Math 3070 § 1. **Final Exam Given 4-30-01** Name: Sample
Treibergs $\alpha\tau$ December 6, 2004

(1.) A standard deck of cards has 52 cards, thirteen cards $\{2-10, J, Q, K, A\}$ in each of four suits ♣, ♢, ♡ and ♠. Five cards are drawn at random without repalcement. What is the probability drawing a royal flush? (10,J,Q,K,A of the same suit.) What is the probability of drawing a flush? (All five cards of the same suit.) What is the probability of of drawing a full house? (three of one kind and a pair of another, e.g. $\{8\clubsuit, 8\heartsuit, 8\spadesuit, K\diamondsuit, K\heartsuit\}$.)

We assume that each hand is equally likely. There are a total of $N = \binom{52}{5} = 2,598,960$ five card hands (order not important), drawn without replacement. There are $n_1 = 4$ royal flushes, only one $A, K, Q, J, 10$ per suit, so the probability of drawing a royal flush is $P = n_1/N = 1.539 \times 10^{-6}$. The number of flushes is $n_2 = 4\binom{13}{5} = 5,148$, the number of choices of suit times number of five card hands in a suit so the probability of a flush is $P = n_2/N = .00198$. The number of hull houses is $n_3 = 13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 3744$, the number of ways of choosing a kind for the three times the number of subsets of three in four suits times the number of remaining kinds for the pair times the number of pairs in four suits, so the probability of a hull house is $n_3/N = .00144$.

(2.) A study of the ability of a cylindrical piece of metal to be formed into the head of a bolt or screw, "headability" did impact tests on specimens of aluminum killed steel and specimens of silicon killed steel. Random samples of the two types of metal gave the following sample statistics. Assume that the headability distributions are approximately normal. Do you agree with the authors that the difference in headability ratings is significant at the 5% level? State the null and altrenative hypotheses. State the test statistic why it is appropriate. State the rejection region for the null hypothesis. Compute and draw a conclusion.

	Number	Sample Mean $\overline{x_i}$	Sample Std. Dev. s_i
Aluminum killed steel:	41	7.14	1.20
Silicon killed steel:	42	7.87	1.32

Let μ_1, μ_2 be means for the means of headability ratings for aluminum killed steel and for the carbon killed steel, resp. The null hypothesis $\mathcal{H}_0 : \mu_1 = \mu_2$ and the alternate is $\mathcal{H}_0 : \mu_1 \neq \mu_2$. Since $n_2 > n_1 > 40$ we may regard this as a large sample, although since it is borderline large, the results will be approximate. Thus we use the z-test with $\sigma_i \approx s_i$. Thus we reject the null hypothesis if $z \le -z_{.025}$ or $z \ge z_{.025} = 1.960$. The statistic is

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{7.14 - 7.87}{\sqrt{\frac{1.20^2}{41} + \frac{1.32^2}{42}}} = -2.710.$$

Thus we reject the null hypothesis: the difference of headability is significant.

(3.) Two candidates AA and GW are running for office in a certain state. It is believed that 0.5 of the voters favor AA. Suppose you conduct a poll to test the alternative hypothesis that more than 0.5 of the voters favor AA. In a random sample of 15 voters, let X denote the number who favor AA. Suppose that the rejection region for the null hypothesis is $X \ge 9$.

a. What is the probability of making a Type I error, that is, of rejecting $\mathcal{H}_0$ even though it is true?
b. What is the probability of a Type II error, that is of accepting $\mathcal{H}_0$ even though the alternative is true, that actually 0.7 of the voters favor AA?
c. How many voters should you poll in order to be sure that the probability of both errors be at most .05?

The null hypothesis $\mathcal{H}_0 :\ p = .5$ and the alternativs $\mathcal{H}_1 :\ p > .5$. The statistic is x/n, the number who favor AA over the number polled. x is approximately a binomial variable. If $\mathcal{H}_0$ is true, $p = .5$. The probability of type-I error, that of rejecting the null hypothesis when it is true is

$$\alpha = P(x \ge 9|p = .5) = 1 - P(x \le 8|p = .5) = 1 - \mathrm{Bin}(8; .5, 15) = 1 - .6964 = .3036.$$

The probability of a type-II error, accepting $\mathcal{H}_0$ even if it's false, $p = .7$, is

$$\beta(.7) = P(x \le 8|p = .7) = \mathrm{Bin}(8; .7, 15) = .1311.$$

To make sure that both α and β are less than .05, assume that the sample will be large and compute β in the z-distribution for the given α. Thus, the statistic is

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0/n}}$$

and $\mathcal{H}_0$ is rejected if $z \ge z_{.05} = 1.645$, or $\hat{p} \ge p_0 + z_\alpha\sqrt{\frac{p_0 q_0}{n}}$. Thus the probability of type-II error, that is of accepting $\mathcal{H}_0$ when it is false, $p = .7 = p'$, is using $\sigma' = \sqrt{p'q'/n}$ and standardizing

$$P(z \le -z_\beta) = \beta = P\left(\hat{p} \le p_0 + z_\alpha\sqrt{\frac{p_0 q_0}{n}}\,\middle|\, p = p'\right) = P\left(z = \frac{\hat{p} - p'}{\sigma'} \le \frac{p_0 + z_\alpha\sqrt{\frac{p_0 q_0}{n}} - p'}{\sigma'}\right).$$

Thus, equating arguments,

$$-z_\beta\sqrt{\frac{p'q'}{n}} = -\sigma' z_\beta = p_0 + z_\alpha\sqrt{\frac{p_0 q_0}{n}} - p'$$

we solve for n to find

$$n = \left(\frac{z_\beta\sqrt{p'q'} + z_\alpha\sqrt{p_0 q_0}}{p' - p_0}\right)^2 = \left(\frac{1.645\sqrt{(.7)(.3)} + 1.645\sqrt{(.5)(.5)}}{.7 - .5}\right)^2 = 62.1$$

so $n = 63$ would suffice.

(4.) Let X be be uniformly distributed in the interval $0 \le X \le 1$. (i.e. the pdf for X is $f(x) = 1$ if $0 \le x \le 1$ and $f(x) = 0$ otherwise.) Suppose $X_1, X_2, \ldots, X_n$ is a random sample taken from this distribution.

a. Let $\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$ be the sample mean of the random sample. Find the expected value $E(\bar{X})$ and variance $V(\bar{X})$.

b. What is the probability that the mean of a random sample of 100 such variables satisfies $\bar{X} \le .45$? Why can you use a large sample approximation?

A single uniform random variable has mean and variance

$$\mu = E(x) = \int_0^1 x\,dx = \frac{1}{2}, \qquad \sigma^2 = E(X^2) - \mu^2 = \int_0^1 x^2\,dx - \mu^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Thus the linear combination, for independent uniform r.v.'s X_i (we're assuming a random sample!)

$$E(\bar{X}) = \frac{1}{n}E(X_1) + \cdots + \frac{1}{n}E(X_n) = \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2},$$
$$V(\bar{X}) = \left(\frac{1}{n}\right)^2 V(X_1) + \cdots + \left(\frac{1}{n}\right)^2 V(X_n) = \frac{1}{12n^2} + \cdots + \frac{1}{12n^2} = \frac{n}{12n^2} = \frac{1}{12n}$$

By the central limit theorem ($n \ge 30$) the distribution of $\bar{X}$ approximates normal, so standardizing,

$$P(\bar{X} \le .45) \approx P\left(Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le \frac{.45 - \mu}{\sigma/\sqrt{n}}\right) = \Phi\left(\frac{.45 - .5}{1/\sqrt{12 \cdot 100}}\right) = \Phi(-1.732) = .0416,$$

writing $\Phi(z) = P(Z \le z)$, for the cumulative normal distribution function.

(5.) The weights of seven people who followed a certain diet were recorded before and after a 2-week period. Compute a two-sided 95% confidence interval for the mean difference of the weights. Assume that the distribution of weight differences is approximately normal.

Person	1	2	3	4	5	6	7
Weight before:	129	133	136	152	141	238	225
Weight after:	130	121	128	147	121	232	220

This is a paired t-test. Computing the differences $d_i = (x_2)_i - (x_1)_i$ we get

$d_i = 1, -12, -8, -5, -20, -6, -5$ with $n = 7$, $\bar{d} = -7.857$ and $s = 6.619$. Thus with $\nu = n - 1 = 6$ d.f., so $t_{.025,6} = 2.447$, the two-sided .05 confidence interval is given by

$$-14.487 = -7.857 - (2.447)\frac{6.619}{\sqrt{6}} = \bar{d} - t_{\alpha/2,\nu}\frac{s}{\sqrt{n}} \le \mu_d \le \bar{d} + t_{\alpha/2,\nu}\frac{s}{\sqrt{n}} = -7.857 + (2.447)\frac{6.619}{\sqrt{6}} = -1.263.$$

(6.) The desired percentage of SiO_2 in a certain type of cement is 7.85. To test whether the true average percentage is 7.85 for a particular production facility, 25 independent samples are analyzed. Suppose that the percentage of SiO_2 is approximately normally distributed with $\sigma = .500$ and that $\bar{x} = 8.02$. (a.)Does this indicate conclusively that the true average differs from 7.85? State the null and altrenative hypotheses. State the test statistic and why it is appropriate. State the rejection region for the null hypothesis. Compute the P-value and draw a conclusion. (b.)If the true average percentage is $\mu = 8.10$ and a level $\alpha = .05$ test based on a test with $n = 25$ is used, what is the probability of detecting this departure from $\mathcal{H}_0$? (c.)What value of n is required to satisfy $\alpha = .05$ and $\beta(8.10) \le .05$?

This is a test of mean where σ, the population standard deviation is known. The null hypothesis $\mathcal{H}_0 : \ \mu = \mu_0 = 7.85$; the alternative $\mathcal{H}_1 : \ \mu \ne \mu_0$. Since σ is known, $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ is normally distributed for all n and so, for a two-tailed test, we reject $\mathcal{H}_0$ if $z \ge z_{\alpha/2}$ or $z \le -z_{\alpha/2}$. In this case, $\alpha = .05$ so $z_{\alpha/2} = z_{.025} = 1.960$.

$$P = 2P\left(Z \ge |z|\right) = 2P\left(Z \ge \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right) = 2P\left(Z \ge \frac{8.02 - 7.85}{.500/\sqrt{25}}\right) = 2P(Z \ge 1.70) = 2(.0446) = .0892.$$

This is greater than α so not significant: we accept the null hypothesis that the true average is 7.85.

If $\mathcal{H}_0$ is false and $\mu = \mu' = \mu_0 + d = 8.10$, then the probability of accepting $\mathcal{H}_0$ is, by standardizing,

$$\begin{aligned}
\beta =& P\left(\mu_0 - \frac{\sigma z_{\alpha/2}}{\sqrt{n}} \le \bar{x} \le \mu_0 + \frac{\sigma z_{\alpha/2}}{\sqrt{n}} \,\middle|\, \mu = \mu'\right) \\
=& P\left(\frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} - z_{\alpha/2} \le Z = \frac{\bar{x} - \mu'}{\sigma/\sqrt{n}} \le \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} + z_{\alpha/2}\right) \\
P(Z \le -z_\beta) =& P\left(Z \le \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} + z_{\alpha/2}\right) - P\left(Z \le \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} - z_{\alpha/2}\right)
\end{aligned}$$

Since $\mu' > \mu_0$ the last term is $\Phi(\text{very neg.}) \approx 0$ so we neglect it. Equating arguments we can solve for n,

$$-z_\beta \approx \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} + z_{\alpha/2} \qquad \Longrightarrow \qquad n \approx \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{(\mu' - \mu_0)^2} = \frac{(1.96 + 1.645)^2 (.500)^2}{(8.10 - 7.85)^2} = 51.98$$

so taking $n = 52$ will give $\alpha, \beta = .05$.

(7.) A study of ionizing radiation as a method of preserving horticultural products reported that 147 of 186 irradiated garlic bulbs were marketable after 240 days of irradiation whereas only 132 of 186 untreated bulbs were marketable after this length of time. Does this data suggest that ionizing radiation is beneficial as far as marketability is concerned? State the null and altrenative hypotheses. State the test statistic and why it is appropriate. Compute the P-value and draw a conclusion.

This is a test comparing two proportions. Let p_1 be the proportion of irradiated bulbs marketable after 240 days and p_2 the proportion of marketable unirradiated bulbs. The studies have the data $n_1 = 186$, $\hat{p}_1 = 147/n_1 = .790$ so $\hat{q}_1 = 1 - \hat{p}_1 = .210$, $n_2 = 186$, $\hat{p}_2 = 132/n_2 = .710$ so $\hat{q}_2 = 1 - \hat{p}_2 = .290$. Since $\hat{p}_1 n_1 \ge \hat{q}_1 n_1 = 39 \ge 10$ and $\hat{p}_2 n_2 \ge \hat{q}_2 n_2 = 54 \ge 10$ we may use the large sample approximation. The null hypothesis is $\mathcal{H}_0 : \ \mu_1 = \mu_2$ and the alternative is $\mathcal{H}_1 : \mu_1 > \mu_2$, one-tailed, since we wish to see if there is strong evidence that irradiating increases marketability. By the null hypothesis $\mathcal{H}_0 : p_1 = p_2$ we get a better esimate of proportion by pooling

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{147 + 132}{186 + 186} = .750 \qquad \Longrightarrow \qquad \bar{q} = 1 - \bar{p} = .250.$$

The pooled z-score is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{.790 - .710}{\sqrt{(.75)(.25)\left(\frac{1}{186} + \frac{1}{186}\right)}} = 1.782.$$

For one tailed tests, the P-value is $P = P(Z \ge z) = P(Z \le -1.782) = .0373$. This is significant at the 5% level, so we reject the null hypothesis: irradiating helps marketability.

(8.) Suppose the continuous variables X and Y satisfy the joint probability distribution

$$f(x,y) = \begin{cases} K(x^2+y^2), & \text{if } -1 \le x \le 1 \text{ and } -1 \le y \le 1. \\ 0, & \text{otherwise.} \end{cases}$$

(a.)Find K for f to be a pdf. (b.)Are X and Y independent? (c.)Are X and Y correlated?

To find K we compute the total probability $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} K(x^2+y^2)\,dxdy = 2K\int_{-1}^{1}\int_{-1}^{1} x^2\,dxdy = \frac{8K}{3}$, so $K = 3/8$. To be independent, we have to check that $f(x,y) = g(x)h(y)$ where $g(x) = h(x)$ (by symmetry) are the marginal probabilities. Thus

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \frac{3}{8}\int_{-1}^{1} x^2+y^2\,dy = \dfrac{3x^2+1}{4} & \text{if } -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f(x,y) \ne g(x)h(y)$ so X, Y are not independent. Now, by symmetry,

$\mu_X = \mu_Y = \int_{-\infty}^{\infty} xg(x)\,dx = \int_{-1}^{1} \dfrac{3x^3+x}{4}\,dx = 0$ since the integrand is odd. The covariance is then

$$\operatorname{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy - \mu_X\mu_Y = \frac{3}{8}\int_{-1}^{1}\int_{-1}^{1} xy(x^2+y^2)\,dxdy = 0$$

since the integrand is odd. This implies the variables are not correlated: their correlation coefficient $\rho = \operatorname{Cov}(X,Y)/(\sigma_X\sigma_Y) = 0$.

THE FOLLOWING ARE QUESTIONS FOCUSING ON THE LAST QUARTER OF THE SEMESTER.

(1.) A plan for an executive traveller's club has been developed by Useless Airlines on the premise that 5% of its current customers would qualify for a membership. Of a random sample of 500 customers, 39 were found to qualify.

(a.)With this data, test at the .05 level of significance the null hypothesis that 5% is correct against the alternative that 5% is not correct.

(b.)What is the probability, that when the test in part (a.) were used, the company's premise will be judged correct when in fact 10% of all current customers qualify?

(c.)How large a random sample is required to be sure that the probability is at most 5% that when the test in part (a.) were used, the company's premise will be judged correct when in fact 10% of all current customers qualify?

This is a test based on a single proportion. The alternate hypothesis is $\mathcal{H}_1 : \ p \ne .05 = p_0$. Since $n \cdot p_0 = 500 \cdot .05 = 25 \ge 10$ and $n(1-p_0) \ge 10$ this is a large sample and the test statistic z is approximately normally distributed. The rejection region for $\mathcal{H}_0$ at the $\alpha = .05$ significance level is $\{z : \ z \ge z_{\alpha/2}$ or $z \le -z_{\alpha/2}\}$ where $z_{\alpha/2} = 1.960$. The estimator is $\hat{p} = 39/500 = .078$. The test statistic is

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{.078 - .05}{\sqrt{(.05)(.95)/500}} = 2.873$$

thus we reject the null hypothesis.

The probability of Type II error, that of accepting $\mathcal{H}_0$ given $p = .10 = p'$ is

$$\begin{aligned}
\beta(.10) =& P\left(p_0 - z_{\alpha/2}\sqrt{p_0(1-p_0)/n} < \hat{p} < p_0 + z_{\alpha/2}\sqrt{p_0(1-p_0)/n} \;\middle|\; p = p'\right) \\
=& P\left(\hat{p} < p_0 + z_{\alpha/2}\sqrt{p_0(1-p_0)/n}\right) - P\left(\hat{p} \le p_0 - z_{\alpha/2}\sqrt{p_0(1-p_0)/n}\right) \\
\approx& \Phi\left(\frac{p_0 + z_{\alpha/2}\sqrt{p_0(1-p_0)/n} - p'}{\sqrt{p'(1-p')/n}}\right) - \Phi\left(\frac{p_0 - z_{\alpha/2}\sqrt{p_0(1-p_0)/n} - p'}{\sqrt{p'(1-p')/n}}\right) \\
=& \Phi(-2.303) - \Phi(-5.151) = .0105 - .0000 = .0105.
\end{aligned}$$

Finally, to find the number required we set $\beta(.10) = .05$ we solve for n in the equation above. It can't be solved exactly. Since the second Φ contributes a very small amount (since $p' > p$ it represents the probability below the lower cutoff for the rejection region) we simply neglect that term and solve $\beta = \Phi(-z_\beta) = \Phi(?)$. Thus, for $\beta = .05$ so $z_\beta = 1.645$, we solve

$$-z_\beta = \frac{p_0 + z_{\alpha/2}\sqrt{p_0(1-p_0)/n} - p'}{\sqrt{p'(1-p')/n}}$$

to get

$$n \approx \left[\frac{z_{\alpha/2}\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p'(1-p')}}{p' - p_0}\right]^2 = \left[\frac{1.96\sqrt{.05\cdot .95} + 1.645\sqrt{.1\cdot .9}}{.1 - .05}\right]^2 = 339.1$$

so we need about $n = 340$. (As expected since for $n = 500$, $\beta < .05$.)

(2.) The mean annual snowfall in Juneau, Alaska is believed to be 100 in. per year. Assuming that the snowfall distribution is approximately normal, test the hypothesis that the snowfall is 100 in. against the alternative that it is more at the .05 level of significance, based on the random sample readings. Does the data strongly suggest that the snowfall is higher? What is the P-value? 93.10, 115.93, 138.16, 124.01, 116.40, 128.76, 150.13, 95.74, 107.66, 108.17, 125.61, 104.89, 99.01.

The null hypothesis is $\mathcal{H}_0 :\ \mu = \mu_0 = 100.00$ and the alternative $\mathcal{H}_0 :\ \mu > \mu_0$. Since the sample is small $n = 13$, with $\nu = n - 1 = 12$ degrees of freedom, since we don't know σ, the population standard deviation, and since we have assumed that the temperatures are a random sample of approximately normal variables, the appropriate statistic is $T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$. At the $\alpha = .05$ significance we reject $\mathcal{H}_0$ if $T > t_{\nu,\alpha} = 1.782$. The sample mean and standard deviation are $\bar{x} = 115.97$ and $s = 16.934$ whence

$$t = \frac{115.97 - 100.00}{16.934/\sqrt{13}} = 3.261 > t_{12,.05}.$$

Thus we reject $\mathcal{H}_0$: the data suggests the snowfall is higher. The P-value is $P = P(T \ge t) = .003$ (from Table A8 of t-curve tail areas.)

(3.) The true average breaking strength of ceramic insulators of a certain type is supposed to be at least 10psi. They will be used in a particular application unless data indicates conclusively that this specification has not been met. A test of hypothesis using $\alpha = .01$ is to be based on a random sample of 10 insulators. Assume that the breaking strength's distribution is normal with unknown standard deviation.

(a.) If the true standard deviation is .80, how likely is it that insulators will be judged satisfactory when the true average breaking strength is only 9.5? 9.0?

(b.) What sample size would be necessary to have a 75% chance of detecting that the true average breaking strength is 9.5 when the true standard deviation is .80?

The null hypothesis is $\mathcal{H}_0 :\ \mu = \mu_0 = 10.0$ psi and $\mathcal{H}_a :\ \mu < \mu_0$. Since the population is assumed to be normal, σ is unknown and the sample size $n = 10$ is small, the appropriate statistic is $T = \dfrac{\bar{x} - \mu_0}{S/\sqrt{n}}$ and $\mathcal{H}_0$ is rejected at $\alpha = .01$ significance if $T < -t_{9,.01} - 2.821$. Suppose that the true breaking strength is $\mu' < \mu_0$ and that the population standard deviation is $\sigma = .80$ then the probability of the Type II error $\beta(\mu') = P(T > -t_{9,.01} | \mu = \mu')$ can be found in Table A17. Computing

$$d(9.5) = \frac{|\mu_0 - \mu'|}{\sigma} = \frac{|10.0 - 9.5|}{.80} = .645; \qquad d(9.0) = \frac{|10.0 - 9.0|}{.80} = 1.25$$

We read from the $v = n - 1 = 9$, one tailed, $\alpha = .01$ power curve that $\beta(d = .65, \alpha = .01, \nu = 9) \approx .6$ so $\beta(9.5) \approx .6$ and since $\beta(d = 1.25, \alpha = .01, \nu = 9) \approx .2$ so $\beta(9.0) \approx .2$. To deduce the n required is to find n so that $\beta(9.5) = .25$, that is, with .75 probability that $\mathcal{H}_0$ is rejected when $\mu = 9.5$. Looking in Table A17, for one tail tests, the point $(d, \beta) = (.65, .25)$ lies between the $\nu = 19$ and $\nu = 29$ curves, closer to the latter, so $\nu(\alpha = .01, \beta = .25, d = .65) \approx 25$, approximately. The on-line power calculator (at `http://calculators.stat.ucla.edu/`) yields $n = 25.819$, $\beta(9.0) \approx .156$ and $\beta(9.5) \approx .738$.

(4.) Suppose $X_1, \ldots, X_n$ are independent Poisson variables, each with parameter λ. When n is large, the sample mean $\overline{X}$ has approximately a normal distribution with $\mu(\overline{X}) = \lambda$ and $\sigma^2 = V(\overline{X}) = \lambda/n$. This implies that

$Z = \dfrac{\overline{X} - \lambda}{\sqrt{\lambda/n}}$ has approximately a standard normal distribution. For testing $\mathcal{H}_0 :\ \lambda = \lambda_0$, we can replace λ by λ_0 in Z to obtain a test statistic. This statistic is actually preferred to the large sample statistic with denominator $S/\sqrt{n}$ when X_i's are Poisson because it is taylored explicitly to the Poisson assumption. If the number of requests for consulting for MyCo during a five day workweek has a Poisson distribution and the total number of consulting requests during a 36 week period is 160, does this suggest that the true average number of weekly requests exceeds 4.0? Test using $\alpha = .02$.

The alternative hypothesis is $\mathcal{H}_a :\ \lambda > \lambda_0 = 4.0$. The average number of consulting requests in $n = 36$ trials is $\bar{x} = 160/36$. At the $\alpha = .02$ significance, $\mathcal{H}_0$ is rejected if $z \ge z_{.02} = 2.054$ (Interpolate Table A5) Computing

$$z = \frac{\bar{x} - \lambda_0}{\sqrt{\lambda_0/n}} = \frac{160/36 - 4.0}{\sqrt{4.0/36}} = 1.333$$

thus the null hypothesis is accepted.

(5.) It is claimed that the resistance of electric wire can be reduced by more than .050 ohm by alloying. If a random sample of 32 standard wires yielded a sample average of 0.136 and $s = 0.004$ ohm and a random sample of 32 alloyed wires yielded a sample average 0.038 ohm and $s_2 = 0.005$ ohm, at the .05 level of significance, does this support the claim? What is the P-value? What is the probability of a Type II error if the actual improvement is .100 ohm?

Let X denote the r.v. giving standard wire resistance and Y the alloyed wire resistance. Both are assumed to be approximately normally distributed with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. Let $\theta = \mu_1 - \mu_2$. It is estimated by $\hat{\theta} = \bar{X} - \bar{Y}$ which is the difference of independent random samples of sizes n_1, n_2 resp. The null hypothesis is $\mathcal{H}_0 :\ \theta = \theta_0 = .050$ and the alternative $\mathcal{H}_1 :\ \theta > \theta_0$. The sample sizes are small according to our rule of thumb (≤ 40). The σ_i's are unknown, we use the two sample t-statistic. The degrees of freedom is calculated from

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2} = \frac{\left(\frac{(.004)^2}{32} + \frac{(.005)^2}{32}\right)^2}{\frac{1}{31}\left(\frac{(.004)^2}{32}\right)^2 + \frac{1}{31}\left(\frac{(.005)^2}{32}\right)^2} = 59.1$$

which gets rounded down to $\nu = 59$. At the $\alpha = .05$ level of significance, $\mathcal{H}_0$ is rejected if $t \ge t_{59,.05} = 1.671$. Computing

$$t = \frac{\hat{\theta} - \theta_0}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \frac{(.136 - .038) - .050}{\sqrt{\dfrac{(.004)^2}{32} + \dfrac{(.005)^2}{32}}} = 42.4$$

so the null hypothesis is rejected. Since for $\nu = 60$, $P(T \ge 3.46) = .0005$, the P-value for $\nu = 59$ is $P(T \ge t) \approx .000$ from Table A8. The β cannot be calculated easily because it depends on both s_1 and s_2. However, by using the power calculator at `http://calculators.stat.ucla.edu/` [p. 370] we find for $\theta' = .100$ that $\beta = .0000$.

(6.) Find a two tailed 95% confidence interval for the difference of true average ultimate load (kN) of two types of beams, fiberglass grid and carbon grid from the following random samples. Assume that the underlying distributions are normal.

Fiberglass 34.4	29.9	34.2	34.9	32.6	31.2	32.6	32.6	33.2	32.4	33.1
grid: 36.3	33.3	36.5	30.4	36.7	34.8	36.5	31.3	29.2	31.9.	
Carbon 50.0	36.8	39.2	41.9	36.1	40.3	43.3	50.5	40.1	33.0	
grid: 34.6	39.4	50.6	51.0	45.0	45.9	42.6	45.7	45.6.		

Let X denote the population of fiberglass beam ultimate loads, and Y the population of carbon beam ultimate loads. Both are assumed to be approximately normal with with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. Let $\theta = \mu_1 - \mu_2$. It is estimated by $\hat{\theta} = \bar{X} - \bar{Y}$ which is the difference of independent random samples of sizes n_1, n_2 resp. Since the sample sizes are small $n_1 = 21,\ n_2 = 19 \le 40$ and that the σ_i's are unknown, we use the two sample t-statistic. Computing, we find $\bar{x} = 33.24$, $s_1 = 2.205$, $\bar{y} = 42.72$, $s_2 = 13.883$, The degrees of freedom is calculated from

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2} = \frac{\left(\frac{(2.205)^2}{21} + \frac{(13.883)^2}{19}\right)^2}{\frac{1}{20}\left(\frac{(2.205)^2}{21}\right)^2 + \frac{1}{18}\left(\frac{(13.883)^2}{19}\right)^2} = 18.82$$

which is rounded down to $\nu = 18$. The two sided confidence interval for $\alpha = .05$ significance depends on the critical value $t_{18,.025} = 2.101$ are given by

$$\hat{\theta} \pm t_{18,.025}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = (33.24 - 42.72) \pm (2.101)\sqrt{\frac{(2.205)^2}{21} + \frac{(13.883)^2}{19}}$$

so with 95% confidence, $-16.25 < \mu_1 - \mu_2 < -2.71$.

(7.) Two observers measure cardiac output of 23 patients using Doppler endocardiography. Is there strong evidence that the two observers are measuring differently? Choosing significance $\alpha = .05$, and assuming that the differences are normally distributed, test the null hypothesis that there is no difference in measurements between the two observers.

Patient	Obs A	Obs B									
1	4.9	5.9	6	6.4	5.6	12	7.6	7.3	18	8.2	11.6
2	5.1	5.9	7	6.6	5.8	13	7.9	7.6	19	9.1	10.7
3	5.2	6.2	8	5.8	8.4	14	8.0	8.3	20	9.8	10.1
4	6.4	4.2	9	6.4	7.7	15	8.4	8.2	21	9.8	11.2
5	5.9	6.0	10	6.8	7.7	16	8.8	8.1	22	11.3	9.0
			11	7.5	6.7	17	8.5	10.0	23	11.3	10.0

Let X denote the cardiac output readings made by observer A and Y the readings from B. Since we wish to eliminate the influence of patient differences on the readings, we use a paired test. We estimate $\mu_D = \mu_X - \mu_Y$ the mean of the differences. The differences yield the data $d_k = x_k - y_k$. Thus $d_1 = 4.9 - 5.9 = -1.0$, $d_2 = 5.1 - 5.9 = -0.8$, etc. Since the differences are assumed to be normally distributed, since we don't know the standard deviation of the differences, and since the sample is small $n = 23$ we use a t-test. The null hypothesis is $\mathcal{H}_0 :\ \mu_D = \mu_0 = 0$ and the alternative is $\mathcal{H}_a :\ \mu_D \neq \mu_0$. The appropriate statistic for $\nu = n - 1 = 22$ degrees of freedom is $T = \dfrac{\bar{D} - \mu_0}{S_D/\sqrt{n}}$ and the null hypothesis is rejected at $\alpha = .05$ significance if $t \ge t_{\nu,\alpha/2} = 2.074$ or $t \le -t_{\nu,\alpha/2}$. Computing, we find $\bar{d} = -.283$ and $s_D = 1.399$ and

$$t = \frac{\bar{d} - \mu_0}{s_D/\sqrt{n}} = \frac{-.283 - 0}{1.399/\sqrt{23}} = -.969$$

thus $\mathcal{H}_0$ is accepted: there is no strong evidence that the observers read differently.

(8.) A sample of 300 urban adult residents of Ohio revealed 63 who favored increasing the highway speed limit from 55 to 65 mph, whereas a sample of 180 rural residents yielded 75 who favored the increase. Does this data indicate the sentiment for increasing the speed limit is different for the two groups of residents? (a.)Test $\mathcal{H}_0 :\ p_1 = p_2$ versus $\mathcal{H}_a :\ p_1 \neq p_2$ using $\alpha = .05$, where p_1 refers to the urban population. (b.)If the true proportion favoring the increase are actually $p_1 = .20$ (urban) and $p_2 =, 40$ (rural), what is the probability that $\mathcal{H}_0$ will be rejected using a level of .05 test with $m = 300$ and $n = 180$?

This is a test for difference of proportions. The proportion of urbanites who favor increasing the speed limit is $\hat{p}_1 = 63/300 = .21$ and the proportion of rurals is $\hat{p}_2 = 75/180 = .417$. Since $\hat{p}_1 n_1 = 63$, $\hat{q}_1 n_1 = 237$, $\hat{p}_2 n_2 = 75$ $\hat{q}_2 n_2 = 105$ all are at least 10, we may use the large sample z statistics for differences of proportion. At the $\alpha = .05$ significance, the null hypothesis is rejected if $z \ge z_{\alpha/2} = 1.960$ or $z \le -1.960$. If $\mathcal{H}_0$ holds, then $p = p_1 = p_2$ so the best estimator for p is the pooled proportion

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{63 + 75}{300 + 180} = .2875; \qquad \hat{q} = 1 - \hat{p} = .7125.$$

The corresponding z-statistic is (assuming $\mathcal{H}_0 :\ p_1 = p_2$)

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\,(1/n_1 + 1/n_2)}} = \frac{.21 - .417}{\sqrt{(.2875)(.7125)(1/300 + 1/180)}} = -4.843$$

which tells us to reject the null hypothesis: the proportions are different.

To compute the probability of Type II error, in case the true proportions are $p_1' = .20$ and $p_2' = .40$ with the same $n_1 = 300$ and $n_2 = 180$, the difference $\hat{p}_1 - \hat{p}_2$ now has the standard deviation

$$\bar{\sigma} = \sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1'q_1'}{n_1} + \frac{p_2'q_2'}{n_2}} = \sqrt{\frac{(.2)(.8)}{300} + \frac{(.4)(.6)}{180}} = .0432.$$

Then approximating $\hat{p} \approx \bar{p} = (n_1p_1' + n_2p_2')/(n_1+n_2) = (300(.2)+180(.4))/(300+180) = .275$, $\hat{q} \approx \bar{q} = 1 - \bar{p} = .725$, standardizing so that $E(p_1 - p_2) = p_1' - p_2' = .2 - .4 = -.2$.

$$\begin{aligned}
\beta(p_1', p_2') &= P\left(-z_{\frac{\alpha}{2}}\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} < p_1 - p_2 < z_{\frac{\alpha}{2}}\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} \,\middle|\, p_1 = p_1',\ p_2 = p_2' \right)\\
&\approx \Phi\left(\frac{z_{\frac{\alpha}{2}}}{\bar{\sigma}}\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} - \frac{p_1'-p_2'}{\bar{\sigma}}\right) - \Phi\left(-\frac{z_{\frac{\alpha}{2}}}{\bar{\sigma}}\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} - \frac{p_1'-p_2'}{\bar{\sigma}}\right)\\
&= \Phi\left(\frac{1.960}{.0432}\sqrt{(.275)(.725)\left(\tfrac{1}{300}+\tfrac{1}{180}\right)} - \frac{.20-.40}{.0432}\right) - \Phi\left(-\frac{1.960}{.0432}\sqrt{(.275)(.725)\left(\tfrac{1}{300}+\tfrac{1}{180}\right)} - \frac{.20-.40}{.0432}\right)\\
&= \Phi(6.540) - \Phi(2.720) = 1.0000 - .9967 = .0033.
\end{aligned}$$

(9.) The sample standard deviation of sodium concentration in whole blood (mEq/L) for $n_1 = 20$ marine eels was found to be $s_1 = 40.5$ whereas the sample standard deviation of $n_2 = 20$ freshwater eels was $s_2 = 32.1$. Assuming normality of the two concentration distributions, construct a confidence interval for σ_1^2/σ_2^2 at level .05 to see whether the data suggests any difference between the concentration variances for the two types of eels.

Let X and Y be the r.v.'s giving sodium concentrations of saltwater and freshwater eels, which we assume to be approximately normal with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. We consider the ratio of sample variances from two random samples of size $m = n = 20$. The statistic is $F = \dfrac{S_1^2\sigma_2^2}{S_2^2\sigma_2^2}$ satisfies an F-distribution with $\nu_1 = n_1 - 1 = 19$ and $\nu_2 = n_2 - 1 = 19$ degrees of freedom. At the $\alpha = .05$ significance, we have confidence bounds $F_{1-\alpha/2,\nu_1,\nu_2} = 1/F_{\alpha/2,\nu_2,\nu_1} < F < F_{\alpha/2,\nu_1,\nu_2}$. Finding an f-table for $\alpha = .025$ (e.g. in text of Levine, Ramsey & Schmid) $F(.025, 15, 19) = 2.62$ and $F(.025, 20, 19) = 2.51$. Interpolating, $F(.025, 19, 19) = 2.49$ so $F(.975, 19, 19) = .401$. Multiplying through $\sigma_1^2\sigma_2^{-2}$ we have, we find

$$.639 = \frac{(40.5)^2}{(2.49)(32.1)^2} = \frac{1}{F(.025,19,19)}\frac{s_1^2}{s_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < F(.025,19,19)\frac{s_1^2}{s_2^2} = \frac{(2.49)(40.5)^2}{(32.1)^2} = 3.96.$$

Thus, since 1 is within the confidence interval, there is no strong evidence against $\sigma_1 = \sigma_2$.

(10.) A sample of 50 lenses used in eyeglasses yields a sample mean thickness of 3.05 mm and a sample standard deviation of .34 mm. The desired true average thickness of such lenses is 3.20 mm. (a.)Does the data strongly suggest that the average thickness of sauch lenses is other than what is desired? Test using $\alpha = .05$. (b.)Suppose the experimenter had believed before collecting the data that the value of σ was approximately .30. If the experimeter wished the probability of Type II error to be .05 when $\mu = 3.00$, was the sample size of 50 unnecessarily large?

The sample size $n = 50$ is large so that we may use z-test. Let X be the r.v. giving lens thicknesses. It has a population mean μ and a population standard deviation σ. The null hypothesis is $\mathcal{H}_0:\ \mu = \mu_0 = 3.20$ and the alternative $\mathcal{H}_a:\ \mu \neq \mu_0$. At $\alpha = .05$ significance, the null hypothesis is rejected if $z \geq z_{\alpha/2} = 1.960$ or $z \leq -z_{\alpha/2}$. Thus

$$z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{3.05 - 3.20}{.34/\sqrt{50}} = -3.120$$

thus we reject the null hypothesis: the thicknesses is other than desired.

The probability of Type II error is, by standardizing z, $\beta(\mu') =$

$$P\left(\mu_0 - \frac{z_{\alpha/2}\sigma}{\sqrt{n}} < z < \mu_0 + \frac{z_{\alpha/2}\sigma}{\sqrt{n}} \,\middle|\, \mu = \mu'\right) = P\left(Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) - P\left(Z \leq z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right).$$

We cannot solve for n exactly. However, if $\mu' = \mu_0 - d = 3.00 < \mu_0$ then the first term, corresponding to the upper cutoff value is very close to unity, therefore we may approximate it by 1. But then $\beta = 1 - P(Z \leq z_\beta)$ implies

$z_\beta = -z_{\alpha/2} + \dfrac{\mu_0 - \mu'}{\sigma/\sqrt{n}}$ which can be solved for n to yield in case $\beta = .05$, $\sigma = .30$

$$n \approx \left\{ \frac{\sigma(z_\beta + z_{\alpha/2})}{\mu_0 - \mu'} \right\}^2 = \left\{ \frac{(.30)(1.645 + 1.960)}{.320 - .300} \right\}^2 = 2924.11.$$

The sample size was not excessive. In fact, for $n = 50$, $\mu' = .3$, $\sigma = .30$, $\beta = P(Z \le 2.431) - P(Z \le -1.489) = .9925 - .0680 = .9245$ so the probability of Type II error is large.