Math 3010 § 1. **First Midterm Exam** Name: Solutions
Treibergs February 5, 2025

1. *For each mathematician fill in their principal location from the list and write a short statement of their mathematical contribution.*

Mathematician	Location	Mathematical Contribution
Thales 624 – 547 BC	Miletus	Proved earliest geometric theorems, e.g., base angles of an isoceles triangle are equal..
Pythagoras 580 – 497 BC	Croton	Figurate numbers, Pythagorean triples Irrationality of $\sqrt{2}$, theory of harmony
Hippocrates 460 – 380 BC	Chios	Wrote early geometry text, quadrature of the lune
Eudoxus 408 – 355 BC	Cnidus	Resolved Xeno's questions, Theory of proportion, exhaustion anticipated calculus.
Aristotle 384 – 322 BC	Athens	Developed a theory of rigorous argument via axiomatic approach. Described syllogisms.
Euclid 330 – 270 BC	Alexandria	Organized all mathematics in *Elements,* a model of mathematical exposition.
Diophantus 218 – 292 AD	Alexandria	His *Arithmetica* used algebraic notation, sought integer solns of underdetermined eqns

Locations (Several may be in the same location.)

Alexandria, Athens, Chios, Cnidos, Croton, Elia, Miletus, Syracuse

2. *Divide 371 by 18.*

   (a) *Express in sexagesimal and use the Babylonian method and sexagesimal arithmetic to compute the quotient. (Other methods receive zero credit.)*

   (b) *Using the Egyptian method of doubling, find the quotient. (You may use usual decimals with proper n-th parts. Other methods receive zero credit.)*

   (a) $N = 371 = 360 + 11 = 6 \cdot 60 + 11$ so in sexagesimal, $N = 6, 11$. To divide, the Babylonians multiplied by the reciprocal.

$$\frac{1}{18} = \frac{1}{3} \cdot \frac{1}{6} = \frac{20}{60} \cdot \frac{10}{60} = \frac{200}{60^2} = \frac{3 \cdot 60 + 20}{60^2} = 0; 3, 20.$$

   Compute $6, 11 \times 0; 3, 20$.

```
     6,  11
x          ;    3,  20
---------------------
           ;  120, 220
       18 ;   33
---------------------
        2 ;   3                    220   =   3 x 60   +   40
          ;   0    40              120   =   2 x 60   +    0
       18 ;  33
---------------------
       20 ;  36,   40
```

   Check: $\frac{371}{18} = 20\frac{11}{18} = 20 + \frac{55}{90}$ and $20; 36, 40 = 20 + \frac{36}{60} + \frac{40}{3600} = 20 + \frac{6}{10} + \frac{1}{90} = 20 + \frac{55}{90}$.

   (b) Egyptian method of doubling finds how many times 18 must be doubled to reach N.

1	$\times$	18	=	18	
2	$\times$	18	=	36	
4	$\times$	18	=	72	$\surd$
8	$\times$	18	=	144	
16	$\times$	18	=	288	$\surd$
$\overline{2}$	$\times$	18	=	9	$\surd$
$\overline{9}$	$\times$	18	=	2	$\surd$

   Subtracting from the the right column until zero is left and checking the rows

```
   371
-  288
---------
    83
-   72
---------
    11
-    9
---------
     2
-    2
---------
     0
```

Thus the quotient is the sum from the checked rows

$$q = 16 + 4 + \overline{2} + \overline{9} = 20\ \overline{2}\ \overline{9}.$$

This is the same as above because

$$20\ \overline{2}\ \overline{9} = 20 + \frac{1}{2} + \frac{1}{9} = 20\frac{11}{18}.$$

3. (a) *Determine whether the following statements are true or false.*
   i. *The Egyptians could calculate the area of a circle of diameter* $d = 36$.
   TRUE.
   ii. *The Babylonians had their own way to find two numbers whose sum is* 30 *and whose product is* 104.
   TRUE.
   iii. *There are exactly five Platonic solids.*
   TRUE.
   iv. *The Greeks could double the cube using just straightedge and compass.*
   FALSE.

   (b) *Give a detailed explanation of ONE of your answers (i)–(iv) above.*
   i. The Egyptians used the formula

$$A = \left(d - \frac{1}{9}d\right)^2$$

   where d is the diameter of the circle. In this case, $d = 36$ so

$$A = \left(36 - \frac{1}{9} \cdot 36\right)^2 = (36 - 4)^2 = 32^2 = 1024.$$

   ii. The Babylonians solved simultaneous equations of this type. Let x and y be the numbers. The equations are

$$x + y = 30$$
$$xy = 104.$$

   They set $x = 15 + a$ and $y = 15 - a$ so that the first equation holds. The second becomes

$$104 = xy = (15 + a)(15 - a) = 225 - a^2$$

   so

$$a^2 = 225 - 104 = 121$$

   so $a = 11$, $x = 26$ and $y = 4$.
   iii. There are five Platonic Solids, which are the regular polyhedra. All of their faces are regular polygons and there are the same number of faces meeting at each vertex. The only possibilities are tetrahedron, octohedron, icosahedron, cube and dodecahedron. The first three have triangular faces, the cube has square faces and the dodecahedron has pentagonal faces.
   The argument that these are the only possibilities is based on considering which polygons can possibly meet at a vertex. There must be at least three faces at a vertex, otherwise two faces meeting come together would be two polygons in a plane on top of each other and the polyhedron would not be three dimensional. The sums of the angles at a vertex must also be less than 360° for otherwise the

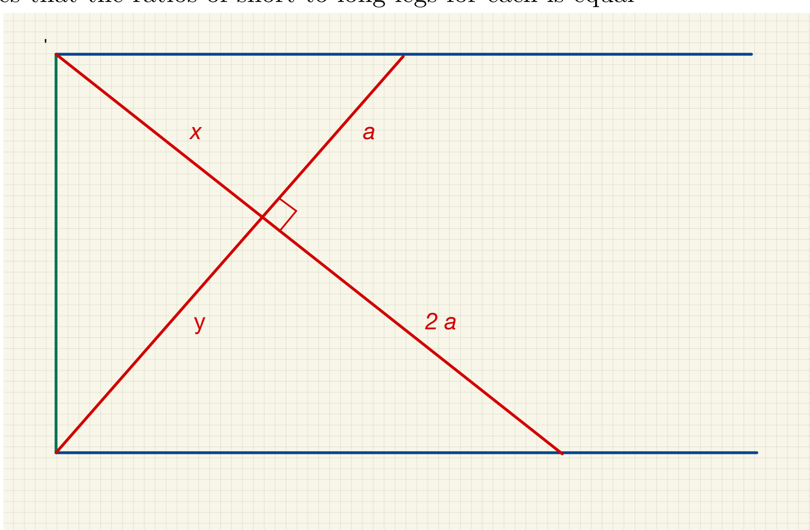
boundary of the polyhedron would not be convex at that vertex. Since the hexagon and polygons with more than six sides have interior angles at least $120°$, three such polygons would total at least $360°$ meeting at a vertex, which is too much for a convex corner. Since the interior angles of a triangle, square and pentagon, respectively are $60°$, $90°$ and $108°$, the only possibilities with total angle less that $360°$ are

$$3 \times 60°, \quad 4 \times 60°, \quad 5 \times 60°, \quad 3 \times 90°, \quad 3 \times 108°,$$

All of these possibilities occur as polyhedra, thus make the complete list: tetrahedron, octahedron, icosahedron, cube and dodecahedron.

iv. The Greeks were unable to double the cube using only straightedge and compass. It was proved in the nineteenth century that such constructions are impossible. However, using more complicated gadgets, the Greeks found ways to do all the Delian problems.

For example, one method of doubling of the cube was attributed to Plato himself. He uses a right angled cross with one leg of length a and the second of length $2a$. The cross is rotated so the two ends slide along parallel lines until the third and fourth legs intersect the parallel lines at a perpendicular bisector. Labelling their length x and y, then the solution to the doubling problem is $x^3 = 2a^3$. We see that the three triangles are all similar since they have the same angles. This implies that the ratios of short to long legs for each is equal



$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}.$$

It follows from the second and then the first equations that

$$2a^3 = 2a \cdot a^2 = \frac{y^2}{x} \cdot a^2 = \frac{(ay)^2}{x} = \frac{(x^2)^2}{x} = x^3.$$

4. (a) *Use Pythagoras method to show* $\sqrt{2}$ *is irrational.*

We argue by contradiction. Assuming that $\sqrt{2}$ is rational, we may write it in lowest terms as a ratio of positive integers

$$\sqrt{2} = \frac{p}{q}$$

where p, q have no common factors other than one. Then

$$2 = \frac{p^2}{q^2}$$

or

$$p^2 = 2q^2.$$

This says 2 divides p^2. But since 2 is prime, it must divide p. Hence $p = 2k$ for some integer k. Inserting

$$(2k)^2 = 2q^2$$

yields

$$q^2 = 2k^2.$$

As before, this says that 2 divides q. We have reached a contradiction. We showed that 2 is a common factor to both p and q, contrary to our choice that p and q have no nontrivial common factors. Thus the contrary statement was false proving instead that $\sqrt{2}$ is not rational.

(b) *Use the Babylonian algorithm to approximate $\sqrt{2}$. Start from $x_1 = 1$, compute the next two approximations x_2 and x_3. (You don't need to use sexagesimal arithmetic.)*

The Babylonian method is to find an improvement to any approximation x_n of $\sqrt{N}$. Starting from the x_n compute the discrepancy

$$b = N - x_n^2.$$

Then the improved value is given by

$$x_{n+1} = x_n + \frac{b}{2x_n}.$$

Here $N = 2$. Starting from $x_1 = 1$ we find $b = N - x_1^2 = 2 - 1^2 = 1$ so

$$x_2 = x_1 + \frac{b}{2x_1} = 1 + \frac{1}{2 \cdot 1} = \frac{3}{2}.$$

From $x_2 = \frac{3}{2}$ we find

$$b = N - x_2^2 = 2 - \left(\frac{3}{2}\right)^2 = \frac{8}{4} - \frac{9}{4} = -\frac{1}{4}.$$

so

$$x_3 = x_2 + \frac{b}{2x_2} = \frac{3}{2} + \frac{-\frac{1}{4}}{2 \cdot \frac{3}{2}} = \frac{17}{12}.$$

The Babylonian algorithm is equivalent to Newton's Method for square root, namely the divide and average procedure. Starting from x_1 the iteration proceeds by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right).$$

5. (a) *Determine if there are integer solutions, and if there are any, find them all*

$$244x + 100y = 8$$

The equation is equivalent to the one we get by dividing the common factor 4.

$$61x + 25y = 2 \tag{1}$$

Now run the Euclidian Algorithm to find $\gcd(61, 25)$.

$$\begin{aligned} 61 &= 2 \cdot 25 + 11 \\ 25 &= 2 \cdot 11 + 3 \\ 11 &= 3 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

Thus $g = \gcd(61, 25) = 1$. We run the Euclidean algorithm backwards

$$\begin{aligned} 1 &= 3 - 2 \\ &= 3 - (11 - 3 \cdot 3) = 4 \cdot 3 - 11 \\ &= 4 \cdot (25 - 2 \cdot 11) - 11 = 4 \cdot 25 - 9 \cdot 11 \\ &= 4 \cdot 25 - 9 \cdot (61 - 2 \cdot 25) = 22 \cdot 25 - 9 \cdot 61 \end{aligned}$$

Thus $x = -9$ and $y = 22$ satisfies

$$61x + 25y = 61(-9) + 25(22) = 1.$$

Doubling these gives an integer solution of (1). Thus all solutions are given by

$$x = -18 + 25k, \qquad y = 44 - 61k \qquad \text{where } k \in \mathbb{Z} \text{ is any integer.}$$

(b) *Give your favorite proof of the Pythagorean Theorem.*
The Greeks knew many proofs. We saw three of them in class, one possibly by Pythagoras and two due to Euclid, the last was your homework!