Math 3010 § 1. **Second Midterm Exam** Name: Solutions

Treibergs Oct. 30, 2024

1. *For each mathematician write a short statement of their mathematical contribution.*

Mathematician	Mathematical Contribution
Apollonius 250 – 175 BC Alexandria	Wrote *Conics,* the culmination of Greek mathematics on conic sections.
Liu Hui 223 – 293 AD Louyang, Cao Wei	Wrote *Commentaries on Nine Chapters* and *Sea Island* *Manual* on arithmetic and geometry. Approximated π.
Brahmagupta 598 – 665 Bhinmal	Quadratic formula. Euclidean algorithm. Solved linear Diophantine and Pell's equations.
Al-Khowarizmi 780 – 850 Baghdad	House of Wisdom. Wrote on arithmetic, astronomy and … *al-Jabr wal muqabala* on quadratic equations.
Thabit ibn Qurra 863 – 901 Baghdad	Translated Greek and Indian works to Arabic. Amicable numbers. Generalized Pythagorean Theorem.
Bhaskara II 1114 – 1185 Ujjian	Wrote *Lilivati* on algebra. Negative roots of quadratic eqns. General solution of Diophantine and Pell's eqn.
Chin Chu-shao 1202 - 1261 Hangzhou	Wrote *Mathematical Teatise in Nine Sections.* High order equations. Binomial theorem. Ta-Yen rule for congruences.

2. (a) *Find positive integers x and y such that* $x^2 = 17y^2 + 1.$

Observe that $4^2 = 17 \cdot 1^2 - 1$ so $(x, y, k) = (4, 1, -1)$ in Brahmagupta's method. Then the thunderbolt

$$(x_1, y_1, k_1) \oplus (x_2, y_2, k_2) = (x_1x_2 + Dy_1y_2,\ x_1y_2 + x_2y_1,\ k_1k_2)$$

gives

$$(4, 1, -1) \oplus (4, 1, -1) = (4 \cdot 4 + 17 \cdot 1^2,\ 4 \cdot 1 + 1 \cdot 4,\ (-1)(-1)) = (33, 8, 1)$$

which is the desired solution $33^2 = 17 \cdot 8^2 + 1$.

(b) *Find all integers N satisfying the simultaneous congruences*

$$\begin{aligned} N &\equiv 2 \mod 7 \\ N &\equiv 7 \mod 8 \\ N &\equiv 1 \mod 9 \end{aligned}$$

Note that $7, 8, 9$ are pairwise relatively prine so that the answer is modulo $7 \cdot 8 \cdot 9 = 504$. The first method is to solve two Diophantine equations. The number satisfies $N = 7i + 2 = 8j + 7 = 9k + 1$ for some $i, j, k \in \mathbb{Z}$. Solving the first two

$$7i + 2 = 8j + 7$$

or

$$7i - 8j = 5.$$

We see that $7 \cdot 1 - 8 \cdot 1 = -1$. Multiplying by -5 yields $7 \cdot (-5) + 8 \cdot (-5) = 5$ Hence all solutions are

$$i = -5 + 8p, \qquad j = -5 + 7p, \qquad \text{for some } p \in \mathbb{Z}.$$

Then we solve the the last congruence

$$N = 7i + 2 = 7(-5 + 8p) + 2 = -35 + 56p + 2 = 9k + 1$$

or

$$56p - 9k = 34.$$

Observing that $56 \cdot 1 - 9 \cdot 6 = 2$, multiplying by 17 yields $56 \cdot 17 - 9 \cdot 102 = 34$. Thus all solutions are

$$p = 17 + 9q, \qquad k = 102 + 56q, \qquad \text{for some } q \in \mathbb{Z}.$$

This means that

$$N = 9k + 1 = 9(102 + 56q) + 1 = 919 + 504q, \qquad \text{for some } q \in \mathbb{Z}.$$

$N = 919 - 504 = 415$ is another solution.

For the second method using Chin Chu-shao's Ta-Yen Rule, we see that

$$\begin{array}{llll} N_1 = 8 \cdot 9 = 72 & \equiv 2 \mod 7, & N_1 \equiv 0 \mod 8, & N_1 \equiv 0 \mod 9 \\ N_2 = 7 \cdot 9 = 63 & \equiv 7 \mod 8, & N_2 \equiv 0 \mod 7, & N_2 \equiv 0 \mod 9 \\ N_3 = 7 \cdot 8 = 56 & \equiv 2 \mod 9, & N_3 \equiv 0 \mod 7, & N_3 \equiv 0 \mod 8 \end{array}$$

N_1 and N_2 are as desired. Also

$$5 \cdot N_3 = 5 \cdot 56 = 280 \equiv 5 \cdot 2 \mod 9 \equiv 1 \mod 9$$

Thus the solution of the simultaneous congruences is

$$N = N_1 + N_2 + 5 \cdot N_3 = 72 + 63 + 280 = 415$$

modulo 504.

3. (a) *Determine whether the following statements are true or false. Give a detailed explanation of ONE of your answers i.–iv. above.*

   i. STATEMENT: *Omar Khayyam showed that the intersection of the parabola $y^2 + cx = ac$ and the hyperbola $xy = c$ yields the solution of the cubic equation $x^3 + c = ax^2$.*

   TRUE. In his book *Treatise on Demonstrations of Problems of al-Jabra...,* Khayyam made geometric constructions to solve cubic equations. In this case, if (x, y) is the point of intersection, $xy = c$ implies $x = \frac{c}{y}$ so that substituting into the parabola,

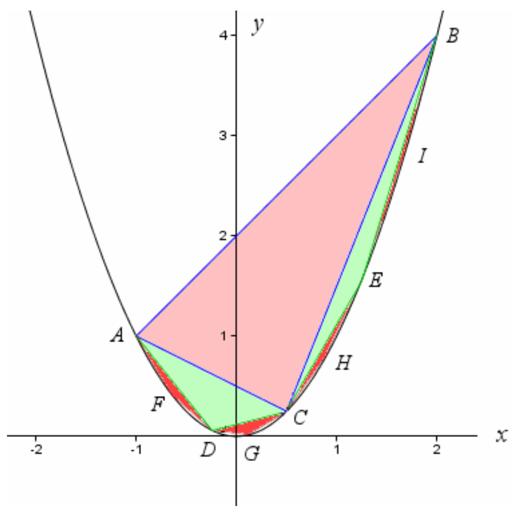
$$\frac{c^2}{x^2} + cx = ac$$

   Multipying by $\frac{x^2}{c}$ yields the desired cubic equation

$$c + x^3 = ax^2.$$

   ii. *Archimedes found the area of a parabolic segment.*

   TRUE. In his book *Quadrature of the Parabola,* Archimedes used the method of exhaustion by triangles to compute the area between a parabolic arc and its chord. The area of the approximating polygon is obtained as a finite geometric sum, which he computed. In his book *The Mechanical Method* he found the area another way by balancing it with the area of a known triangle.



   In the exhaustion method, the first pink triangle ΔACB of area T_0 connects the endpoints with the vertex on the parabola below the midpoint of the chord. Then two green triangles ΔADC and ΔCEB of area T_1 are added in the gaps with vertices at their midpoints. The area of each green triangle is one eighth that of the previous triangle $T_1 = \frac{1}{8}T_0$. Then four red triangles of area T_2 are added, with areas $T_2 = \frac{1}{8}T_1$ and so on. This means that the area of an nth stage triangle is $T_n = \frac{1}{8^n}T_0$. Hence after the nth stage, the total of the area is

$$A_n = \sum_{k=0}^{n} 2^k T_k = \sum_{k=0}^{n} 2^k \frac{T_0}{8^k} = \sum_{k=0}^{n} \frac{T_0}{4^k} = \frac{T_0\left(1 - \dfrac{1}{4^{n+1}}\right)}{1 - \dfrac{1}{4}} \to \frac{4T_0}{3} \quad \text{as } n \to \infty.$$

iii. *Heron could compute the area of the 13-20-21 triangle.*
TRUE. Heron developed a formula for the area. The semiperimeter is

$$s = \frac{a+b+c}{2} = \frac{13+20+21}{2} = 27.$$

Then the square of the area is

$$A^2 = s(s-a)(s-b)(s-c) = 27 \cdot 14 \cdot 7 \cdot 6 = 15876 = 3^4 \cdot 2^2 \cdot 7^2$$

so the area is $A = 9 \cdot 2 \cdot 7 = 126$.

iv. *The Chinese found better and better approximations to π.*
TRUE. Liu Hui in his 23AD *Commentaries on the Nine Chapters of the Mathematical Art* developed a recursion sequence to compute the perimeter and area of $6 \cdot 2^{n-1}$-gon inscribed in the unit circle. Computing up to the 3072-gon, his estimate was $\pi \approx 3.14159$. Tsu Chung-Chi found $\pi \approx 3.1415926$ in 480AD.
Liu's recursion is as follows. If a_n is the inradius and c_n the length of a side of the $6 \cdot 2^{n-1}$-gon and b_n the total area of the $6 \cdot 2^n$-gon, starting from $c_1 = 1$ the recursions are

$$a_n = \sqrt{1 - \left(\frac{c_n}{2}\right)^2}, \qquad c_{n+1} = \sqrt{\left(\frac{c_n}{2}\right)^2 + (1-a_n)^2}, \qquad b_n = 6 \cdot 2^{n-2} c_n.$$

Not part of your answer but just for fun, here is an **R**© code to execute this recursion.

```
> c=1:10;a=1:10;b=1:10
> for(i in 1:9){
    a[i] = sqrt(1 - c[i]^2/4);
    c[i+1] = sqrt(c[i]^2/4 + (1 - a[i])^2);
    b[i] = c[i]*6*2^(i-2)
    }
> cbind(a,c,b)
   i         a[i]        c[i]       b[i]
 [1,]  0.8660254 1.000000000  3.000000
 [2,]  0.9659258 0.517638090  3.105829
 [3,]  0.9914449 0.261052384  3.132629
 [4,]  0.9978589 0.130806258  3.139350
 [5,]  0.9994646 0.065438166  3.141032
 [6,]  0.9998661 0.032723463  3.141452
 [7,]  0.9999665 0.016362279  3.141558
 [8,]  0.9999916 0.008181208  3.141584
 [9,]  0.9999979 0.004090613  3.141590
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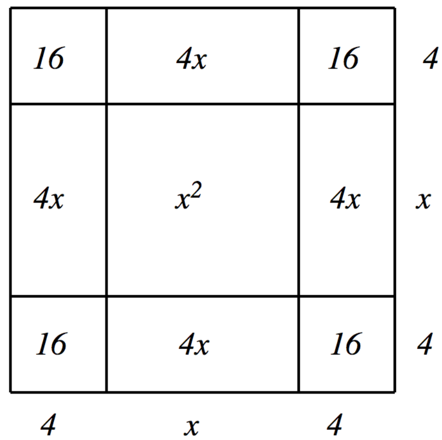
For the $6 \cdot 2^9 = 3072$-gon, the area is 3.141590.

4. (a) *Using al-Khowarizmi's method, label the edges of the diagram with lengths and find x which solves the equation* $x^2 + 16x = 260$.

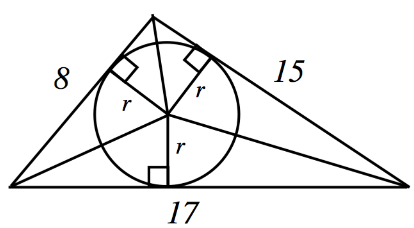


One completes the square geometrically. The area of the big square equals the sum of the areas of the middle square plus four rectangular flaps plus four corners.

$$(x+8)^2 = x^2 + 4 \cdot 4x + 4 \cdot 16 = 260 + 64 = 324 = 18^2$$
$$x + 8 = \pm 18.$$

Thus $x = 10, -26$. Of course, al-Khowarizmi didn't give the negative solution.

(b) *Solve the following problem from Nine Chapters: Find the diameter of the largest circle that can be inscribed in the right triangle with legs 8 and 15 and hypotenuse 17.*

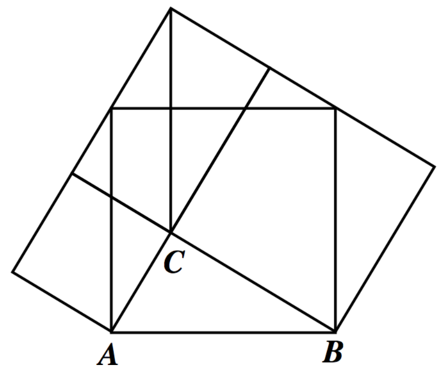


The triangle is right because $8^2 + 15^2 = 64 + 225 = 289 = 17^2$. The total area is the sum of the areas of the three triangles touching the sides.

$$A = \frac{1}{2} \cdot 8 \cdot 15 = \frac{1}{2} \cdot 8 \cdot r + \frac{1}{2} \cdot 15 \cdot r + \frac{1}{2} \cdot 17 \cdot r$$
$$60 = \frac{(8+15+17)r}{2} = 20r$$

so $r = 3$.

5. (a) *Give a proof of the Pythagorean Theorem that comes from China, India or the Islamic world.*



Here is an argument from Thabit ibn Qurra. The area of the pentagon is the sum of the area of the large square plus areas of three triangles. It is also the sum of areas of the two smaller squares plus areas of three other triangles of the same size.

$$\text{Area of Pentagon} = c^2 + T_1 + T_2 + T_3 = a^2 + b^2 + T_4 + T_5 + T_6.$$

Thus $a^2 + b^2 = c^2$. There are many other solutions to this problem.

(b) *Use the Chinese method to find the the square root of $N = 17424$.*

Look for a square root of the form $x = 100a + 10b + c$ where a, b and c are integers from 0 to 9. Observe that

$$100^2 = 10,000 < 17424 < 40,000 = 200^2$$

so $a = 1$ and $x = 100 + 10b + c$. Try to find the largest b so that

$$(100 + 10b)^2 = 10,000 + 2000b + 100b^2 \leq 17424$$

or

$$2000b + 100b^2 \leq 17424 - 10000 = 7424$$

We have $2000 \cdot 3 + 100 \cdot 3^2 = 6900$ and $2000 \cdot 4 + 100 \cdot 4^2 = 9600$ so $b = 3$ because $b = 4$ is too large. Thus

$$17424 - 130^2 = 7424 - 6900 = 524.$$

Finally

$$(130 + c)^2 = 130^2 + 260c + c^2 \leq 17424$$

implies

$$260c + c^2 \leq 524.$$

Since $260 \cdot 2 + 2^2 = 524$ we have $c = 2$ so the square root of $N = 17424$ is $x = 132$.