Math 3010 § 1. **Second Midterm Exam** Solutions
Treibergs March 13, 2024

1. *For each mathematician fill in their principal location from the list and write a short statement of their mathematical contribution.*

Mathematician	Location	Mathematical Contribution
Archimedes 287 – 212 BC	Syracuse	Engineer. Math of levers, pulleys. Master of exhaustion, mechanical method. Computed π
Apollonius 250 – 175 BC	Alexandria	Wrote the most advanced Greek work on conic sections
Hipparchus 190 – 120 BC	Bithynia	Astronomer who developed trigonometry
Heron 24 – 105 AD	Alexandria	Invented steam engine and vending machine Formula for area of triangle.
Ptolemy 100 – 178 AD	Alexandria	Astronomer and geometer. His *Almagest* is the Euclid *Elements* of astronomy and trigonometry
Liu Hui 223 – 293 AD	Cao-Wei	Commented on *Nine Chapters* Wrote *Sea Island Manual*
Li Chunfeng 601 – 670 AD	Chang-an	Corrected and annotated old texts in *Ten Mathematical Classics* used in University

Locations (Several may be in the same location.)

Alexandria, Athens, Beijing, Bithynia, Chang-an, Cao Wei, Syracuse

2. (a) *Using the Euclidean Algorithm, find* $\gcd(315, 120)$.

$$
\begin{aligned}
315 &= 2 \cdot 120 + 75 \\
120 &= 1 \cdot 75 + 45 \\
75 &= 1 \cdot 45 + 30 \\
45 &= 1 \cdot 30 + 15 \\
30 &= 2 \cdot 15 + 0.
\end{aligned}
$$

Thus $g = \gcd(315, 120) - 15$.

(b) *Find all integer solutions of* $\quad 315x + 120y = 15.$

We run the Euclidean algorithm backwards

$$\begin{aligned} 15 &= 45 - 30 & \\ &= 45 - (75 - 45) & &= 2 \cdot 45 - 75 \\ &= 2 \cdot (120 - 75) - 75 & &= 2 \cdot 120 - 3 \cdot 75 \\ &= 2 \cdot 120 - 3 \cdot (315 - 2 \cdot 120) & &= 8 \cdot 120 - 3 \cdot 315 \end{aligned}$$

One solution is of the form $x = -3$ and $y = 8$. All solutions have the form

$$x = -3 + \frac{120j}{g} = -3 + 8j, \qquad y = 8 - \frac{315j}{g} = 8 - 21j,$$

where $j \in \mathbb{Z}$ is an arbitrary integer.

3. (a) *Determine whether the following statements are true or false.*

   i. STATEMENT. *Euclid gave a geometric proof that* $x(b-x) + (\frac{b}{2} - x)^2 = (\frac{b}{2})^2$ *and used it to solve* $bx - x^2 = c$.
   TRUE.

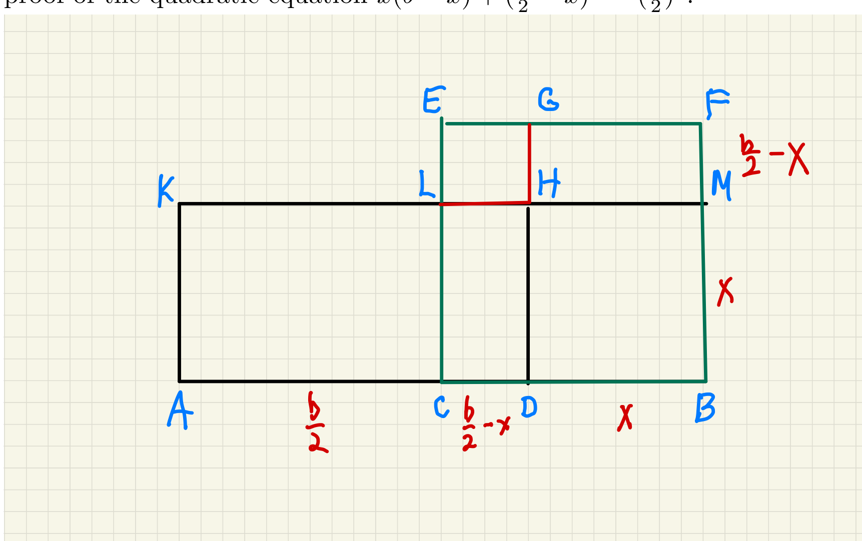
   ii. STATEMENT. *Archimedes found the area of a parabolic segment.*
   TRUE

   iii. STATEMENT. *The only possible non-degenerate intersection curves of an oblique cone and plane are the ellipse, the parabola and the hyperbola.*
   TRUE.

   iv. STATEMENT. *Heron could compute the area of the 13-14-15 triangle.*
   TRUE.

   (b) *Give a detailed explanantion of ONE of your answers i.– iv. above.*

   i. This is from Euclid's Elements Propositions II-5 and II-14. He gave a geometric proof of the quadratic equation $x(b-x) + (\frac{b}{2} - x)^2 = (\frac{b}{2})^2$.



The segment AB has length b. C is the midpoint so CB has length $\frac{b}{2}$. D cuts the segment with DB of length x. The perpendiculars AK, CL, DH and BM have length x and their extensions CE, DG, BF have length $\frac{b}{2}$. The square $BCEF$ has area $\left(\frac{b}{2}\right)^2$ and equals the sum of areas of the rectangles $ADHK$ and $HGEL$

of areas $x(b-x)$ and $\left(\frac{b}{2}-x\right)^2$, resp. One cuts off the rectangle $ACLK$ and pastes it over $BFDG$ of the same dimensions.
The algebraic argument is

$$x(b-x)+\left(\frac{b}{2}-x\right)^2 = bx - x^2 + \left(\frac{b}{2}\right)^2 - bx + x^2 = \left(\frac{b}{2}\right)^2.$$

To solve $bx - x^2 = c$, one uses the formula

$$c = bx - x^2 = x(b-x) = \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}-x\right)^2$$

which gives

$$\left(\frac{b}{2}-x\right)^2 = \left(\frac{b}{2}\right)^2 - c.$$

It follows that

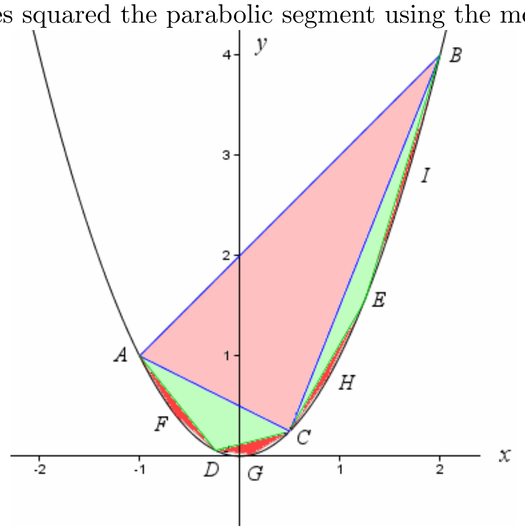
$$\frac{b}{2} - x = \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

or

$$x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

which, of course, is the quadratic formula to solve $x^2 - bx + c = 0$.

ii. Archimedes squared the parabolic segment using the method of exhaustion.

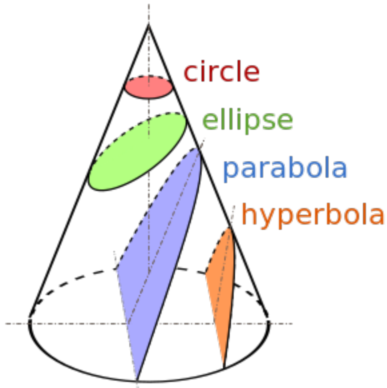


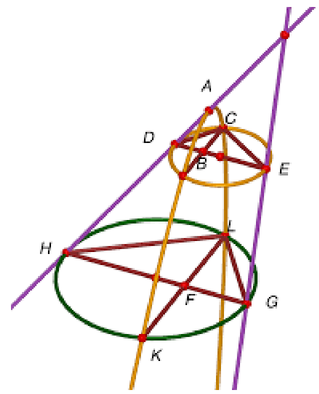
First, by invariance under translation and scaling, the parabola may be moved to the origin and stretched to the standard parabola $y = x^2$. This means that the ratio of areas of the parabolic segment and triangle remain the same after transformation. The left and right endpoints are at A and B. The point whose x-coordinates are halfway between those of A and B is at C. Archimedes showed that the area of the segment between the parabola and the line AB is $\frac{4}{3}$ the area of the triangle ABC.
To approximate the area of the parabolic segment, triangles ACD and BCE are added to fill in the gap between the original triangle and the parabola. The x coordinate of D is half way between those of A and C, the x-coordinate of E is half way between those of C and B. Then four triangles ADF, CDG, CEH and HBI are added to fill in the gaps with new vertices at the midpoints, and so on.

Archimedes showed that the area of each two triangles ACD and BCH have an area one eighth of the previous stage ABC of the four triangles ADF have an area on eighth of ADC. Thus at the nth stage, there are 2^{n-1} triangles each having an area $8^{1-n}T$ where T is the area of triangle ACB. Thus the atea of the parabolic segment is total area of the triangles which is using the formula for a geometric series,

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{8^{n-1}} T = \sum_{k=0}^{\infty} \frac{T}{4^k} = \frac{T}{1-\frac{1}{4}} = \frac{4}{3}T.$$

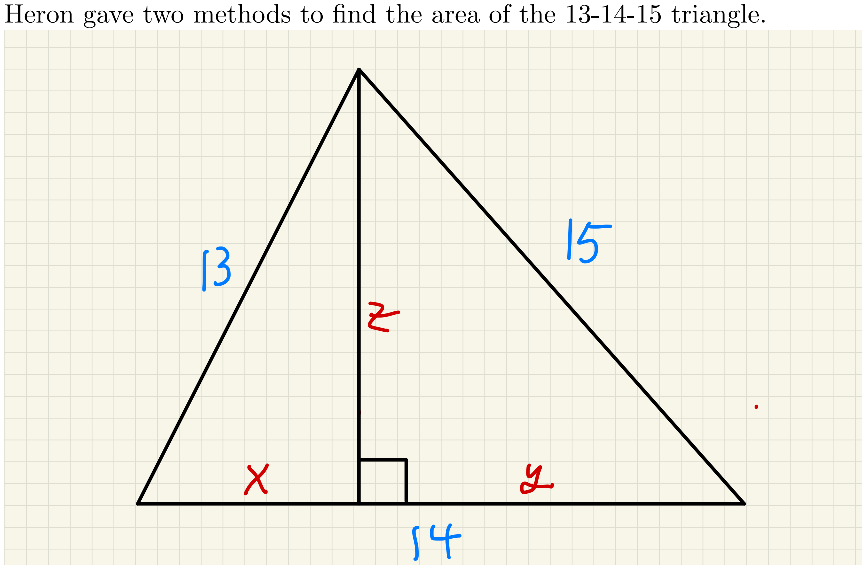
iii. A cone is generated by a circle in a plane and a point P not in the plane. It is the union of lines through P and points of the circle. If P is in the perpendicular line through the circle's center, then the cone is a right cone (the figure on the left). If the point can be anywhere off the plane, then the cone is an oblique cone (the figure on the right.).





Appolonius showed that the only sections of an oblique cone and a plane that cuts the inside of the cone are either an ellipse (including the circle), a parabola or a hyperbola, depending on the angle of the plane. If the plane is parallel to one of the generating lines of the cone then the intersection is a parabola. Otherwise it is an ellipse if the intersection is finite, or the hyperbola if the intersection is unbounded.

iv. Heron gave two methods to find the area of the 13-14-15 triangle.



For the first method, split the triangle into two right triangles as in the diagram.

Then the legs $x + y = 14$ so $y = 14 - x$ or $y - x = 14 - 2x$. Use the Pythagorean Theorem to solve for z^2 in two ways.

$$13^2 - x^2 = z^2 = 15^2 - y^2.$$

Hence

$$14^2 - 28y = 14 \cdot (14 - 2x) = (y + x) \cdot (y - x) = y^2 - x^2 = 15^2 - 13^2 = 225 - 169 = 56$$

Thus

$$140 = 196 - 56 = 14^2 - 56 = 28y$$

so that $x = 5$, $y = 9$ and $z = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = 12$ so the area of the triangle is half of base times height is $7z = 84$.
The second method is to use Heron's Formula. The semiperimiter is

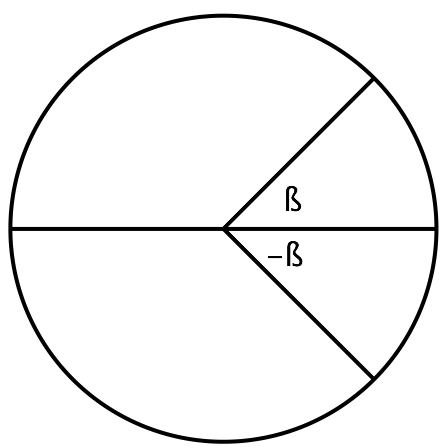
$$s = \frac{a + b + c}{2} = \frac{13 + 14 + 15}{2} = 21.$$

Heron's formula is

$$A^2 = s(s - a)(s - b)(s - c) = 21(21 - 13)(21 - 14)(21 - 15) = 21 \cdot 8 \cdot 7 \cdot 6 = 7056$$

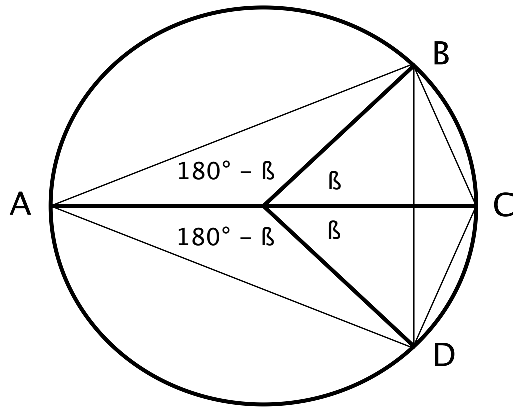
so $A = 84$.

4. (a) *Suppose the radius of circle is R. Define the chord* $\operatorname{crd}(\beta)$ *for an angle β. Find and prove a formula for* $\operatorname{crd}(2\beta)$*, the double angle. Hint:*



Label the vertices A, B, C and D and angles at the origin. The chord is the length of the secant cut by the angle.. Here, for β,

$$\operatorname{crd}(\beta) = BC.$$



Then Ptolemy's theorem applied to the circular quadrilteral says sum of products of opposite sides equals the product of diagonals

$$AB \cdot CD + BC \cdot AD = AC \cdot BD$$

In terms of chords, this is

$$\operatorname{crd}(180^\circ - \beta) \cdot \operatorname{crd}(\beta) + \operatorname{crd}(\beta) \cdot \operatorname{crd}(180^\circ - \beta) = 2R \cdot \operatorname{crd}(2\beta)$$

In other words,

$$R \cdot \operatorname{crd}(2\beta) = \operatorname{crd}(\beta) \cdot \operatorname{crd}(180^\circ - \beta).$$

No surprise. It is the analog of the double angle formula $\sin 2\beta = 2 \sin \beta \cos \beta$.

(b) *Use the Chinese method to find the the square root of* $N = 18496$.

Observe that $100^2 = 10,000 < N < 1000^2 = 1,000,000$, so we look for the square root of the form $x = 100a + 10b + c$. We see that $200^2 = 40,000$ is too large so $a = 1$. Look for the biggest b that satisfies

$$18496 \geq (100 + 10b)^2 = 10,000 + 2 \cdot 100 \cdot 10b + (10b)^2$$

or

$$8,496 = 18,496 - 10,0000 \geq 2,000b + 100b^2.$$

For $b = 3$ the right side is $6,000 + 900 = 6,900$ which is smaller than $N - 100^2$ and For $b = 4$ the right side is $8,000 + 1,600 = 9,600$ which is larger. Thus $b = 3$ and

$$N - 130^2 = 8,496 - 6,900 = 1,596.$$

Finally, we look for the largest c so that

$$N \geq (130 + c)^2 = 130^2 + 2 \cdot 130 \cdot c + c^2$$

or

$$1,596 = N - 130^2 \geq 260c + c^2.$$

Trying $c = 6$ we find $260c + c^2 = 260 \cdot 6 + 36 = 1,560 + 36 = 1,596$ as desired. Thus $x = 136$.

5. (a) *How did Euclid argue that there are infinitely many primes?*

He argued by contradiction: Suppose that there are finitely many primes given by $p_1 = 2$, $p_2 = 3$, $\ldots$, p_k, where p_k is the greatest prime. Consider the numbers $N = p_1 \cdot p_2 \cdot \ldots \cdot p_k$ and $Q = N + 1$. First observe that for any prime p_j from the list we have $p_j | N$ but Q is not divisible by p_j because $p_j | Q$ would imply p_j divides $Q - N = 1$ which is not true. Q is larger than p_k's because N is the product of p_k and at least some other numbers all of which are strictly greater than one. Q is even larger. Either Q is prime in which case it is a prime number not in the list. Or Q is composite and is a product of primes each of which is greater than p_k because Q is not divisible by any of the p_j's in the list. In both cases there are prime numbers not in the list of all prime numbers which is a contradiction. Thus the hypothesis that there are finitely many primes is false.

(b) *Solve the following problem from Nine Chapters: There are nine equal pieces of gold and 11 equal pieces of silver. The two lots weigh the same. If one piece is removed from each lot and put in the other, then the lot containing mainly the gold is found to weigh 13 ounces less than the lot containing mainly silver. Find the weight of each piece of gold and silver.*

This illustrates a problem involving simultaneous limear equations and negative numbers. We solve for x the weight of a gold coin and y the weight of a silver coin in ounces. The two statements give the pair of equations

$$\begin{aligned} 9x &= 11y \\ 8x + y &= 10y + x - 13 \end{aligned}$$

or equivalently

$$\begin{aligned} 9x - 11y &= 0 \\ 7x - 9y &= -13 \end{aligned}$$

Multiplying the equations by 9 and 11 yields

$$\begin{aligned} 81x - 99y &= 0 \\ 77x - 99y &= -11 \cdot 13 = -143 \end{aligned}$$

Subtracting

$$4x = 143$$

so

$$x = \frac{143}{4}, \qquad y = \frac{9}{11}x = \frac{9 \cdot 143}{11 \cdot 4} = \frac{117}{4}.$$