Math 3010 § 1. **First Midterm Exam** Name: Solutions
Treibergs February 7, 2024

1. *For each mathematician fill in their principal location from the list and write a short statement of their mathematical contribution.*

Mathematician	Location	Mathematical Contribution
Thales 624 – 547 BC	Miletus	Proved earliest geometric theorems, e.g., base angles of an isoceles triangle are equal..
Pythagoras 580 – 497 BC	Croton	Figurate numbers, Pythagorean triples Irrationality of $\sqrt{2}$, theory of harmony
Zeno 490 – 425 BC	Elia	Posed paradoxes, e.g., "Achilles," challenging intuition that space / time are discrete.
Plato 429 – 348 BC	Athens	Delian problems were studied in his Academy. Devised way to double the cube.
Eudoxus 400 – 347 BC	Cnidus	Resolved Xeno's questions by a theory of irrationals. Anticipated calculus limits.
Aristotle 384 – 322 BC	Athens	Developed a theory of rigorous argument via axiomatic approach. Described syllogisms.
Euclid 330 – 270 BC	Alexandria	Organized all mathematics in *Elements,* a model of mathematical exposition.

Locations (Several may be in the same location.)

Alexandria, Athens, Cnidus, Croton, Elia, Miletus, Syracuse

2. (a) *Use the Babylonian method and sexigesimal arithmetic to compute the quotient. (Other methods receive zero credit.)*

$1,50 \quad \div \quad 9 \quad =$

Observe that

$$\frac{1}{9} = \frac{1}{3} \times \frac{1}{3} = \frac{20}{60} \times \frac{40}{60} = \frac{400}{3600} = \frac{360+40}{3600} = \frac{6}{60} + \frac{40}{3600} =; 6, 40.$$

Thus we multiply $1,50$ by $\frac{1}{9}$. We expct an answer to be about $110 \times \frac{1}{9} \approx 12$.

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1, 50
     ;   6,   40
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     ;  40, 2000
    6; 300
----------------
    5; 33
     ; 40, 20
    6;  0
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   11; 73, 20
----------------
   12; 13, 20
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where we have used

$$2000 = 1980 + 20 = 33 \times 60 + 20$$

and

$$300 = 5 \times 60.$$

Not necessary as a part of your solution but we check:

$$\begin{aligned} \frac{110}{9} = \frac{108+2}{9} &= 12 + \frac{2}{9} = 12 + \frac{800}{3600} = 12 + \frac{780+20}{3600} \\ &= 12 + \frac{13 \times 60 + 20}{3600} = 12 + \frac{13}{60} + \frac{20}{3600} = 12; 13; 20. \end{aligned}$$

(b) *Using the Egyptian method of doubling, find the product $22 \times 2\overline{6}$. (Other methods receive zero credit.)*

The first method uses $\frac{2}{3} = \overline{\overline{3}}$.

1	$2\,\overline{6}$
2	$4\,\overline{3}$
4	$8\,\overline{\overline{3}}$
8	$17\,\overline{3}$
16	$34\,\overline{\overline{3}}$

where we have used $2 \times \overline{\overline{3}} = 2 \times \frac{2}{3} = \frac{4}{3} = 1\overline{3}$. Now because

$$22 = 16 + 4 + 2$$

we see that $22 \times 2\ \overline{6}$ equals the sum of the corresponding terms in the right column

$$\begin{array}{rr} & 34\ \overline{\overline{3}} \\ & 8\ \overline{\overline{3}} \\ + & 4\ \overline{3} \\ \hline & 43\ \overline{3} \\ + & 4\ \overline{3} \\ \hline & 47\ \overline{\overline{3}} \end{array}$$

In the second method we use $2 \times \overline{3} = \frac{2}{3} = \frac{4}{6} + \frac{3+1}{6} = \overline{2}\ \overline{6}$.

$$\begin{array}{rr} 1 & 2\ \overline{6} \\ 2 & 4\ \overline{3} \\ 4 & 8\ \overline{2}\ \overline{6} \\ 8 & 17\ \overline{3} \\ 16 & 34\ \overline{2}\ \overline{6} \end{array}$$

Now because

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$$\begin{array}{rr} & 34\ \overline{2}\ \overline{6} \\ & 8\ \overline{2}\ \overline{6} \\ + & 4\ \overline{3} \\ \hline & 43\ \overline{3} \\ + & 4\ \overline{3} \\ \hline & 47\ \overline{2}\ \overline{6} \end{array}$$

Not necessary as a part of your solution, but we check:

$$22 \times 2\ \overline{6} = 22 \times 2\frac{1}{6} = 22 \times \frac{13}{6} = \frac{143}{3} = 47\frac{2}{3} = 47\ \overline{2}\ \overline{6} = 47\ \overline{\overline{3}}.$$

3. (a) *Determine whether the following statements are true or false.*
   i. STATEMENT: *The Babylonians could find better and better approximations of* $\sqrt{5}$*.*
      TRUE.
   ii. STATEMENT: *The Babylonians could solve quadratic equations, such as "what are the dimensions of a rectangle whose perimeter is 40 and whose area is 51?"*
      TRUE.
   iii. STATEMENT: *There are exactly five Platonic Solids.*
      TRUE.
   iv. STATEMENT: *The Greeks could double the cube using just straightedge and compass.*
      FALSE.

(b) *Give a detailed explanantion of ONE of your answers (a)–(d) above.*

You only need to supply one of these elaborations.

i. The Babylonians used the divide and average method to improve their estimation of the square root of a natural number N. Starting from a guess, a, they improved their guess as follows. First find

$$b = N - a^2.$$

For the Babylonians who did not have negative numbers, the guess a would have been an underestimate $a^2 < N$ resulting in positive b. Algebraically this doesn't matter for us. Then the correction to the old guess is compute

$$c = \frac{b}{2a}$$

and then $a + c$ is the better approximation. For example to find the square root of $N = 5$, starting from the guess $a = 2$ they found

$$\begin{aligned} b &= N - a^2 = 5 - 4 = 1 \\ c &= \frac{b}{2a} = \frac{1}{2 \cdot 2} = \frac{1}{4} \\ a + c &= 2 + \frac{1}{4} = 2\frac{1}{4}. \end{aligned}$$

This is a much better approximation. We may now repeat the steps, this time starting with improved approximation to $\sqrt{5}$, $a = \frac{9}{4}$.

$$\begin{aligned} b &= N - a^2 = 5 - \frac{81}{16} = -\frac{1}{16} \\ c &= \frac{b}{2a} = \frac{-\frac{1}{16}}{2 \cdot \frac{9}{4}} = -\frac{1}{72} \\ a + c &= \frac{9}{4} - \frac{1}{72} = \frac{161}{72} \approx 2.2361111111. \end{aligned}$$

This is close to the actual value $\sqrt{5} = 2.2360679775$. The procedure may be repeated. The Babylonians might have to handle irregular fractions, complicating their calculations.

ii. The Babylonians knew the quadratic formula for specific types of quadratic problems, such as the one mentioned. For this particular rectangle problem, the semiperimiter is 20. The Babylonians may have solved the problem like this. The sides would have length $10 + a$ and $10 - a$ which adds to the right semiperimiter. Multiplying to get the area

$$\begin{aligned} (10 + a)(10 - a) &= 51 \\ 100 - a^2 &= 51 \\ a^2 &= 49 \\ a &= 7 \end{aligned}$$

so the side lengths are $10 + 7 = 17$ and $10 - 7 = 3$.

iii. The Platonic solids are the regular polyhedra. All of their faces are regular polygons and there are the same number of faces meeting at each vertex. The only possibilities are tetrahedron, octohedron, icosahedron, cube and dodecahedron.

The first three have triangular faces, the cube has square faces and the dodecahedron has pentagonal faces.

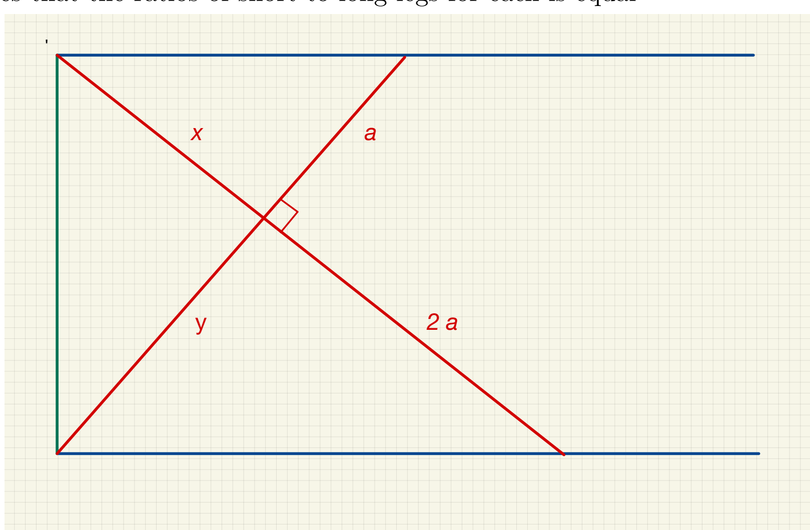
The argument that these are the only possibilities is based on considering which polygons can possibly meet at a vertex. There must be at least three faces at a vertex, otherwise two faces meeting come together would be two polygons in a plane on top of each other and the polyhedron would not be three dimensional. The sums of the angles at a vertex must also be less than 360° for otherwise the boundary of the polyhedron would not be convex at that vertex. Since the hexagon and polygons with more than six sides have interior angles at least 120°, three such polygons would total at least 360° meeting at a vertex, which is too much for a convex corner. Since the interior angles of a triangle, square and pentagon, respectively are 60°, 90° and 108°, the only possibilities with total angle less that 360° are

$$3 \times 60^\circ, \quad 4 \times 60^\circ, \quad 5 \times 60^\circ, \quad 3 \times 90^\circ, \quad 3 \times 108^\circ,$$

All of these possibilities occur as polyhedra, thus make the complete list: tetrahedron, octahedron, icosahedron, cube and dodecahedron.

iv. The Greeks were unable to double the cube using only straightedge and compass. It was proved in the nineteenth century that such constructions are impossible. However, using more complicated gadgets, the Greeks found ways to do all the Delian problems.

For example, one method of doubling of the cube was attributed to Plato himself. He uses a right angled cross with one leg of length a and the second of length $2a$. The cross is rotated so the two ends slide along parallel lines until the third and fourth legs intersect the parallel lines at a perpendicular bisector. Labelling their length x and y, then the solution to the doubling problem is $x^3 = 2a^3$. We see that the three triangles are all similar since they have the same angles. This implies that the ratios of short to long legs for each is equal



$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}.$$

It follows from the second and then the first equations that

$$2a^3 = 2a \cdot a^2 = \frac{y^2}{x} \cdot a^2 = \frac{(ay)^2}{x} = \frac{(x^2)^2}{x} = x^3.$$

4. (a) *Use Pythagoras method to show $\sqrt{5}$ is irrational.*

We argue by contradiction. Assuming that $\sqrt{5}$ is rational, we may write it in lowest terms as a ratio of positive integers

$$\sqrt{5} = \frac{p}{q}$$

where p, q have no common factors other than one. Then

$$5 = \frac{p^2}{q^2}$$

or

$$p^2 = 5q^2.$$

This says 5 divides p^2. But since 5 is prime, it must divide p. Hence $p = 5k$ for some integer k. Inserting

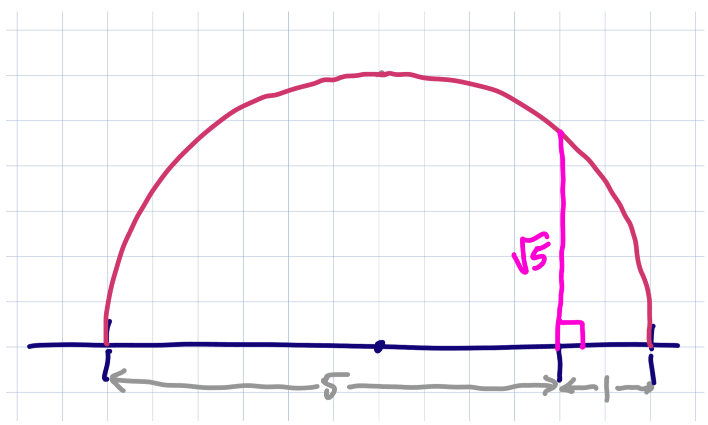
$$(5k)^2 = 5q^2$$

yields

$$q^2 = 5k^2.$$

As before, this says that 5 divides q. We have reached a contradiction. We showed that 5 is a common factor to both p and q, contrary to our choice that p and q have no nontrivial common factors. Thus the contrary statement was false proving instead that $\sqrt{5}$ is not rational.

(b) *Show that $\sqrt{5}$ may be constructed using just straightedge and compass.*



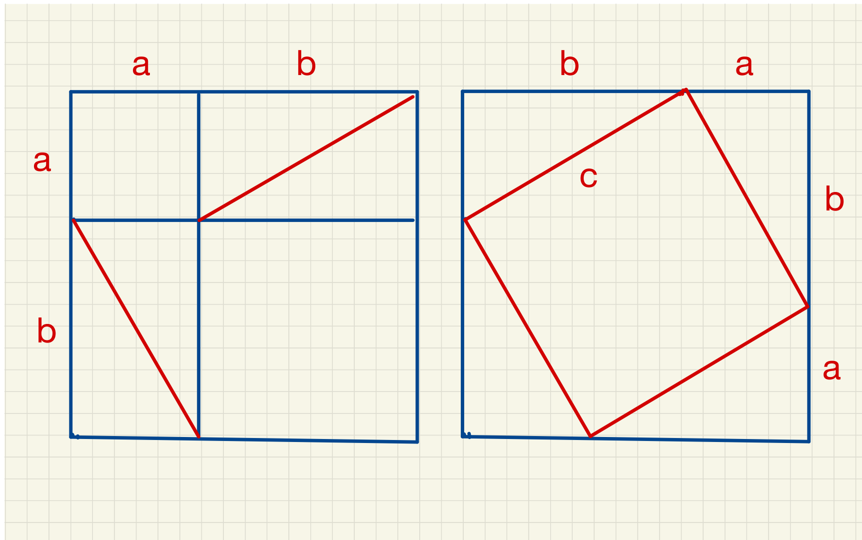
5. (a) *Was the Pythagorean Theorem and its proof known to the Babylonians, Egyptians and Greeks? Compare and contrast.*

The Babylonians knew the Pythagorean Theorem. It was used in computing triangles in practical situations. But nothing like a general proof was given by them.

The Pythagorean Theorem was not known to the Egyptians. The story that the rope stretchers, that is, surveyors, used a knotted rope knotted at points to divide the total length into parts of ratios 3 to 4 to 5, which would then be used to form a right triangle, is not confirmed in any document.

The Greeks knew the theorem and gave many proofs. We have already encountered at least two from Euclid in class.

(b) *Give your favorite proof of the Pythagorean Theorem.*



Removing four triangles from the big square leaves the same area. On the left the remainder is $a^2 + b^2$ and on the right it is c^2.