Math 3010 § 1. **First Midterm Exam** Name: Solutions

Treibergs September 17, 2025

1. *For each mathematician fill in their principal location from the list and write a short statement of their mathematical contribution.*

Mathematician	Mathematical Contribution
Thales 622 – 547 BC Miletus	First Greek to have written proofs and have a theorem named after him.
Pythagoras 585 – 500 BC Croton	Stressed geometry, arithmetic, astronomy and music. "All is number." Figurate numbers. Incommensurability of $\sqrt{2}$.
Zeno 490 – 425 BC Elia	Zeno's paradoxes conflicted with ancient intuitive notions of the infintely small *e.g.,* "Achilles and the Tortoise."
Hippocrates 460 – 380 BC Chios	Wrote early logical treatise about geometry, including the quadrature of the lune.
Plato 429 – 348 BC Athens	Academy trained many. Advocated the deductive method in algebra and straightedge compass constructions.
Eudoxus 408 – 355 BC Cnidos	Method of proportion handles incommensurate quantities. Method of exhaustion. Cosmological model.
Euclid 323 – 285 BC Alexandria	His 13 book "Elements" summarized mathematics. An influential model of logic and deductive method.

2. (a) *Use the Babylonian method and compute using sexagesimal arithmetic to find the quotient. (Other methods receive zero credit.)*

   $5,03 \quad \div \quad 36 \quad =$

   Babylonians multiplied by the reciprocal expressed in sexagesimal. Thus

$$\frac{1}{36} = \frac{100}{3600} = \frac{1 \cdot 60 + 40}{3600} =;1,40.$$

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    5,  3                              5, 3  =  5 x 60   +  2  =  303
x         ;    1,  40
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          ;  200, 120
       5  ;    3
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       3  ;   2                         120  =   2 x 60  +   0
          ;  20     0                   200  =   3 x 60  +  20
       5  ;   3
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       8  ;  25
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Check: $\frac{303}{36} = 8\frac{15}{36} = 8\frac{5}{12} = 8\frac{25}{60} = 8;25$.

(b) *Using the Egyptian method of doubling, find the quotient. Make sure that your solution is expressed with a proper Egyptian unit fraction. (Other methods receive zero credit.)*
699 $\div$ 33 $=$

1	$\times$	33	$=$	33	$\surd$
2	$\times$	33	$=$	66	
4	$\times$	33	$=$	132	$\surd$
8	$\times$	33	$=$	264	
16	$\times$	33	$=$	528	$\surd$
$\overline{3}$	$\times$	33	$=$	11	
$\overline{6}$	$\times$	33	$=$	$5\overline{2}$	$\surd$
$\overline{66}$	$\times$	33	$=$	$\overline{2}$	$\surd$

Subtracting from the the right column until zero is left and checking the rows

$$\begin{array}{r} 699 \\ \underline{-528} \\ 171 \\ \underline{-132} \\ 39 \\ \underline{-33} \\ 6 \\ \underline{-5\overline{2}} \\ \overline{2} \\ \underline{-\overline{2}} \\ 0 \end{array}$$

Thus the quotient is the sum from the checked rows

$$q = 16 + 4 + 1 + \overline{6} + \overline{66} = 21\ \overline{6}\ \overline{66}.$$

An alternative expression is gotten by using

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

so with $n = 11$,

$$\frac{2}{11} = \frac{1}{11} + \frac{1}{11} = \frac{1}{11} + \frac{1}{12} + \frac{1}{132},$$

which yields

$$21\frac{6}{33} = 21\frac{2}{11} = 21\ \overline{11}\ \overline{12}\ \overline{132}.$$

Check:

$$\frac{699}{33} = 21\frac{6}{33} = 21\frac{2}{11}.$$

3. (a) *Determine whether the following statements are true or false.*

   i. *The Babylonians had a general formula for generating Pythagorean Triples.*
   TRUE.

   ii. *The Babylonians had a method to solve problems such as "The length and width of a canal are together 6;30 GAR. The area of the canal is 7;30 SAR. What are the length and width?"*
   TRUE.

   iii. *Eight times a triangular number plus one unit is a square number. (This was noted by Plutarch, the historian.)*
   TRUE.

   iv. *The Greeks could double the cube using just straightedge and compass.*
   FALSE.

   (b) *Give a detailed explanantion of ONE of your answers (i)–(iv) above.*

   i. The large cuneiform tablet Plimpton 322 lists 15 huge Pythagorean triples as discussed in section 2.6. Burton argues that the Babylonians must have had general formulas to generate the entries in the table. With choices of increasing m and n, the entries follow the formulae

   $$x = 2mn, \qquad y = m^2 - n^2, \qquad z = m^2 + n^2$$

   for which $x^2 + y^2 = z^2$. This scheme gives infinitely many Pythagorean triples.

   ii. Let x and y denote the length and width. They satisfy

   $$\begin{aligned} x + y &= 6\tfrac{1}{2} \\ xy &= 7\tfrac{1}{2} \end{aligned}$$

   Put $x = 3\frac{1}{4} + a$ and $y = 3\frac{1}{4}$. Then the first equation holds. The second says

   $$\frac{169}{16} - a^2 = \left(\frac{13}{4} + a\right)\left(\frac{13}{4} - a\right) = \frac{15}{2}$$

   so

   $$a^2 = \frac{169}{16} - \frac{120}{16} = \frac{49}{16}$$

   so $a = \frac{7}{4}$. Thus $x = \frac{13}{4} + \frac{7}{4} = 5$ and $y = \frac{13}{4} - \frac{7}{4} = \frac{3}{2}$.

   iii. The first few triangular numbers are $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$, $t_5 = 15$. Triangular numbers have the form

   $$t_n = \frac{n(n+1)}{2},$$

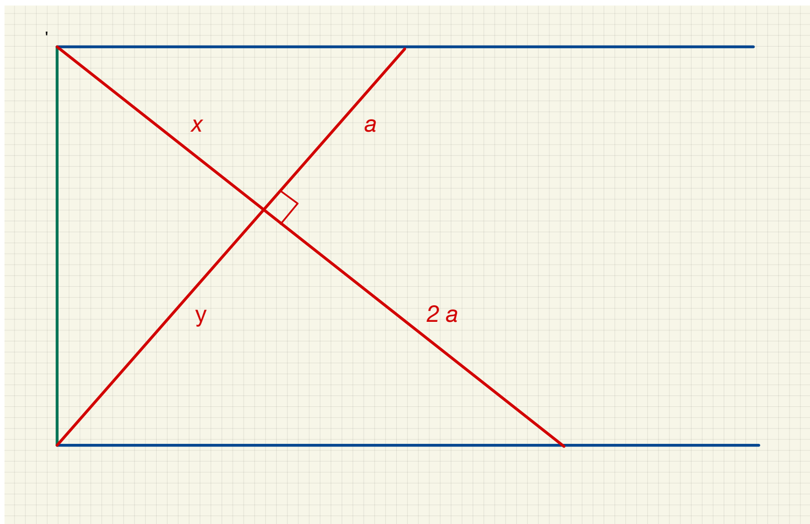
   which can be proved by induction. Computing the Plutarch's formula,

   $$8t_n + 1 = 8\left(\frac{n(n+1)}{2}\right) + 1 = 4n(n+1) + 1 = (2n+1)^2.$$

   Thus eight triangular numbers plus one makes a square number.

iv. The Greeks were unable to double the cube using only straightedge and compass. It was proved in the nineteenth century that such constructions are impossible. However, using more complicated gadgets, the Greeks found ways to do all the Delian problems.

For example, one method of doubling of the cube was attributed to Plato himself. He uses a right angled cross with one leg of length a and the second of length $2a$. The cross is rotated so the two ends slide along parallel lines until the third and fourth legs intersect the parallel lines at a perpendicular bisector. Labelling their length x and y, then the solution to the doubling problem is $x^3 = 2a^3$. We see that the three triangles are all similar since they have the same angles. This implies that the ratios of short to long legs for each is equal



$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}.$$

It follows from the second and then the first equations that

$$2a^3 = 2a \cdot a^2 = \frac{y^2}{x} \cdot a^2 = \frac{(ay)^2}{x} = \frac{(x^2)^2}{x} = x^3.$$

4. (a) *Use Pythagoras method to show* $\sqrt{7}$ *is irrational.*

Here is the proof recorded by Aristotle and found in Euclid. Suppose that $x = \sqrt{7}$ were rational for contradiction. Then there are integers p, q without a common factor that satisfy

$$x = \frac{p}{q}.$$

Substituting into $x^2 = 7$ we see that

$$p^2 = 7q^2.$$

Hence 7 divides p^2, $7|p^2$. But since 7 is prime, we have $7|p$. Thus there is an integer k such that $p = 7k$. Substituting for p,

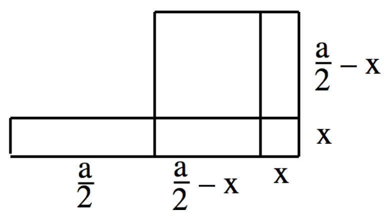
$$49k^2 = (7k)^2 = 7q^2$$

or

$$7k^2 = q^2.$$

Hence $7|q^2$, but since 7 is prime, we have $7|q$. He have reached a contradiction: 7 is a factor of both p and q contrary to our choice that p and q don't have common factors. Thus the contradiction statement must have been false, therefore "$\sqrt{7}$ is irrational" must be true.

(b) *Suppose* $a > 0$ *and* $\frac{a}{2} > x > 0$. *Euclid's Proposition VI-28 essentially says that* $x(a-x) = \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - x\right)^2$. *Explain why this can be deduced from the following diagram. If* $0 < c < \frac{a^2}{4}$ *show how VI-28 can be used to solve* $x(a-x) = c$.



The first two lower rectangles together have a total area $x(a-x)$. This is the same as the area of the big square minus the upper left square, whose area is $2(\frac{a}{2} - x)x + x^2 = x(a-x)$. In other words

$$x(a-x) = \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - x\right)^2.$$

Substituting $x(a-x) = c$ we have

$$c = \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2} - x\right)^2.$$

so

$$\left(\frac{a}{2} - x\right)^2 = \frac{a^2}{4} - c.$$

If $0 < c < \frac{a^2}{4}$ then the right side is positive, and we can take square roots to find the solution

$$\frac{a}{2} - x = \pm\sqrt{\frac{a^2}{4} - c}.$$

which amounts to the quadratic formula

$$x = \frac{a}{2} \mp \sqrt{\frac{a^2}{4} - c}.$$

5. (a) *Use the Euclidean Algorithm to find* gcd(350, 264).
(Other methods receive zero credit.)

(b) *Determine if there are integer solutions, and if there are, find one.*

$$350x + 264y = 2$$

The equation is equivalent to the one we get by dividing the common factor 2.

$$175x + 132y = 1 \qquad (1)$$

Now run the Euclidian Algorithm to find gcd(175, 132).

$$\begin{aligned} 175 &= 1 \cdot 132 + 43 \\ 132 &= 3 \cdot 43 + 3 \\ 43 &= 14 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0 \end{aligned}$$

Thus $\gcd(175, 132) = 1$. We run the Euclidean algorithm backwards

$$\begin{aligned} 1 &= 43 - 14 \cdot 3 \\ &= 43 - 14 \cdot (132 - 3 \cdot 43) = 43 \cdot 43 - 14 \cdot 132 \\ &= 43 \cdot (175 - 132) - 14 \cdot 132 = 43 \cdot 175 - 57 \cdot 132 \end{aligned}$$

Thus $x = 43$ and $y = -57$ satisfies

$$175x + 132y = 1$$

and also (1). Not needed for the answer, but all solutions are then given by

$$x = 43 + 132k, \qquad y = -57 - 175k \qquad \text{where } k \in \mathbb{Z} \text{ is any integer.}$$