefore, there must be infinitely many primes.

(c) *There are only finitely many Pythagorean Triples.*

FALSE. Pythagorean triples are positive integer solutions of the Diophantine equation $x^2 + y^2 = z^2$. Euclid showed that all solutions are of the form

$$x = (p^2 - q^2)r, \qquad y = 2pqr \qquad z = (p^2 + q^2)r$$

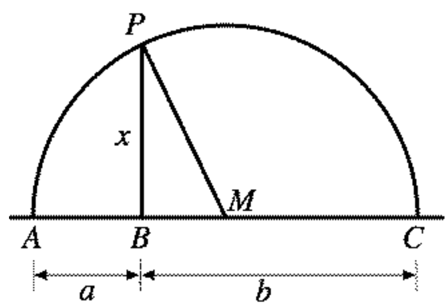
where p, q, r are integers, which range over infinitely many choices of (p, q, r). One infinite family is given by multiples of one solution, e.g., $(x, y, z) = (3r, 4r, 5r)$. Another less trivial family is given by Euclid's formuula with $q = r = 1$ and p tending to infinity. Then the ratios y/x tend to zero showing that all of the triples in this family are not multiples of one another.

(d) *All real numbers are rational.*

FALSE. The numbers $\sqrt{p}$ are not rational for p any prime. Arguing by contradiction, suppose that it were rational so $\sqrt{p} = \dfrac{m}{n}$ where m and n are integers with no common factors. Squaring one finds $pn^2 = m^2$ so $p \mid m^2$. Because p is prime, this means that $p \mid m$, or in other words $m = p\ell$ for some integer ℓ. But this means that $pn^2 = p^2\ell^2$ or $n^2 = p\ell^2$. As before, this means $p \mid n^2$ so $p \mid n$. We have reached a contradiction: p is a common factor to both m and n. Thus $\sqrt{p}$ could not have been rational.

4. *The* geometric mean *of two positive magnitudes a and b is given by $\mathcal{G}(a, b) = \sqrt{ab}$. Show that the geometric mean can be constructed using straightedge and compass. Explain why your construction gives the geometric mean. The arithmetic mean is given by $\mathcal{A}(a, b) = \dfrac{a + b}{2}$. Show how to see from your diagram that $\mathcal{G}(a, b) \le \mathcal{A}(a, b)$.*

What famous problems resisted solution by straightedge and compass by the Greeks? Were these problems ever solved by straightedge and compass? Were they ever solved by by other means?



Here is the construction of the geometric mean of a and b. Put three points A, B and C on the line so that the lengths $L(AB) = a$ and $L(BC) = b$. Construct a circle whose diameter is AC. That is, find the midpoint M between A and C so that $r = L(AM) = L(MC) = \dfrac{a + b}{2}$. Draw a perpendicular to the line at B. The perpendicular intersects the circle at P. The geometric mean is $x = L(BP)$.

To see it, recall that the triangles $\triangle(ABP)$ and $\triangle(PBC)$ are similar right triangles. The ratio of side lengths are the same

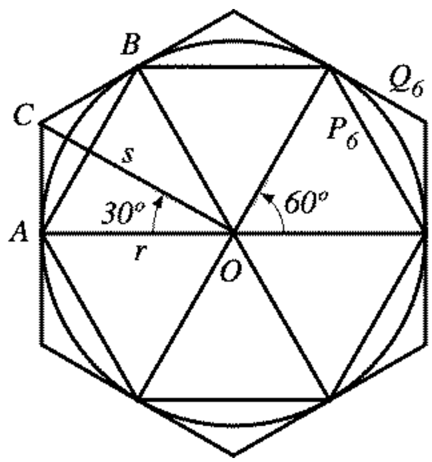
$$\frac{x}{a} = \frac{b}{x} \qquad \text{so } x^2 = ab \text{ or } x = \sqrt{ab} = G(a, b).$$

The hypotenuse of triangle $\triangle(MBP)$ of length $r = A(a, b)$ is longer than either leg: $r \ge x$ or $A(a, b) \ge G(a, b)$.

Three famous problems that Greeks couldn't solve using only straightedge and compass are the *doubling of the cube, trisecting an angle* and *squaring the circle.* It's been proved that these problems can't be solved using straightedge and compass only. However, the Greeks were able to find solutions to these problems by using other graphic devices.

5. *Even if the ratio* $\frac{a}{b}$ *were irrational, how would Eudoxus show that it equals* $\frac{c}{d}$ *using only rational numbers? Compute the area of the inscribed hexagon which approximates the area of a circle of radius* r*. How close is the approximation to the area? (You may use* $30°-60°-90°$ *triangles but may not use your knowledge of trigonometric functions or* π*.)*

In Eudoxus's theory of proportions, he says that $\frac{a}{b}$ equals $\frac{c}{d}$ if whenever there are rational numbers p and q such that $p < \frac{a}{b} < q$, then $p < \frac{c}{d} < q$.



Let A denote the area of the circle C. Label the origin O and two neighboring vertices A and B of the hexagon. The height of the triangle $\triangle(OAB)$ is $\frac{\sqrt{3}}{2}r$. This can be seen from the Pythagorean theorem applied to a right triangle whose hypotenuse length is r and the leg lengths are $\frac{r}{2}$ and the height. The area of a single triangle

$$A(\triangle(AOB)) = \frac{1}{2} \cdot \text{height} \cdot \text{base} = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} r \cdot r = \frac{\sqrt{3}\, r^2}{4}.$$

Thus the area of the approximating hexagon is six times the area of one triangle, or

$$A(P_6) = 6 \cdot A(\triangle(AOB)) = \frac{3\sqrt{3}\, r^2}{2}$$

Let Q_6 be the circumscribing hexagon. Since $P_6 \subset C \subset Q_6$ we have $A(P_6) \le A \le A(Q_6)$. It follows that the error in estimating the area of the circle by the area of the inscribed hexagon satisfies

$$0 \le A - A(P_6) \le A(Q_6) - A(P_6).$$

The radius of Q_6 is s. It is the hyporenuse of the $30° - 60° - 90°$ triangle $\triangle(OAC)$ so has length $s = \frac{2r}{\sqrt{3}}$. The areas grow like the squares of the radii so that

$$\frac{A(Q_6)}{A(P_6)} = \frac{s^2}{r^2} = \frac{4}{3}.$$

Thus, the error of approximation is no more than

$$0 \leq A - A(P_6) \leq A(Q_6) - A(P_6) = A(P_6)\left(\frac{A(Q_6)}{A(P_6)} - 1\right) = \frac{1}{3}A(P_6) = \frac{\sqrt{3}\,r^2}{2}.$$

This can be seen geometrically too. The area of the region between P_6 and Q_6 may be rearranged to make two triangles.