Math 3070 § 1. **Approximating the Hypergeometric** Name: Example
Treibergs **by Binomial Distribution Example.** June 12, 2011

The hypergeometric distribution describes the probability that if a sample of size n is chosen without replacement from a population of N consisting of M successes and $L = N - M$ failures, that there be x successes in the sample. In **R**, this is given by

$$P(X = x) = \mathrm{h}(x; n, M, N) = \texttt{dhyper}(\texttt{x}, \texttt{M}, \texttt{L}, \texttt{n}).$$

Devore's rule of thumb, is that it's O.K. to approximate if the sample is small compared to the population.

> In an experiment where each trial results in S or F, but the sampling is without replacement from a population of size N, if the sample size n is at most 5% of the population size, then the experiment may be analyzed as if it were a binomial experiment.

We print and graph the hypergeometric pmf and its binomial approximation. We see how the errors compare as n/N decreases for several p's

R Session:

```
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> # Devore Rule of Thumb for hypergeometric
> # N = number of trials
> # N = population size
> # n = sample size
> # M = number of successes in pop
> # x = number of successes in sample
> N <- 20
> M <- 8
> n <- 7
```

```
>
> ################ SUBPROGRAM TO LIST HYP AND BINOM PMF'S ###################
> # Use fix(listHB) to write function subprogram.
> listHB <- function(N,M,n)
+               {
+                L<- N-M
+                p<- M/N
+                x <- matrix(numeric(3*(n+1)),ncol=3,
+                dimnames=list(0:n,c("dhyper", "dbinom", "difference")))
+                x[,1]<-dhyper(0:n,M,L,n)
+                x[,2]<-dbinom(0:n,n,p)
+                x[,3]<-x[,1]-x[,2]
+                error <- max(abs(x[,3]))
+                if(n/N <= .05)
+                              {answer <- "IS"}
+                else
+                              {answer<-"IS NOT"}
+                cat("Binomial Approximation to Hypergeometric\n Pop. Size N =",
+                N,"\n No. Successes in Pop., M =",M, "\n No. failures in Pop.,
+                L =",L,"\n Proportion of Successes p = M / N =",p,
+                "\n Sample Size n =",n,"\n Sample Frction of Population,
+                n / N =",n/N, "\n Devore's Rule of Thumb ",answer,
+                " satisfied.\n\n")
+                print(x)
+                cat("\n\n Maximum Error =",error,"\n\n")
+               }

> listHB(12,6,5)
Binomial Approximation to Hypergeometric
 Pop. Size N = 12
 No. Successes in Pop., M = 6
 No. failures in Pop., L = 6
 Proportion of Successes p = M / N = 0.5
 Sample Size n = 5
 Sample Frction of Population, n / N = 0.4166667
 Devore's Rule of Thumb  IS NOT  satisfied.

       dhyper  dbinom  difference
0 0.007575758 0.03125 -0.02367424
1 0.113636364 0.15625 -0.04261364
2 0.378787879 0.31250  0.06628788
3 0.378787879 0.31250  0.06628788
4 0.113636364 0.15625 -0.04261364
5 0.007575758 0.03125 -0.02367424


 Maximum Error = 0.06628788
>
>
> #  .066 error is large, but n is 42% of  N
```

```
> listHB(50,6,5)
Binomial Approximation to Hypergeometric
 Pop. Size N = 50
 No. Successes in Pop., M = 6
 No. failures in Pop., L = 44
 Proportion of Successes p = M / N = 0.12
 Sample Size n = 5
 Sample Frction of Population, n / N = 0.1
 Devore's Rule of Thumb  IS NOT  satisfied.

        dhyper       dbinom    difference
0 5.125677e-01 0.5277319168 -1.516419e-02
1 3.844258e-01 0.3598172160  2.460858e-02
2 9.376239e-02 0.0981319680 -4.369579e-03
3 8.929751e-03 0.0133816320 -4.451881e-03
4 3.115030e-04 0.0009123840 -6.008810e-04
5 2.831845e-06 0.0000248832 -2.205135e-05


 Maximum Error = 0.02460858

> listHB(50,25,5)
Binomial Approximation to Hypergeometric
 Pop. Size N = 50
 No. Successes in Pop., M = 25
 No. failures in Pop., L = 25
 Proportion of Successes p = M / N = 0.5
 Sample Size n = 5
 Sample Frction of Population, n / N = 0.1
 Devore's Rule of Thumb  IS NOT  satisfied.

      dhyper  dbinom   difference
0 0.02507599 0.03125 -0.006174012
1 0.14926183 0.15625 -0.006988168
2 0.32566218 0.31250  0.013162180
3 0.32566218 0.31250  0.013162180
4 0.14926183 0.15625 -0.006988168
5 0.02507599 0.03125 -0.006174012


 Maximum Error = 0.01316218
```

```
> listHB(500,250,5)
Binomial Approximation to Hypergeometric
 Pop. Size N = 500
 No. Successes in Pop., M = 250
 No. failures in Pop., L = 250
 Proportion of Successes p = M / N = 0.5
 Sample Size n = 5
 Sample Frction of Population, n / N = 0.01
 Devore's Rule of Thumb  IS  satisfied.

      dhyper  dbinom    difference
0 0.03062564 0.03125 -0.0006243624
1 0.15561808 0.15625 -0.0006319228
2 0.31375629 0.31250  0.0012562852
3 0.31375629 0.31250  0.0012562852
4 0.15561808 0.15625 -0.0006319228
5 0.03062564 0.03125 -0.0006243624


 Maximum Error = 0.001256285

> listHB(500,250,50)
Binomial Approximation to Hypergeometric
 Pop. Size N = 500
 No. Successes in Pop., M = 250
 No. failures in Pop., L = 250
 Proportion of Successes p = M / N = 0.5
 Sample Size n = 50
 Sample Frction of Population, n / N = 0.1
 Devore's Rule of Thumb  IS NOT  satisfied.

         dhyper       dbinom    difference
0  5.823448e-17 8.881784e-16 -8.299439e-16
1  3.621547e-15 4.440892e-14 -4.078737e-14
2  1.093725e-13 1.088019e-12 -9.786460e-13
3  2.137882e-12 1.740830e-11 -1.527041e-11
4  3.041505e-11 2.045475e-10 -1.741324e-10
5  3.357821e-10 1.881837e-09 -1.546055e-09
6  2.995144e-09 1.411378e-08 -1.111863e-08
7  2.219176e-08 8.871517e-08 -6.652341e-08
8  1.393520e-07 4.768440e-07 -3.374920e-07
9  7.529897e-07 2.225272e-06 -1.472282e-06
10 3.542996e-06 9.123616e-06 -5.580620e-06
11 1.465436e-05 3.317678e-05 -1.852243e-05
12 5.369232e-05 1.078246e-04 -5.413223e-05
13 1.753678e-04 3.151795e-04 -1.398117e-04
14 5.132843e-04 8.329743e-04 -3.196899e-04
15 1.352206e-03 1.999138e-03 -6.469324e-04
16 3.218140e-03 4.373115e-03 -1.154975e-03
17 6.940505e-03 8.746230e-03 -1.805724e-03
18 1.359978e-02 1.603475e-02 -2.434972e-03
19 2.426455e-02 2.700590e-02 -2.741355e-03
```

```
20 3.949055e-02 4.185915e-02 -2.368598e-03
21 5.871252e-02 5.979878e-02 -1.086264e-03
22 7.983412e-02 7.882567e-02  1.008450e-03
23 9.936850e-02 9.596169e-02  3.406813e-03
24 1.132867e-01 1.079569e-01  5.329846e-03
25 1.183418e-01 1.122752e-01  6.066676e-03
26 1.132867e-01 1.079569e-01  5.329846e-03
27 9.936850e-02 9.596169e-02  3.406813e-03
28 7.983412e-02 7.882567e-02  1.008450e-03
29 5.871252e-02 5.979878e-02 -1.086264e-03
30 3.949055e-02 4.185915e-02 -2.368598e-03
31 2.426455e-02 2.700590e-02 -2.741355e-03
32 1.359978e-02 1.603475e-02 -2.434972e-03
33 6.940505e-03 8.746230e-03 -1.805724e-03
34 3.218140e-03 4.373115e-03 -1.154975e-03
35 1.352206e-03 1.999138e-03 -6.469324e-04
36 5.132843e-04 8.329743e-04 -3.196899e-04
37 1.753678e-04 3.151795e-04 -1.398117e-04
38 5.369232e-05 1.078246e-04 -5.413223e-05
39 1.465436e-05 3.317678e-05 -1.852243e-05
40 3.542996e-06 9.123616e-06 -5.580620e-06
41 7.529897e-07 2.225272e-06 -1.472282e-06
42 1.393520e-07 4.768440e-07 -3.374920e-07
43 2.219176e-08 8.871517e-08 -6.652341e-08
44 2.995144e-09 1.411378e-08 -1.111863e-08
45 3.357821e-10 1.881837e-09 -1.546055e-09
46 3.041505e-11 2.045475e-10 -1.741324e-10
47 2.137882e-12 1.740830e-11 -1.527041e-11
48 1.093725e-13 1.088019e-12 -9.786460e-13
49 3.621547e-15 4.440892e-14 -4.078737e-14
50 5.823448e-17 8.881784e-16 -8.299439e-16


 Maximum Error = 0.006066676
```

```
> listHB(1000,500,50)
Binomial Approximation to Hypergeometric
 Pop. Size N = 1000
 No. Successes in Pop., M = 500
 No. failures in Pop., L = 500
 Proportion of Successes p = M / N = 0.5
 Sample Size n = 50
 Sample Frction of Population, n / N = 0.05
 Devore's Rule of Thumb  IS  satisfied.

          dhyper       dbinom    difference
0  2.446417e-16 8.881784e-16 -6.435368e-16
1  1.356107e-14 4.440892e-14 -3.084785e-14
2  3.667939e-13 1.088019e-12 -7.212247e-13
3  6.451686e-12 1.740830e-11 -1.095661e-11
4  8.298729e-11 2.045475e-10 -1.215602e-10
5  8.322805e-10 1.881837e-09 -1.049556e-09
6  6.775968e-09 1.411378e-08 -7.337809e-09
7  4.604015e-08 8.871517e-08 -4.267502e-08
8  2.663769e-07 4.768440e-07 -2.104671e-07
9  1.332465e-06 2.225272e-06 -8.928071e-07
10 5.831272e-06 9.123616e-06 -3.292343e-06
11 2.253854e-05 3.317678e-05 -1.063824e-05
12 7.753112e-05 1.078246e-04 -3.029343e-05
13 2.388664e-04 3.151795e-04 -7.631303e-05
14 6.625822e-04 8.329743e-04 -1.703921e-04
15 1.662013e-03 1.999138e-03 -3.371256e-04
16 3.783888e-03 4.373115e-03 -5.892274e-04
17 7.843262e-03 8.746230e-03 -9.029683e-04
18 1.484019e-02 1.603475e-02 -1.194567e-03
19 2.568680e-02 2.700590e-02 -1.319104e-03
20 4.074637e-02 4.185915e-02 -1.112782e-03
21 5.932137e-02 5.979878e-02 -4.774123e-04
22 7.935605e-02 7.882567e-02  5.303754e-04
23 9.762858e-02 9.596169e-02  1.666894e-03
24 1.105273e-01 1.079569e-01  2.570396e-03
25 1.151904e-01 1.122752e-01  2.915208e-03
26 1.105273e-01 1.079569e-01  2.570396e-03
27 9.762858e-02 9.596169e-02  1.666894e-03
28 7.935605e-02 7.882567e-02  5.303754e-04
29 5.932137e-02 5.979878e-02 -4.774123e-04
30 4.074637e-02 4.185915e-02 -1.112782e-03
31 2.568680e-02 2.700590e-02 -1.319104e-03
32 1.484019e-02 1.603475e-02 -1.194567e-03
33 7.843262e-03 8.746230e-03 -9.029683e-04
34 3.783888e-03 4.373115e-03 -5.892274e-04
35 1.662013e-03 1.999138e-03 -3.371256e-04
36 6.625822e-04 8.329743e-04 -1.703921e-04
37 2.388664e-04 3.151795e-04 -7.631303e-05
38 7.753112e-05 1.078246e-04 -3.029343e-05
39 2.253854e-05 3.317678e-05 -1.063824e-05
40 5.831272e-06 9.123616e-06 -3.292343e-06
```

```
41 1.332465e-06 2.225272e-06 -8.928071e-07
42 2.663769e-07 4.768440e-07 -2.104671e-07
43 4.604015e-08 8.871517e-08 -4.267502e-08
44 6.775968e-09 1.411378e-08 -7.337809e-09
45 8.322805e-10 1.881837e-09 -1.049556e-09
46 8.298729e-11 2.045475e-10 -1.215602e-10
47 6.451686e-12 1.740830e-11 -1.095661e-11
48 3.667939e-13 1.088019e-12 -7.212247e-13
49 1.356107e-14 4.440892e-14 -3.084785e-14
50 2.446417e-16 8.881784e-16 -6.435368e-16


 Maximum Error = 0.002915208

> listHB(100,10,6)
Binomial Approximation to Hypergeometric
 Pop. Size N = 100
 No. Successes in Pop., M = 10
 No. failures in Pop., L = 90
 Proportion of Successes p = M / N = 0.1
 Sample Size n = 6
 Sample Frction of Population, n / N = 0.06
 Devore's Rule of Thumb  IS NOT  satisfied.

        dhyper   dbinom    difference
0 5.223047e-01 0.531441 -9.136251e-03
1 3.686857e-01 0.354294  1.439171e-02
2 9.645847e-02 0.098415 -1.956531e-03
3 1.182633e-02 0.014580 -2.753674e-03
4 7.055478e-04 0.001215 -5.094522e-04
5 1.902601e-05 0.000054 -3.497399e-05
6 1.761668e-07 0.000001 -8.238332e-07


 Maximum Error = 0.01439171
```

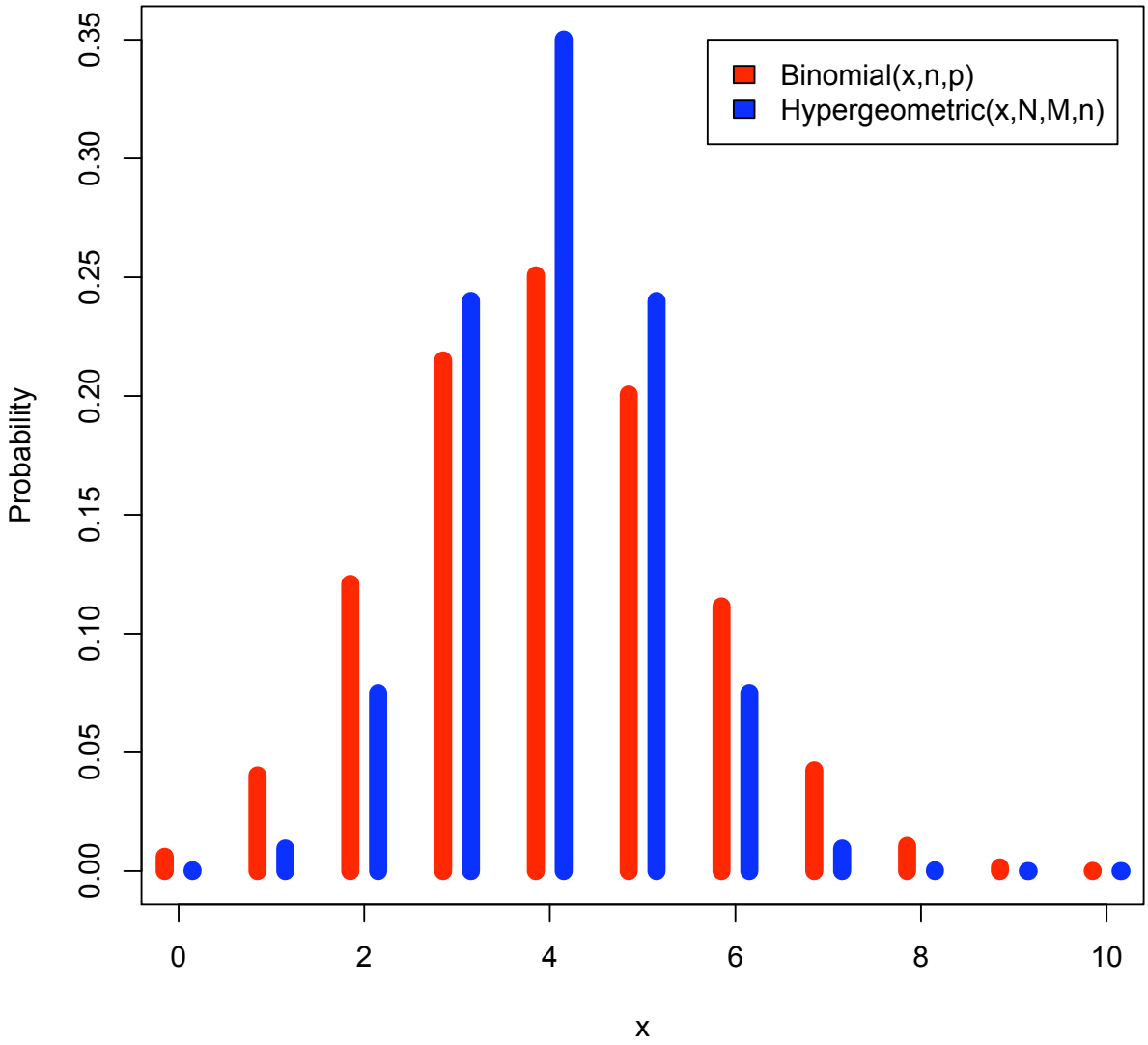
```
> listHB(200,10,6)
Binomial Approximation to Hypergeometric
 Pop. Size N = 200
 No. Successes in Pop., M = 10
 No. failures in Pop., L = 190
 Proportion of Successes p = M / N = 0.05
 Sample Size n = 6
 Sample Frction of Population, n / N = 0.03
 Devore's Rule of Thumb  IS  satisfied.

        dhyper       dbinom    difference
0 7.321404e-01 7.350919e-01 -2.951507e-03
1 2.374509e-01 2.321343e-01  5.316654e-03
2 2.872390e-02 3.054398e-02 -1.820081e-03
3 1.638440e-03 2.143438e-03 -5.049974e-04
4 4.575431e-05 8.460937e-05 -3.885506e-05
5 5.810071e-07 1.781250e-06 -1.200243e-06
6 2.548277e-09 1.562500e-08 -1.307672e-08


 Maximum Error = 0.005316654


> listHB(200,20,6)
Binomial Approximation to Hypergeometric
 Pop. Size N = 200
 No. Successes in Pop., M = 20
 No. failures in Pop., L = 180
 Proportion of Successes p = M / N = 0.1
 Sample Size n = 6
 Sample Frction of Population, n / N = 0.03
 Devore's Rule of Thumb  IS  satisfied.

        dhyper   dbinom    difference
0 5.269439e-01 0.531441 -4.497138e-03
1 3.613329e-01 0.354294  7.038934e-03
2 9.751883e-02 0.098415 -8.961684e-04
3 1.322289e-02 0.014580 -1.357108e-03
4 9.471454e-04 0.001215 -2.678546e-04
5 3.386442e-05 0.000054 -2.013558e-05
6 4.703391e-07 0.000001 -5.296609e-07


 Maximum Error = 0.007038934
```

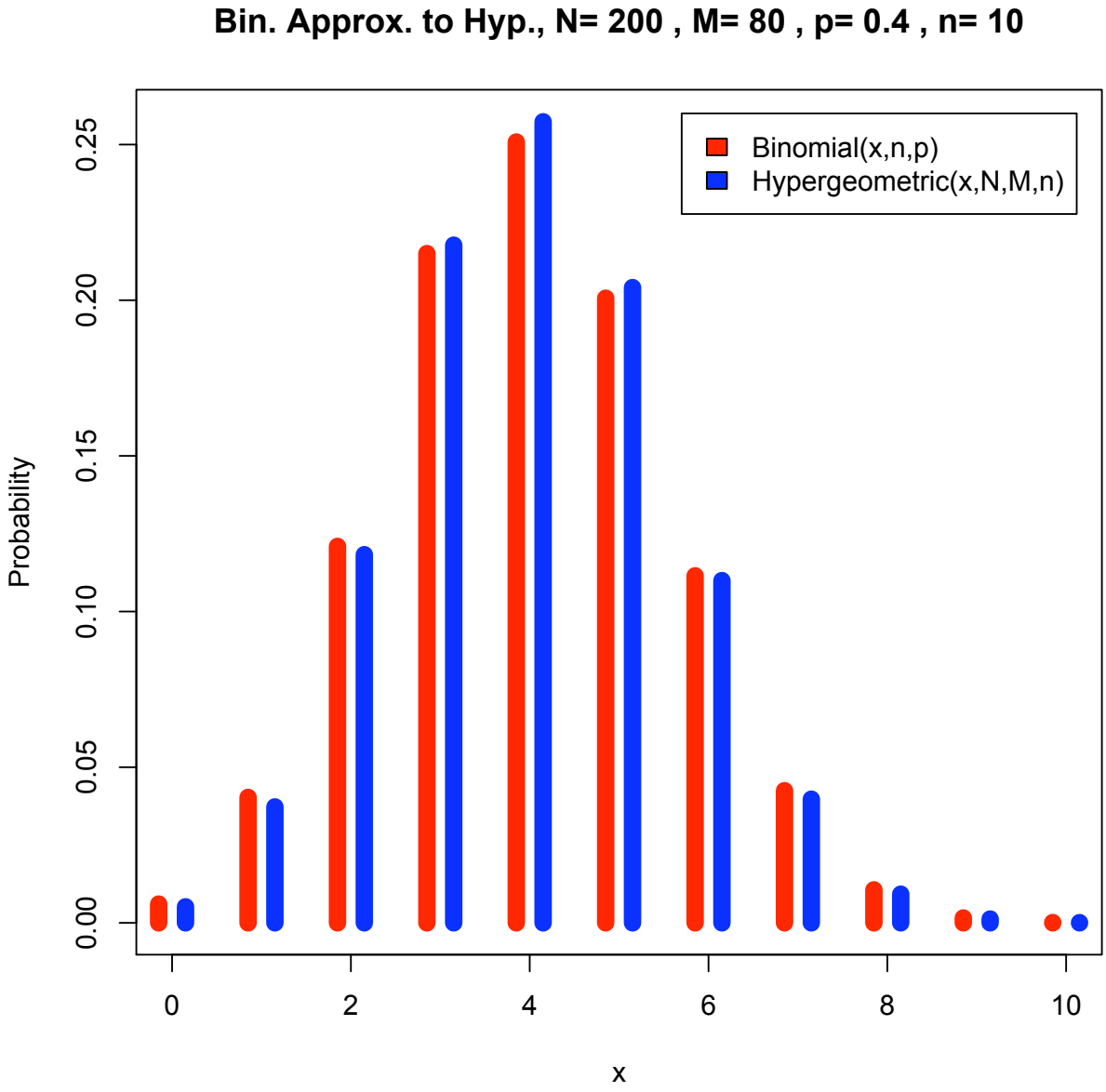
```
> listHB(200,100,6)
Binomial Approximation to Hypergeometric
 Pop. Size N = 200
 No. Successes in Pop., M = 100
 No. failures in Pop., L = 100
 Proportion of Successes p = M / N = 0.5
 Sample Size n = 6
 Sample Frction of Population, n / N = 0.03
 Devore's Rule of Thumb  IS  satisfied.

      dhyper   dbinom   difference
0 0.01446514 0.015625 -0.001159859
1 0.09135879 0.093750 -0.002391215
2 0.23553437 0.234375  0.001159368
3 0.31728341 0.312500  0.004783410
4 0.23553437 0.234375  0.001159368
5 0.09135879 0.093750 -0.002391215
6 0.01446514 0.015625 -0.001159859


 Maximum Error = 0.00478341
```

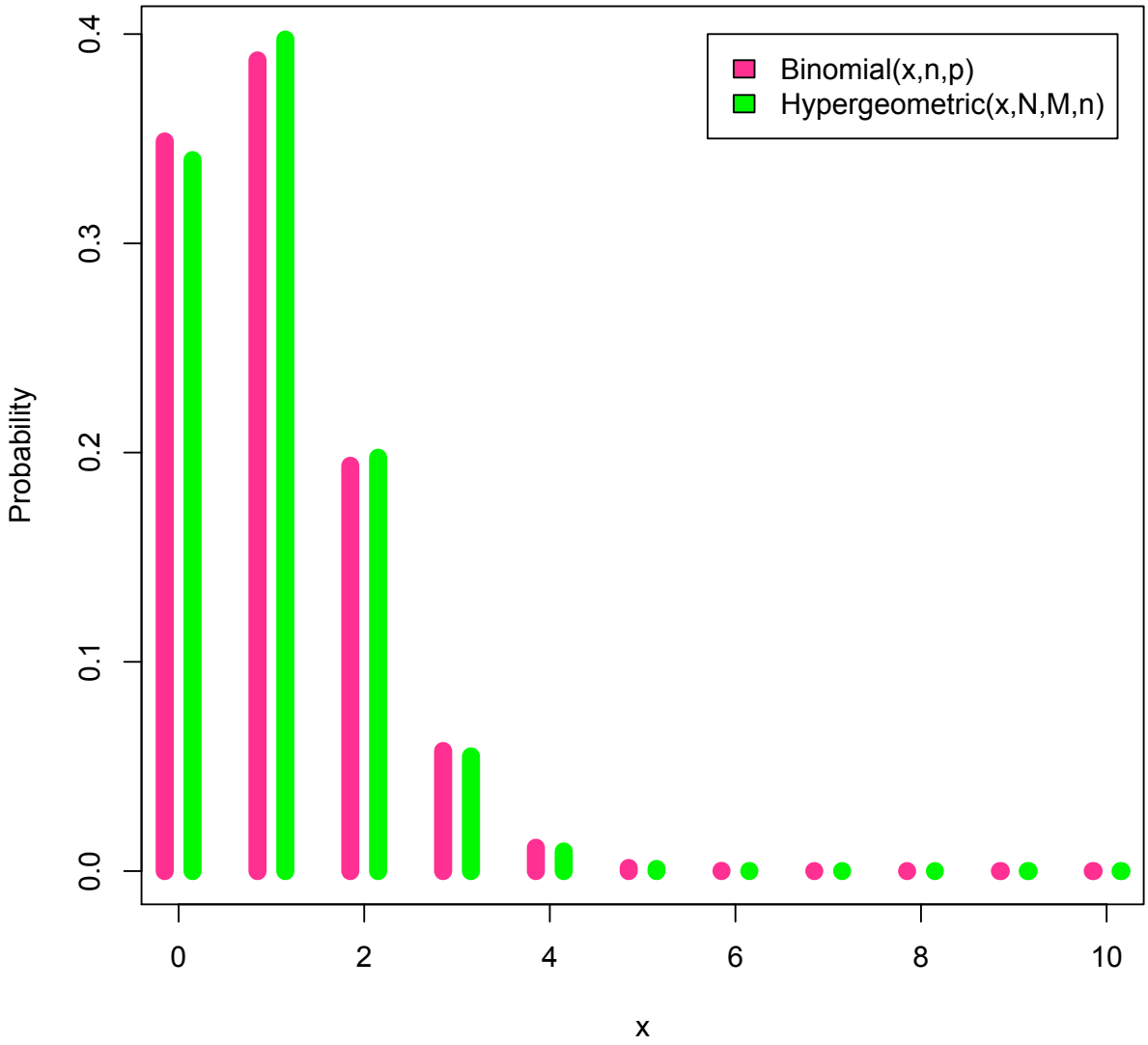
```
> ############## FUNCTION TO COMPUT ERROR OVER ENTIRE LIST ###################
> # To tabulate errors over various choices of N, M, n
> maxd <- function(N,M,n){}
> fix(maxd)
>
> maxd <- function(N,M,n)
+                        {
+                         L <- N-M
+                         p <- M/N
+                         max(abs(dhyper(0:n,M,L,n)-dbinom(0:n,n,p)))
+                        }

> maxd(200,10,6)
[1] 0.005316654


> Ns <- c(200,400,1000,2000)
> Ms <-.2*Ns
> ns <- c(5,10,20,50,100,200)
> Ms
[1]  40  80 200 400
> Ns
[1]  200  400 1000 2000
> ns
[1]   5  10  20  50 100 250

> y <- matrix(numeric(24),ncol=4,dimnames=list(ns,Ns))
> for(i in 1:6){
+               for(j in 1:4){
+                             y[i,j]<- maxd(Ns[j],Ms[j],ns[i])
+                            }
+              }
> y
>
> # n\N  Max Error Table for  p = .2
>
            200         400        1000         2000
5   0.005194780 0.002578546 0.001026953 0.0005127372
10  0.007809532 0.003838583 0.001520028 0.0007574849
20  0.011747070 0.005655574 0.002213311 0.0010987637
50  0.021512405 0.009630446 0.003628964 0.0017804114
100 0.040824680 0.015320001 0.005366314 0.0025784830
200 0.929630401 0.029039258 0.008297400 0.0038046684
```

```
> Ms <-.1*Ns
> for(i in 1:6){
+ for(j in 1:4){
+ y[i,j]<- maxd(Ns[j],Ms[j],ns[i])}}
> y
>
> # n\N  Max Error Table for  p = .1
>

            200         400        1000         2000
5   0.005567055 0.002758405 0.001097423 0.0005477289
10  0.009976442 0.004914302 0.001948439 0.0009713762
20  0.015285460 0.007376067 0.002890347 0.0014354635
50  0.028308379 0.012708635 0.004795623 0.0023537993
100 0.053836860 0.020292772 0.007119921 0.0034226446
200 0.906363689 0.038506820 0.011030217 0.0050604070
> Ms
[1]  20  40 100 200
> Ns
[1]  200  400 1000 2000
> Ms <-.05*Ns
> Ms
[1]  10  20  50 100
> for(i in 1:6){
+ for(j in 1:4){
+ y[i,j]<- maxd(Ns[j],Ms[j],ns[i])}}
> y
>
> # n\N  Max Error Table for  p = .05
>

            200         400         1000         2000
5   0.003823851 0.001893539 0.0007530712 0.0003758172
10  0.011644430 0.005707603 0.0022564285 0.0011238505
20  0.020043329 0.009717930 0.0038181004 0.0018978370
50  0.036886535 0.016458374 0.0061925967 0.0030368332
100 0.072452423 0.027559691 0.0097023964 0.0046684448
200 0.871642627 0.052411041 0.0150905960 0.0069306180
>
>
>
```

```
> ################### PLOT HYPERGEOMETRIC AND BINOMIAL PMF'S ##############
> N <- 20
> M <- 8
> L <- N-M
> p <- M/N; p
[1] 0.4
> n <- 10
> plot(c(0:n,0:n),c(dbinom(0:n,n,p), dhyper(0:n,M,L,n)), type = "n",
+      main = paste("Bin. Approx. to Hyp., N=", N, ", M=", M, ", p=", p,
+      ", n=",n), ylab = "Probability",xlab="x")
> points((0:n)-.15,dbinom(0:n,n,p), type = "h", col = "red", lwd=10)
> points((0:10)+.15,dhyper(0:n,M,L,n), type = "h", col = "blue", lwd=10)
> legend(5.7,.35,legend=c("Binomial(x,n,p)","Hypergeometric(x,N,M,n)"),
+        fill=c("red","blue"),bg="white")
> # M3074ApproxHyp1.pdf
>
> N <- 200
> M <- 80
> L <- N-M
> p <- M/N; p
[1] 0.4
> n <- 10
> plot(c(0:n,0:n),c(dbinom(0:n,n,p),dhyper(0:n,M,L,n)), type = "n",
+      main=paste("Bin. Approx. to Hyp., N=",N,", M=",M,", p=",p,",
+      n=",n),ylab="Probability",xlab="x")
> points((0:n)-.15,dbinom(0:n,n,p), type = "h", col = "red", lwd=10)
>
> points((0:10)+.15,dhyper(0:n,M,L,n), type = "h", col = "blue", lwd=10)
> legend(5.7,.26,legend=c("Binomial(x,n,p)","Hypergeometric(x,N,M,n)"),
+        fill=c("red","blue"),bg="white")
> # M3074ApproxHyp2.pdf
>
> N <- 200
> M <- 20
> L <- N-M
> p <- M/N; p
[1] 0.1
> n <- 10
> plot(c(0:n,0:n), c(dbinom(0:n,n,p), dhyper(0:n,M,L,n)), type = "n",
+      main = paste("Bin. Approx. to Hyp., N=", N, ", M=", M, ", p=", p,
+      ", n=", n), ylab = "Probability", xlab="x")
> points((0:n)-.15,dbinom(0:n,n,p), type = "h", col = clr[12], lwd=10)
> points((0:10)+.15,dhyper(0:n,M,L,n), type = "h", col = clr[5], lwd=10)
> legend(5.7,.4, legend = c("Binomial(x,n,p)", "Hypergeometric(x,N,M,n)"),
+        fill = clr[c(12,5)], bg="white")
> #M3074ApproxHyp3.pdf
```


Bin. Approx. to Hyp., N= 20 , M= 8 , p= 0.4 , n= 10
Binomial(x,n,p)
Hypergeometric(x,N,M,n)
Probability
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0
2
4
6
8
10
x


Bin. Approx. to Hyp., N= 200 , M= 80 , p= 0.4 , n= 10
Binomial(x,n,p)
Hypergeometric(x,N,M,n)
Probability
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
10
x


Bin. Approx. to Hyp., N= 200 , M= 20 , p= 0.1 , n= 10
Binomial(x,n,p)
Hypergeometric(x,N,M,n)
Probability
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
10
x